KITP-UCSB – 27 June, 2012

Recent progress in topological fluid dynamics: from helicity to Jones polynomials

Renzo L. Ricca

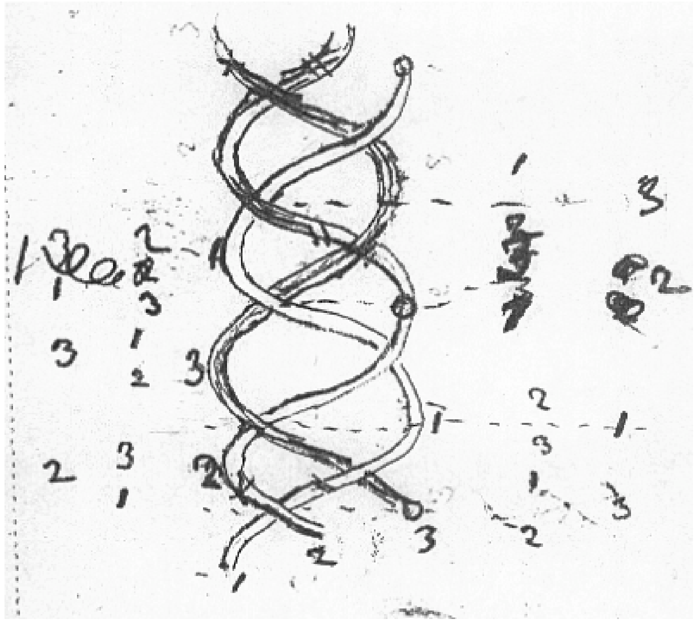
Department of Mathematics & Applications, U. Milano-Bicocca

renzo.ricca@unimib.it

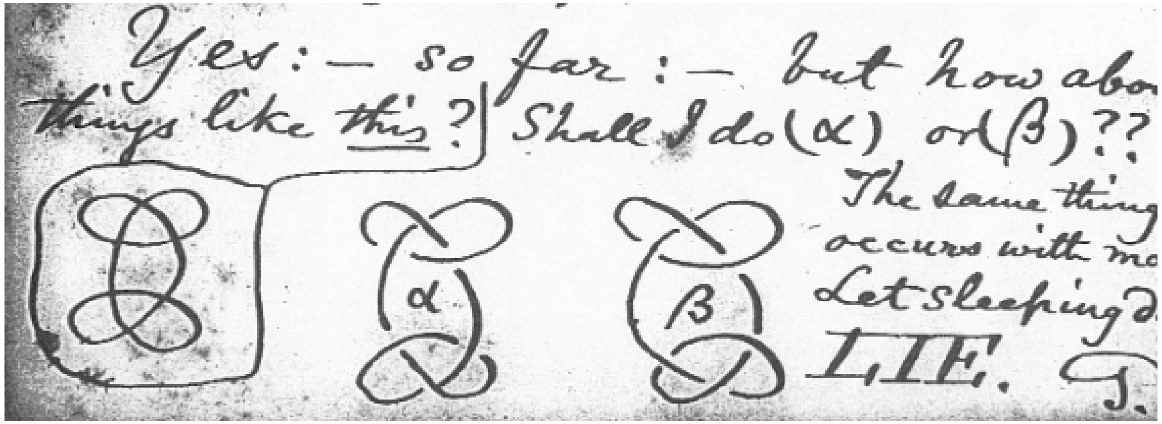
Outline

- *150 years of topological dynamics:*
  - *diffeomorphisms of frozen fields and helicity.*
- *Magnetic fields under energy relaxation:*
  - *from inflexional knots to inflexion-free braids;*
  - *groundstate energy spectrum of first 250 prime knots;*
  - *new lower bounds on energy.*
- *Vortex dynamics:*
  - *torus knot solutions under LIA and Biot-Savart law;*
  - *tangle analysis on energy-complexity relations;*
  - *Jones polynomial for fluid knots from helicity.*

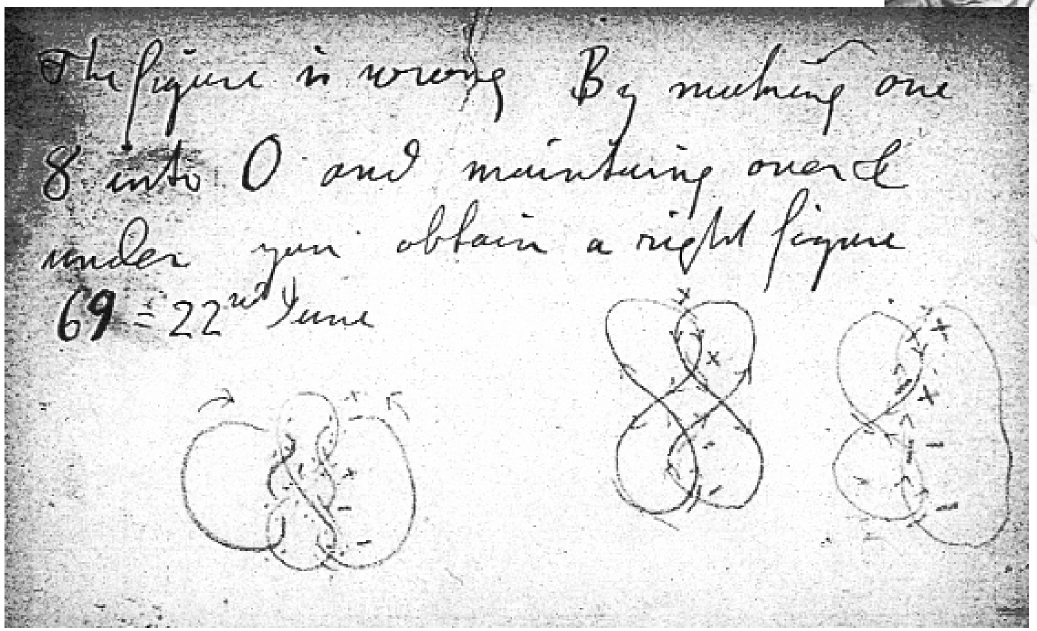
Maxwell behind the scene

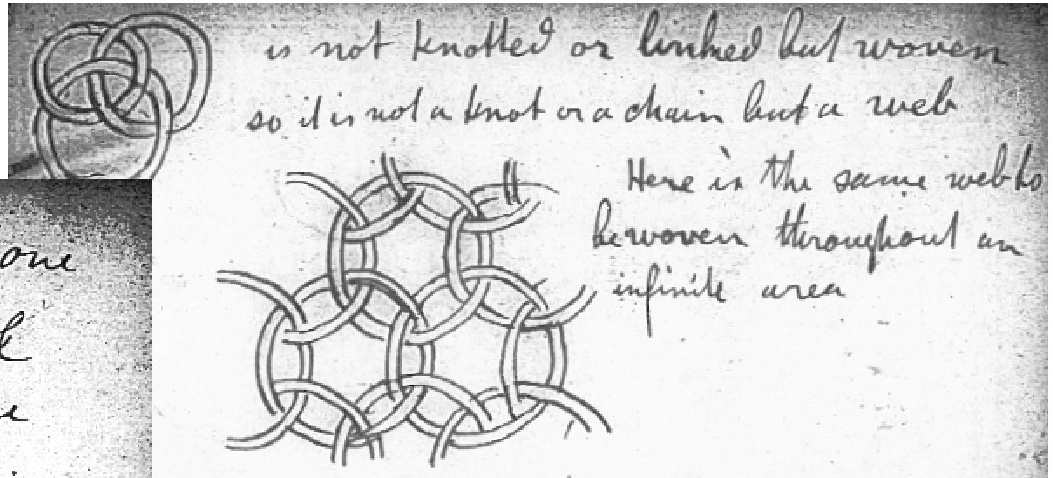


(Lord Kelvin, 1867)



(Tait, 1884)





(Maxwell, 1877)

(Ricca & Weber, in preparation)

150 years of topological dynamics

Linking number formula
(Gauss, 1833)

Applications to magnetic fields
(Maxwell, 1867)

Knot tabulation
(Tait, 1877)

Applications to vortices
(Kelvin, 1867)

"topological fluid mechanics"

- **Knotted solutions to Euler's equations**
  - 3-D fluid topology
  - vortex solutions
  - fluid invariants
  - topological stability
- **Energy relaxation methods**
  - magnetic knots
  - "charged" knots
  - groundstate energy
- **Dynamical systems and μ-preserving flows**
  - ∃ Theorems for vector fields
  - closed and cahotic orbits
  - Hamiltonian structures
- **Change of topology**
  - reconnection mechanisms
  - singularity formation

Diffeomorphisms of frozen fields

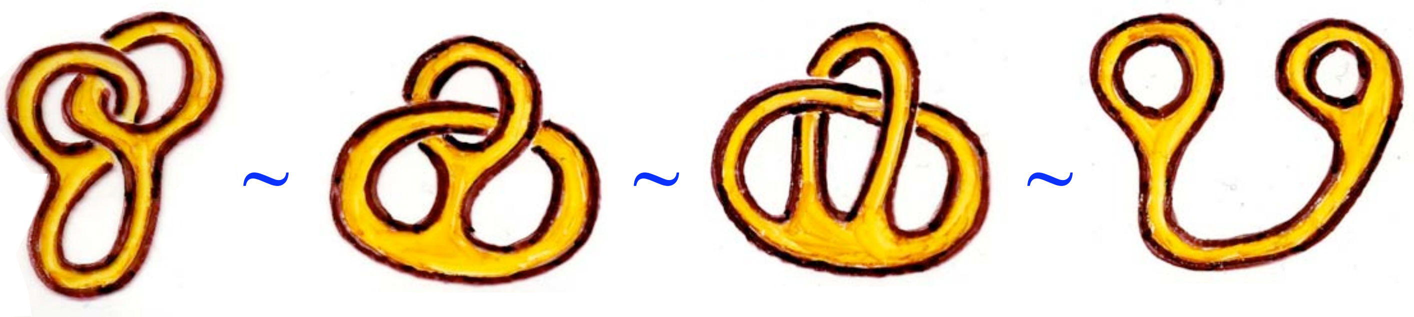
- ***ideal, incompressible*** / ***perfectly conducting*** } ***fluid in*** $\mathbb{R}^3$***:*** $\quad \mathbf{u} = \mathbf{u}(\mathbf{X}, t) \begin{cases} \nabla \cdot \mathbf{u} = 0 & \textit{in } \mathbb{R}^3 \\ \mathbf{u} = 0 & \textit{as } \mathbf{X} \to \infty \end{cases}$

- ***frozen field evolution:***

$$\mathbf{B}(\mathbf{X},t) \in \left\{ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) \ \wedge \ \nabla \cdot \mathbf{B} = 0; L_2 - \mathrm{norm} \right\}$$

- ***topological equivalence class:***

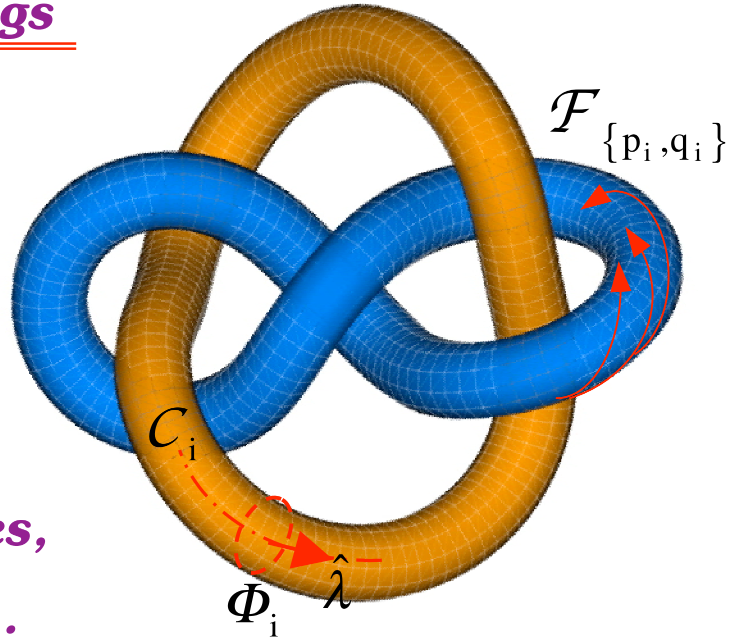
$$B_i(\mathbf{X},t) = B_j(\mathbf{X}_0,0)\frac{\partial X_i}{\partial X_{0j}} \quad : \quad \mathbf{B}(\mathbf{X}_0,0) \sim \mathbf{B}(\mathbf{X},t)$$



Physical knots and links as tubular embeddings

Let $\mathcal{T}_i = \mathcal{S}_i \otimes \mathcal{C}_i$ ***and*** $V_i = V(\mathcal{T}_i)$:

$$\mathcal{T}_i \to \mathcal{K}_i \quad \textit{in} \quad \mathbb{R}^3$$



- ***magnetic embedding:***

$$\mathcal{K}_i := \mathrm{supp}(\mathbf{B})$$

by a standard foliation $\mathcal{F}_{\{p_i,q_i\}}$ ***of the*** **B*-lines,***
such that $\mathbf{B} \cdot \hat{\boldsymbol{\nu}} = 0$ ***on*** $\partial\mathcal{T}_i$ ***(material surface).***

- ***Definition: A physical knot/link is a smooth immersion into*** $\mathbb{R}^3$ ***of finitely many disjoint standard solid tori*** $\mathcal{T}_i$***, such that***

$$\mathrm{supp}(\mathbf{B}) := \bigcup_i \mathcal{K}_i \to \mathcal{L}_n \qquad (\mathrm{i}=1,\ldots,\mathrm{n}) .$$

- ***volume and flux-preserving diffeomorphism:***

$$V = V(\mathcal{L}_n), \qquad \Phi_i = \int_{A(\mathcal{S}_i)} \mathbf{B} \cdot \hat{\lambda} \; \mathrm{d}^2\mathbf{x} \; :$$

signature $\{V, \Phi_i\}$ ***constant.***

Helicity and linking numbers

- ***Magnetic helicity*** $H(t)$ **:**

$$H(t) := \int_{V(\mathcal{L}_\mathrm{n})} \mathbf{A}\cdot\mathbf{B}\ \mathrm{d}^3\mathbf{X}$$

where $\mathbf{B}=\nabla\times\mathbf{A}$, ***with*** $\nabla\cdot\mathbf{A}=0$ ***in*** $\mathbb{R}^3$.

- ***Theorem* (Woltjer 1958). *In ideal fluid magnetic helicity is frozen in the flow, that is***

$$\frac{\mathrm{d}}{\mathrm{d}t} H\left(\mathcal{L}_{\mathrm{n},\varphi}\right) = 0 \quad \Rightarrow \quad H(t) = H\ .$$

- ***Theorem* (Moffatt 1969; Moffatt & Ricca 1992). *Let*** $\mathcal{L}_\mathrm{n}$ ***be an essential magnetic link in an ideal fluid. Then, we have***

$$H = \int_{V(\mathcal{L}_\mathrm{n})} \boldsymbol{A}\cdot\boldsymbol{B}\ \mathrm{d}^3\boldsymbol{X} = \sum_{\mathrm{i}} Lk_{\mathrm{i}}\Phi_{\mathrm{i}}^{\ 2} + 2\sum_{\mathrm{i}\neq\mathrm{j}} Lk_{\mathrm{ij}}\Phi_{\mathrm{i}}\Phi_{\mathrm{j}}$$

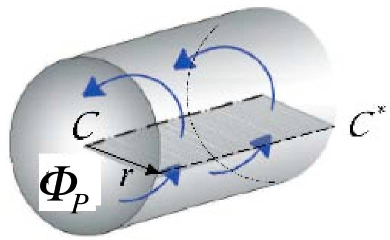
$$= \sum_{\mathrm{i}} (Wr + Tw)\Phi_{\mathrm{i}}^{\ 2} + 2\sum_{\mathrm{i}\neq\mathrm{j}} Lk_{\mathrm{ij}}\Phi_{\mathrm{i}}\Phi_{\mathrm{j}}\ .$$

Lorentz force on magnetic flux tubes in ideal MHD

Ideal magnetohydrodynamics (MHD): tubular knot $K\ :\ V(K)=\pi a^2\cdot L$;

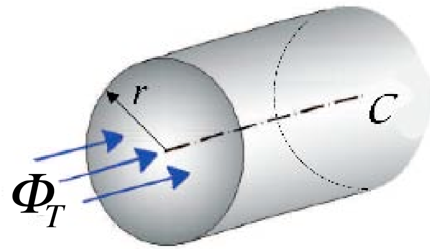
- **magnetic field in cylindrical coordinates** (r,ϑ,s):

$$\mathbf{B}=\mathbf{B}_{\mathrm{m}}+\mathbf{B}_{\mathrm{a}}=\left(0,\ B_{\vartheta}(r),\ B_{s}(r)\right)\ ;$$



in terms of fluxes Φ_P, Φ_T:

$$\mathbf{B}=\left(0,\frac{1}{L}\frac{\mathrm{d}\Phi_P}{\mathrm{d}r},\frac{1}{2\pi r}\frac{\mathrm{d}\Phi_T}{\mathrm{d}r}\right)+\left(0,\frac{\partial\tilde{\psi}}{\partial s},-\frac{\partial\tilde{\psi}}{\partial\vartheta_{\mathrm{R}}}\right)\quad ,$$



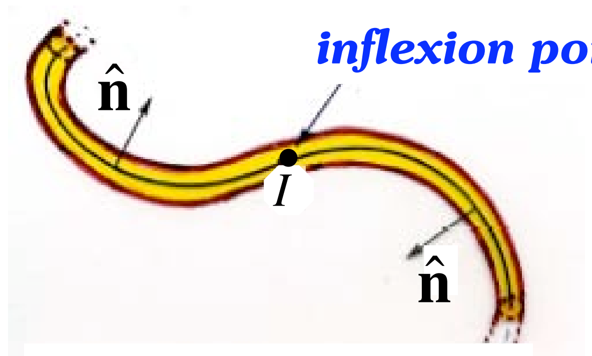
with twist parameter $h=\Phi_P/\Phi_T$.

- **Lorentz force:** $\mathbf{F}=\mathbf{J}\times\mathbf{B}=(\nabla\times\mathbf{B})\times\mathbf{B}$.

If $\mathbf{B}_{\mathrm{m}}\ll\mathbf{B}_{\mathrm{a}}$ **, then:**

$$\mathbf{F}\propto c\ \hat{\mathbf{n}}$$

"curvature flow"



planar configuration

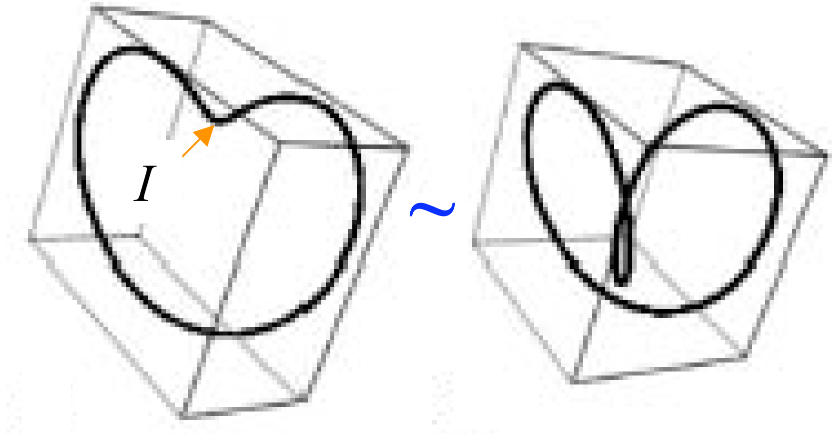
Inflexional configuration and Reidemeister type I move

- ***Inflexion at*** I ***(in isolation):*** $c = 0$

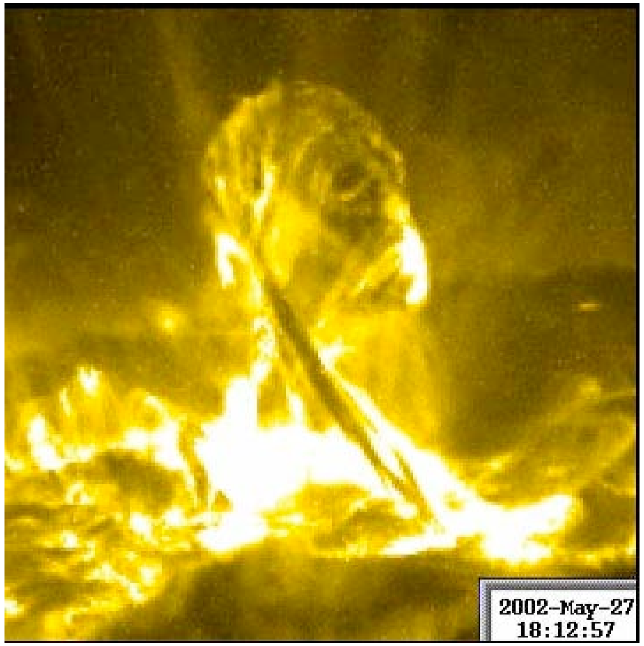
  generic behaviour in $\mathbb{R}^3$***:***

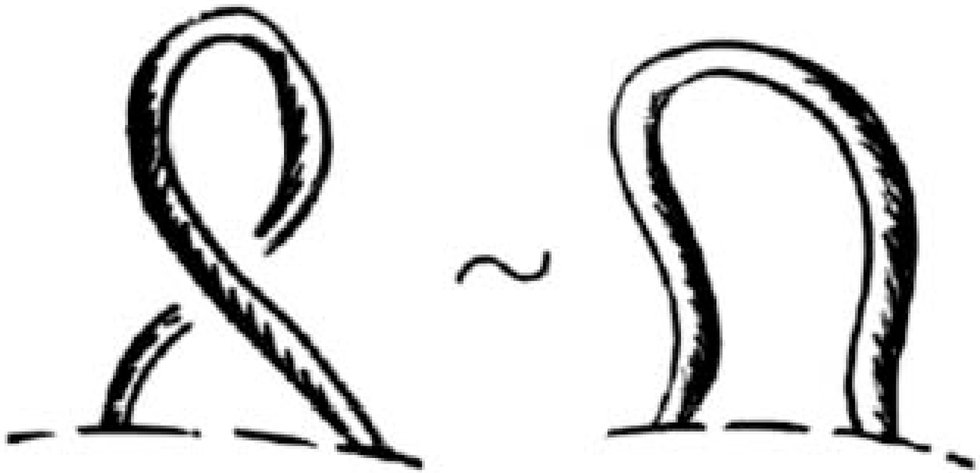
$$\mathbf{X}(s,t) = \left(s - \frac{2}{3}t^2 s^3, -ts^2, s^3 \right)$$

(Ricca & Moffatt, 1992).



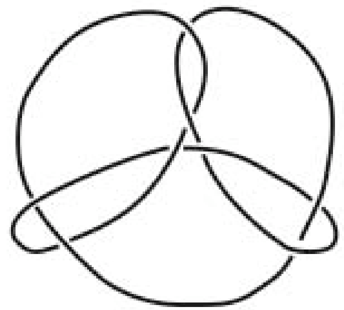
- ***Reidemeister type I move in action:***



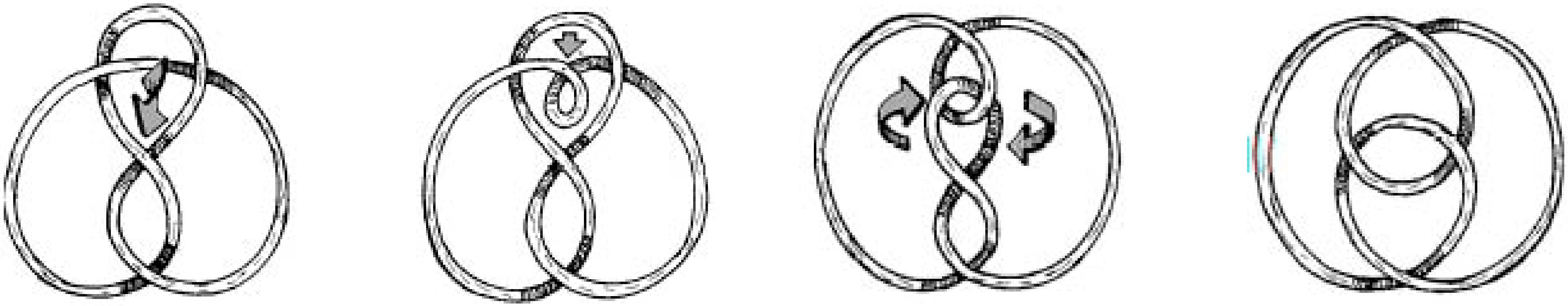


(TRACE, 2002)

From inflexion-free knots to braids

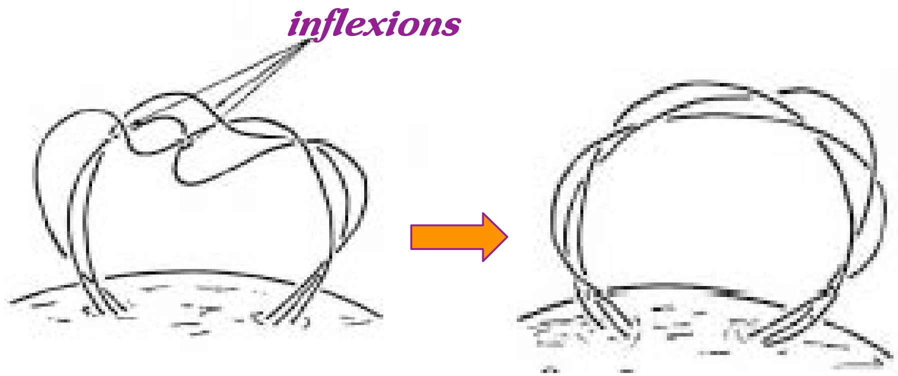


- **Definition.** A **spiral knot** is a knot free from inflexion points in isolation.
- Figure 8 knot in braid presentation:

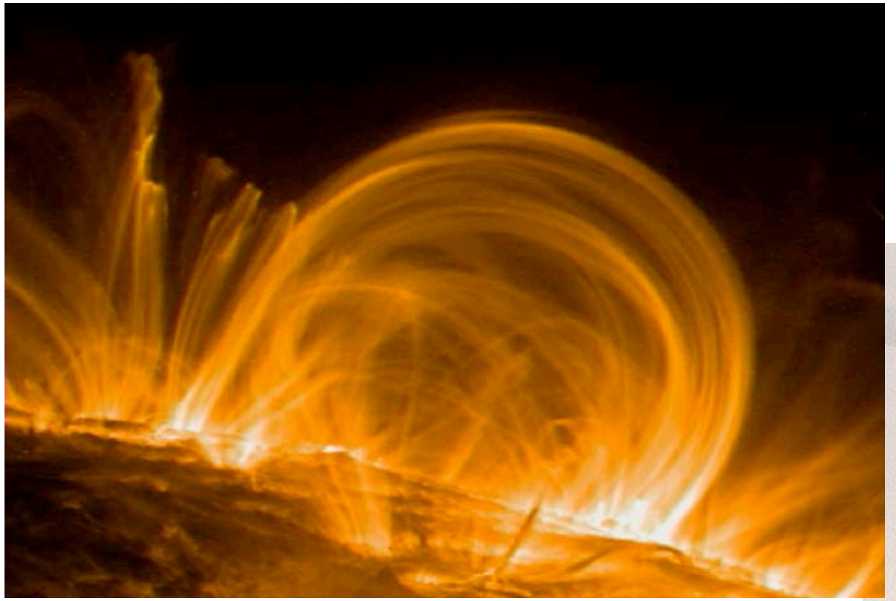


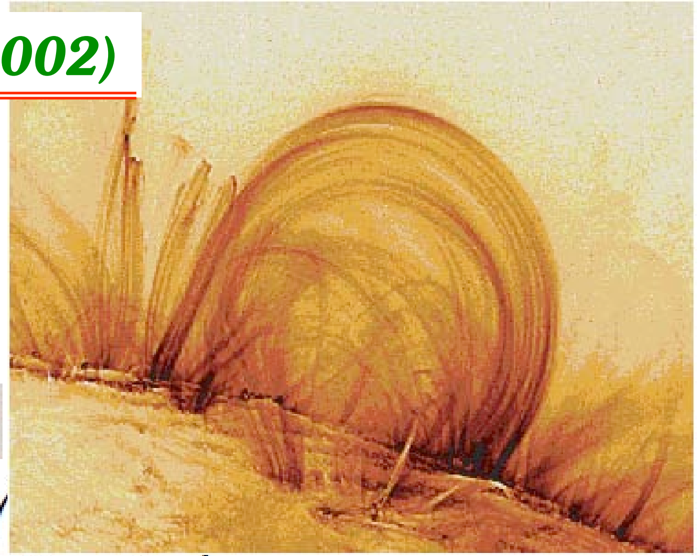
- **Theorem (Ricca, 2005).** Let $\tilde{\mathcal{K}}_{t_0}$ denote a loose magnetic knot in inflexional state. Then $\tilde{\mathcal{K}}_{t_0}$ is in inflexional disequilibrium.

- **Corollary.** $\tilde{\mathcal{K}}_{t_0}$ is naturally isotoped to a spiral knot $\mathcal{K}_t$ for any $t > t_0$.

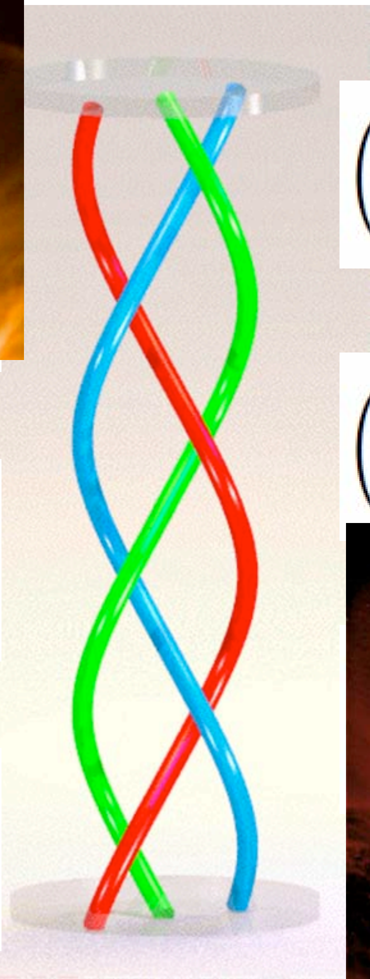


Magnetic braids in the solar corona (TRACE, 2002)





$$\begin{pmatrix} & \\ 0 & 1 \end{pmatrix}$$

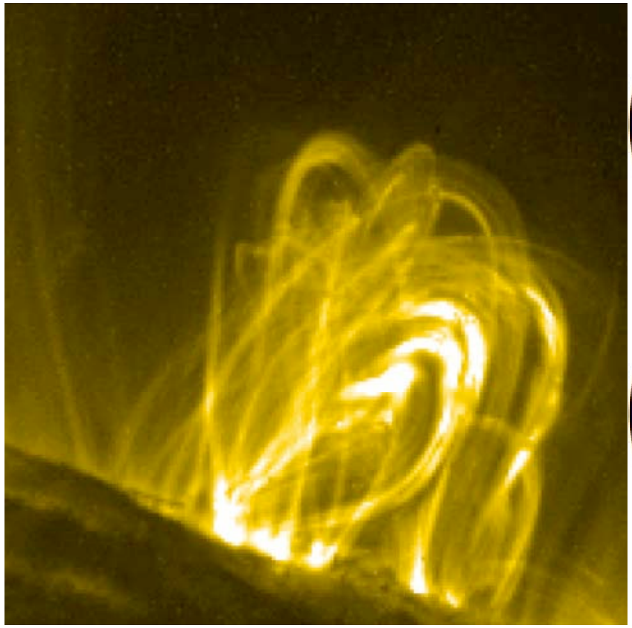


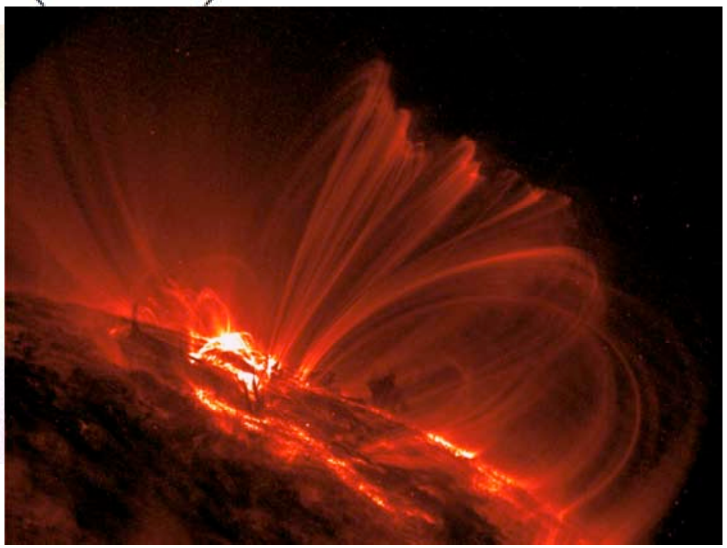
$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Mitch Berger
(U. Exeter)





Magnetic relaxation

Let $\mathcal{L}_n$ *be a zero-framed magnetic link:* $\begin{cases} n \text{ components} & Lk_i = 0 \quad \forall i \\ \text{equal flux} & \Phi_i = \Phi \end{cases}$

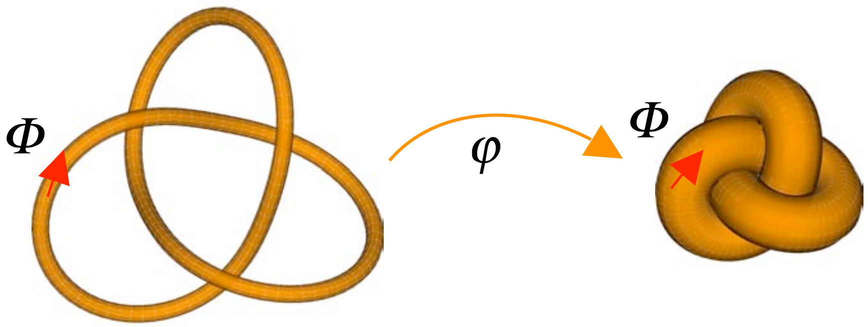
- *Magnetic energy:*

$$M(t) = \frac{1}{2}\int_{V(\mathcal{L}_n)} \|\mathbf{B}\|^2 \, \mathrm{d}^3\mathbf{x} \quad ,$$

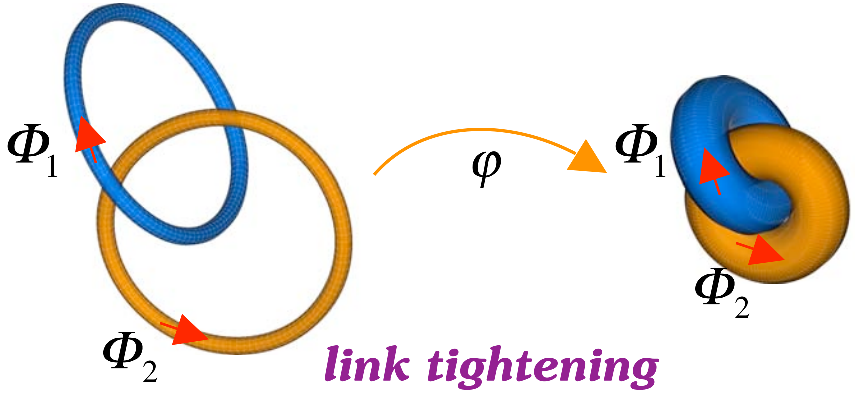
- *Magnetic helicity:*

$$H = \int_{V(\mathcal{L}_n)} \mathbf{A}\cdot\mathbf{B} \, \mathrm{d}^3\mathbf{x} = 2\Phi^2 \sum_{i\neq j} Lk_{ij} \; .$$

- *Magnetic relaxation (Moffatt, 1985) under* $\{V, \Phi_i\}$*-preserving flow:*



knot tightening



link tightening

- *Theorem 1 (Arnold, 1974; Freedman, 1988; Moffatt, 1990; Freedman & He, 1991; Ricca, 2008). Let* $\mathcal{L}_n$ *be a zero-framed link. Then,*

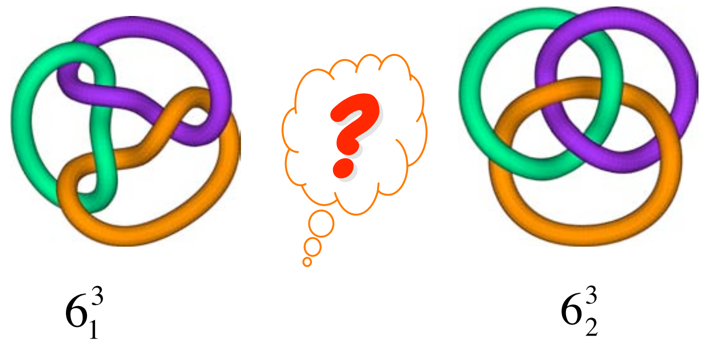
i) $$M(t) \geq \left(\frac{2}{\pi V}\right)^{1/3} |H| \quad ;$$

ii) $$M_{\min} = \left(\frac{2\Phi^6}{\pi V}\right)^{1/3} c_{\min} \quad .$$

Constrained minimization of magnetic energy of knots

Under signature-preserving flows, we have:

$$M_{\min} \propto c_{\min}$$



- **Assumptions:**

- *tubular knot* $\mathcal{K}$: $V(\mathcal{K}) = \pi a^2 L$; **Mercier** (orthogonal) *system:* (r, ϑ, s)
- *magnetic field:* $\mathbf{B} = (0,\ B_\vartheta(r),\ B_s(r))$ $\quad (\nabla \cdot \mathbf{B} = 0)$
- *fluxes* Φ_P, Φ_T ; **twist parameter:** $h = \Phi_P / \Phi_T$

- **Theorem (Maggioni & Ricca, 2009).** *Let us assume that*

  (i) $\{V, \Phi\}$ *invariant* (V=1, Φ=1);

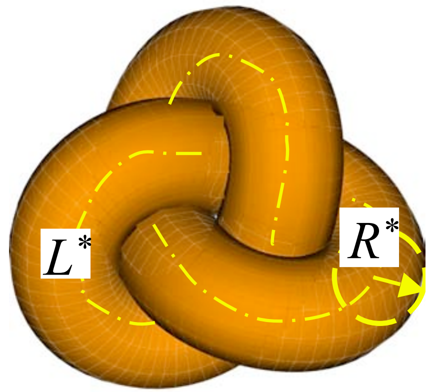
  (ii) circular cross-section independent of arc-length;

  (iii) $\tilde{\psi}$ *independent of arc-length;*

  (iv) L *independent of internal twist.*

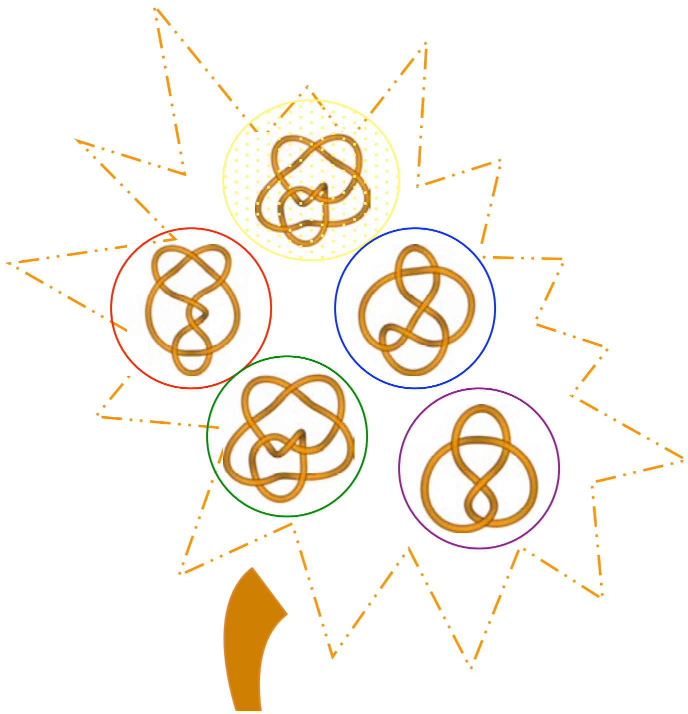
  Then,

$$M_\lambda^*(h) = \frac{(\lambda)^{4/3}}{2\pi^{2/3}} + \frac{\pi^{4/3} h^2}{(\lambda)^{2/3}} .$$



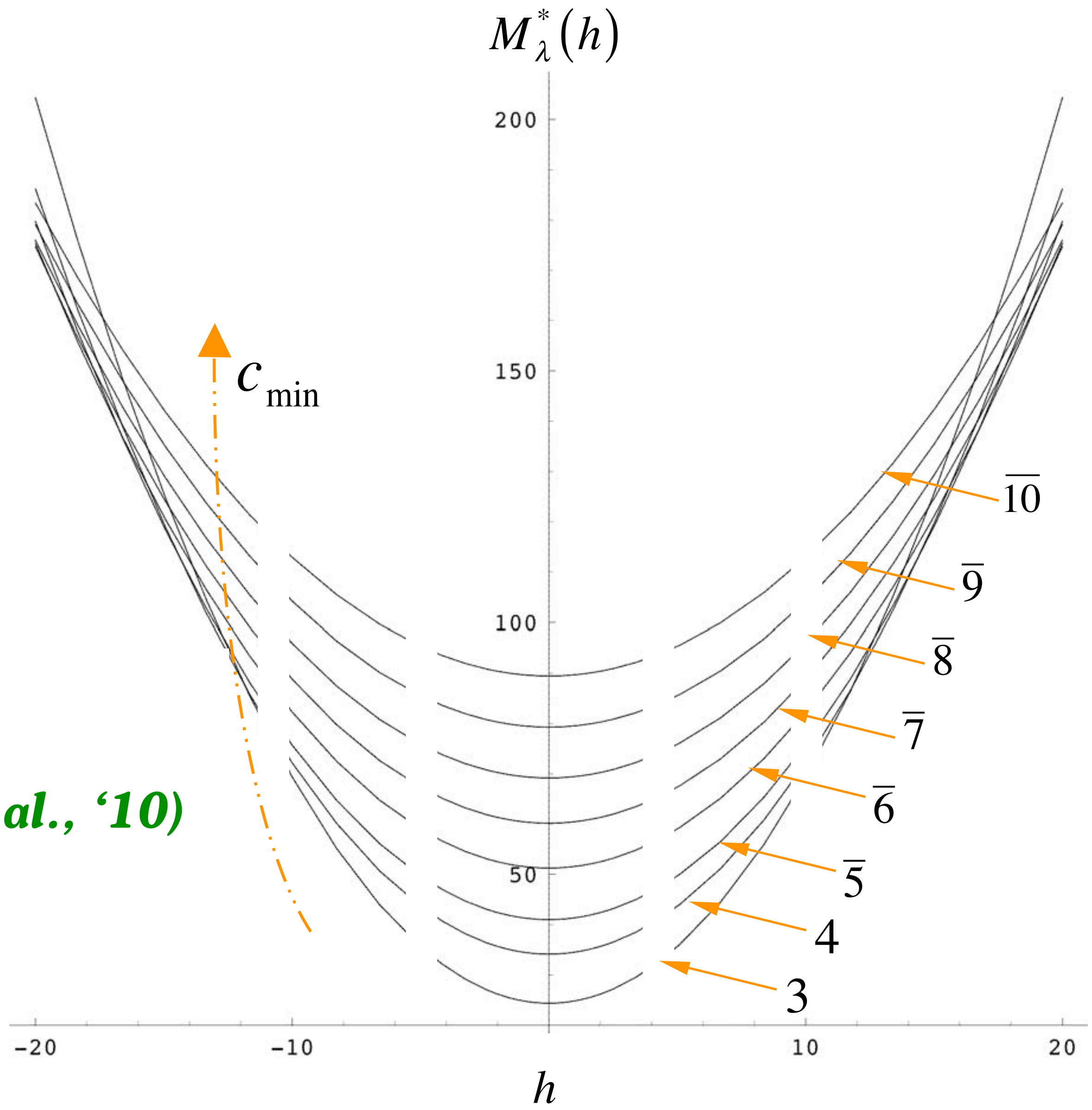
ropelength: $\lambda = L^*/R^*$

Groundstate energy spectrum: averaging over complexity



tightening ***(**$h = 0$**)***
SONO ***(Pieranski, '98)***
RIDGERUNNER ***(Ashton et al., '10)***

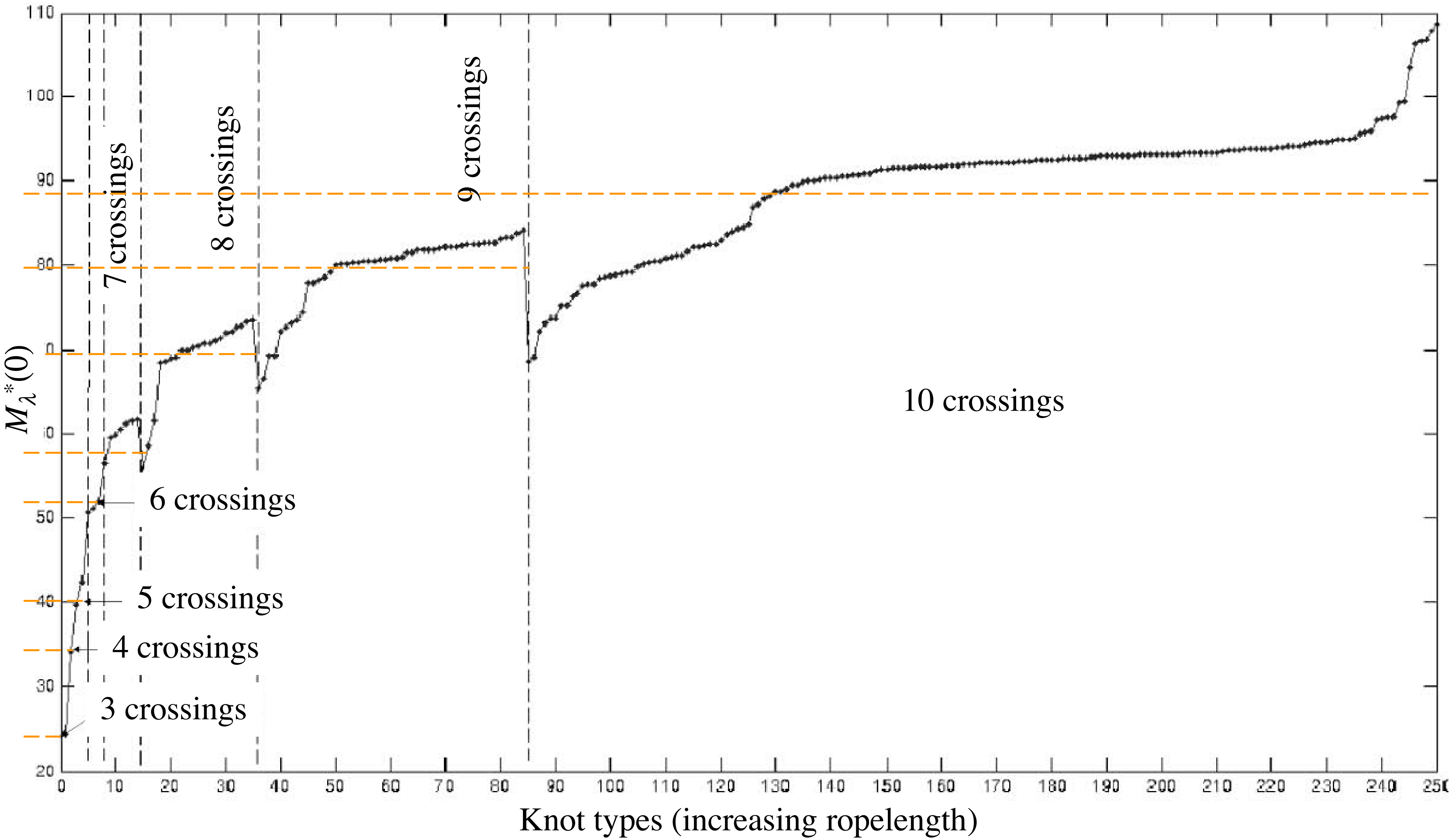
$V = 1$, $\Phi = 1$



(Maggioni & Ricca, Proc. Roy. Soc. A 465, 2009)

Groundstate energy spectrum of first 250 prime knots

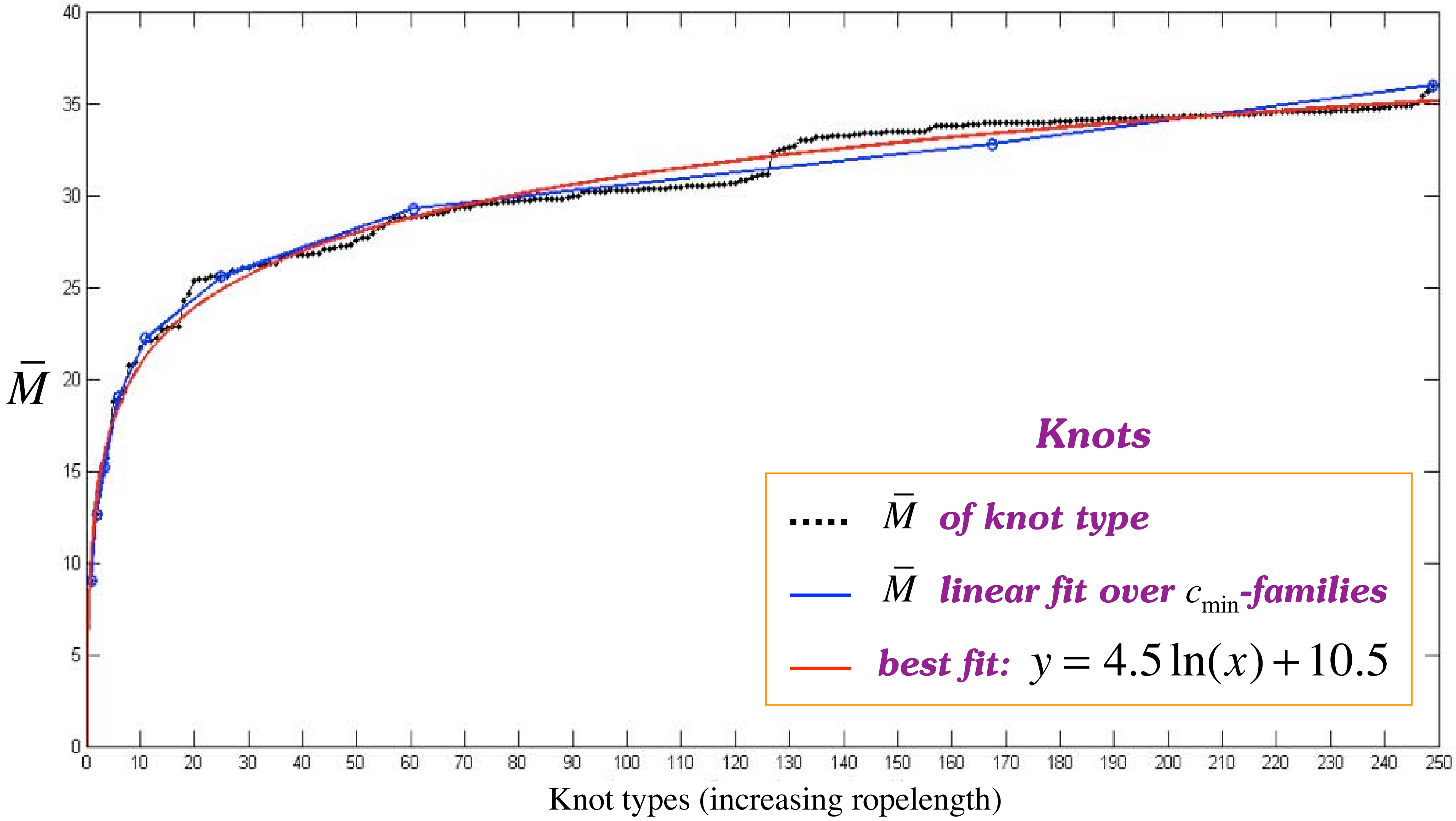
- $V = 1$, $\Phi = 1$, $h = 0$: $M_\lambda^*(0) = \frac{(\lambda)^{4/3}}{2\pi^{2/3}}$



Knot energy spectrum: normalized energy vs. ropelength

- $V = 1$, $\Phi = 1$, $h = 0$
- ***tight unknot:*** $M_o^* = (2\pi^2)^{1/3}$

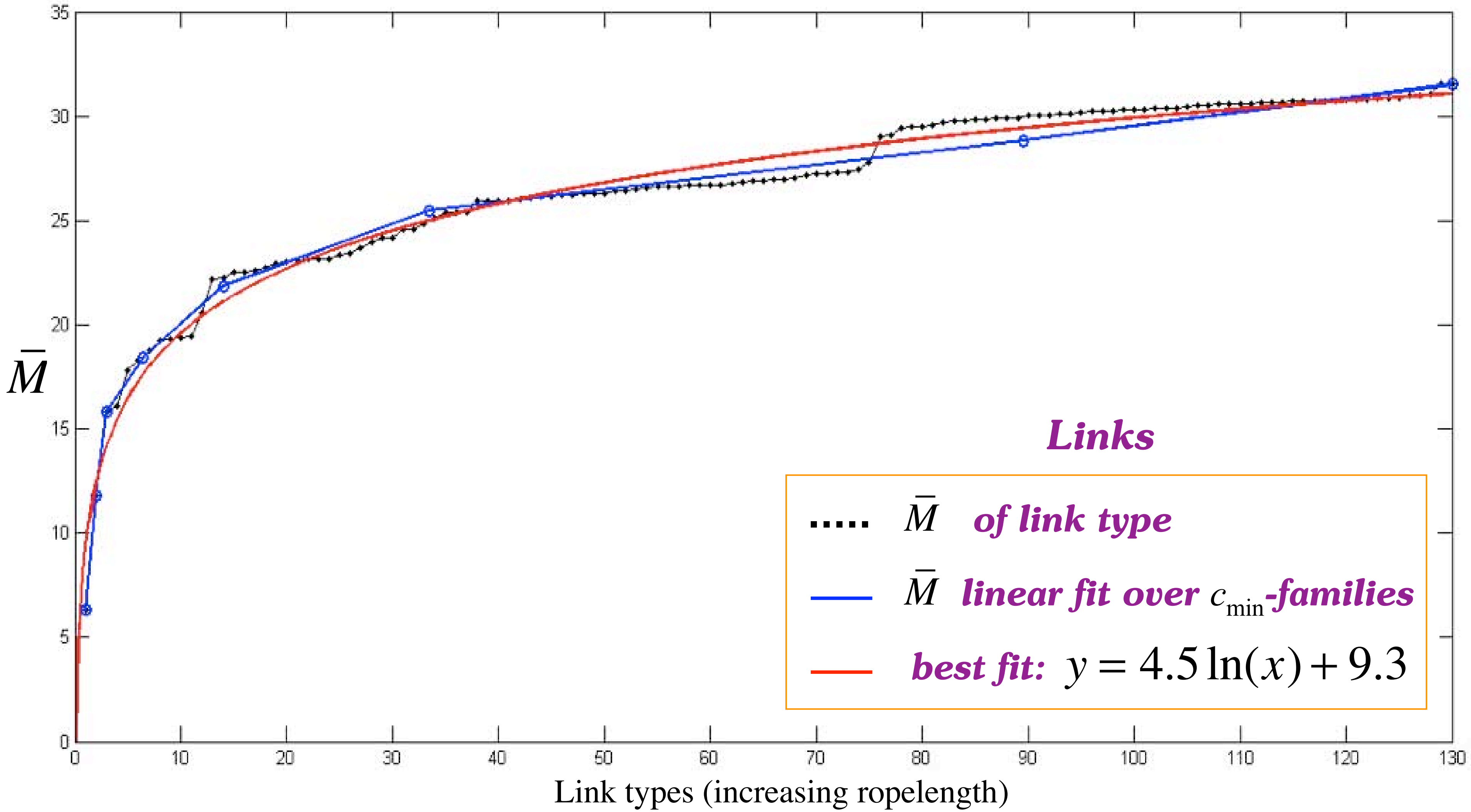
$$\bar{M} = \frac{M_\lambda^*(0)}{M_o^*} = \left(\frac{\lambda}{2\pi}\right)^{4/3}$$



Link energy spectrum: normalized energy vs. ropelength

- $V = 1,\ \Phi = 1,\ h = 0$
- ***tight unknot:*** $M_o^* = \left(2\pi^2\right)^{1/3}$

$$\bar{M} = \frac{M_\lambda^*(0)}{M_o^*} = \left(\frac{\lambda}{2\pi}\right)^{4/3}$$



New lower bounds for tight knots, links and braids

By comparing energy minima, we have:

$$M_\lambda^*(0) = \frac{\lambda^{4/3}}{2\pi^{2/3}} \geq M_{\min} = \left(\frac{2}{\pi}\right)^{1/3} c_{\min} \quad ,$$

hence:

$$\lambda \geq (16\pi)^{1/4}\, c_{\min}^{3/4} \approx \underline{2.66}\; c_{\min}^{3/4} \qquad \forall\, c_{\min} \quad .$$

Best results for constant so far:

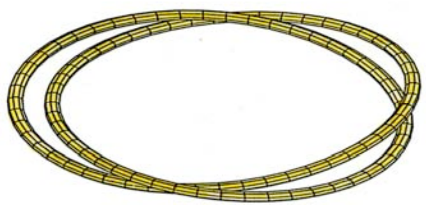
- **for any $c_{\min}$: Buck & Simon (1999):** $(4\pi/11)^{3/4} \approx \underline{1.10}$
- **for small $c_{\min}$: Denne, Diao & Sullivan (2000): …**
  Cantarella, Kusner & Sullivan (2002): …
- **for $c_{\min} = 3$: by SONO Baranska et al. (2004):** $\underline{14.04}$

- **Minimal braids $\mathcal{B}$: by using Ohyama (1993) inequality, we have:**

$$M_{\min} \geq \left(\frac{16}{\pi}\right)^{1/3} \left(b(\mathcal{B}) - 1\right), \qquad b(\mathcal{B})\text{: braid index of } \mathcal{B} .$$

Vortex knots and links

- **Knotted solutions to Euler equations**

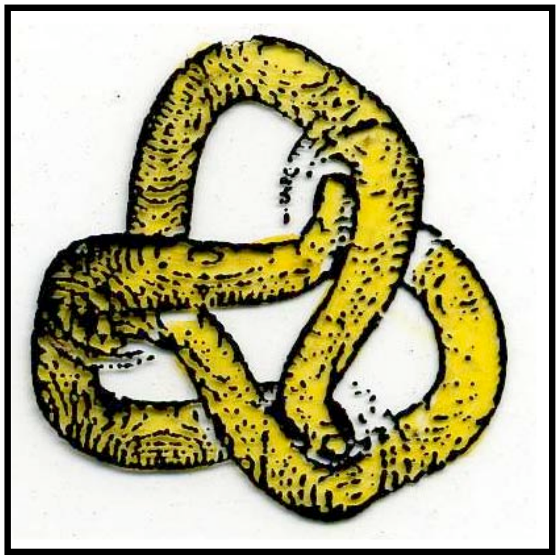


$$\begin{cases} r^2 = F_r(J) \\ \alpha = F_\alpha(E) \\ z = F_z(\Pi) \end{cases}$$

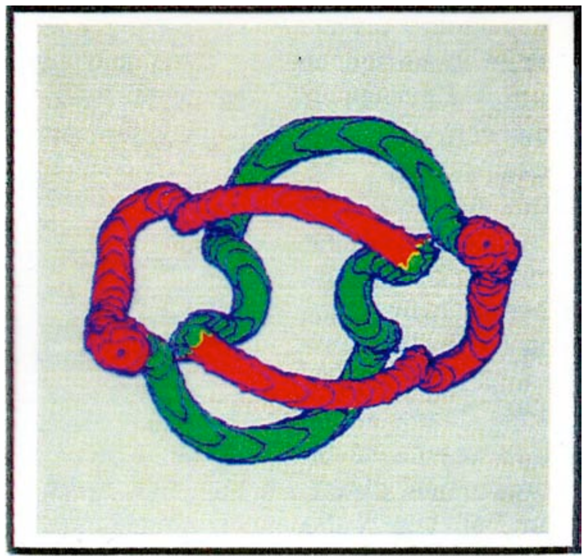
- existence of torus knot solutions
- existence of knotted chaotic orbits
- stability criteria
- fluid invariants, soliton invariants, Hamiltonian structures & knot types

Kida, Keener, Ricca, MacKay, Ghrist, Sullivan, Fuentes, Enciso & Peralta-Salas, …

- **Visiometric approach to knotting and linking**

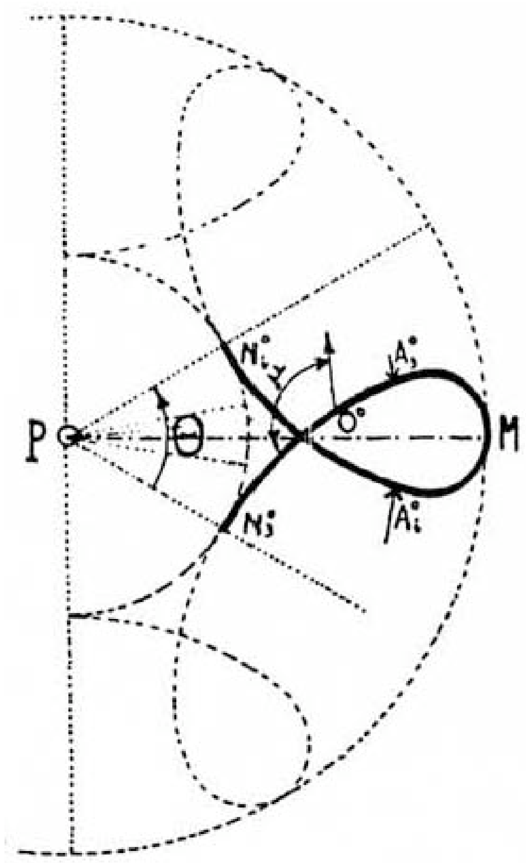


(Kida & Takaoka, 1988)



(Aref & Zawadzki, 1991)

Steady torus knots under LIA ($\mathbf{u}_{\mathrm{LIA}} = c\hat{\mathbf{b}}$)



(Levi-Civita, 1932
Da Rios, 1933)

- ***Kida's class:*** *existence of torus knot solutions* $\mathcal{T}_{p,q}$ *(* $p>1,\ q>1$ *co-prime integers) in terms of incomplete elliptic integrals:*

(Kida, 1981)

$$\begin{cases} r^2 = F_r(J) \\ \alpha = F_\alpha(E) \\ z = F_z(\Pi) \end{cases}$$

- ***Solutions*** $\mathcal{T}_{p,q}$ ***in explicit analytic closed form:***

(Ricca, 1993)

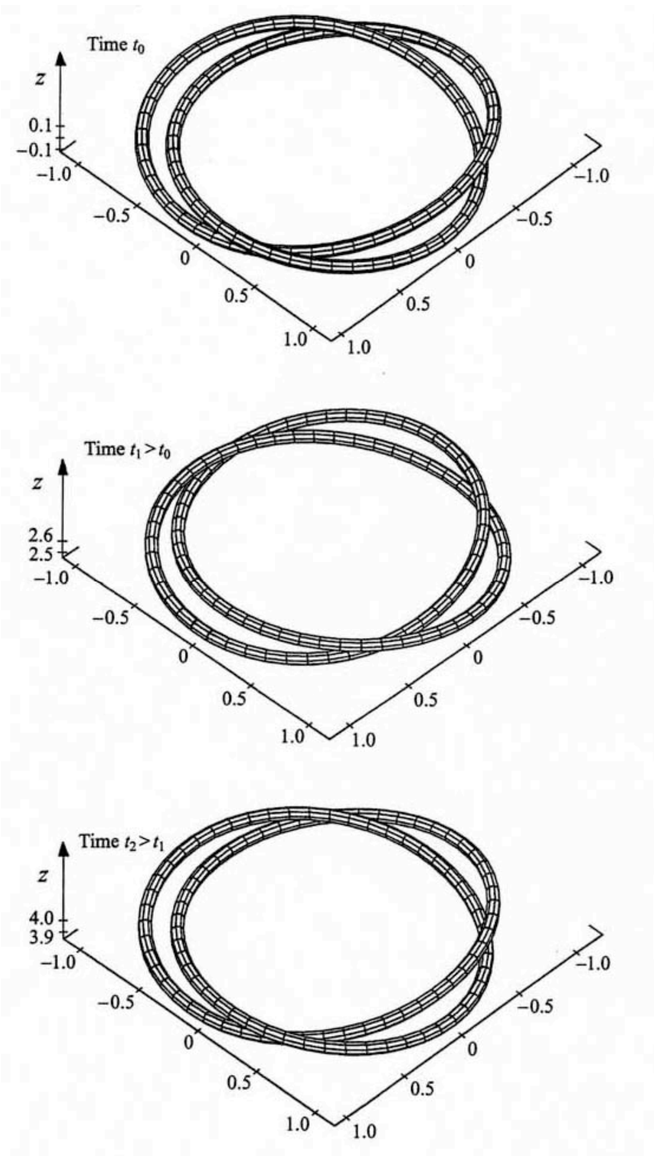
$$\begin{cases} r = r_0 + \varepsilon \sin\dfrac{w\xi}{r_0} \\ \alpha = \dfrac{s}{r_0} + \dfrac{\varepsilon}{wr_0} \cos\dfrac{w\xi}{r_0} \\ z = \dfrac{\bar{t}}{r_0} + \varepsilon\left(1 - \dfrac{1}{w^2}\right)^{1/2} \cos\dfrac{w\xi}{r_0} \end{cases}$$

- ***Linear stability criterium:*** *given* $w = q/p$*, then*

"$\mathcal{T}_{p,q}$ ***steady & stable iff*** $w>1$"

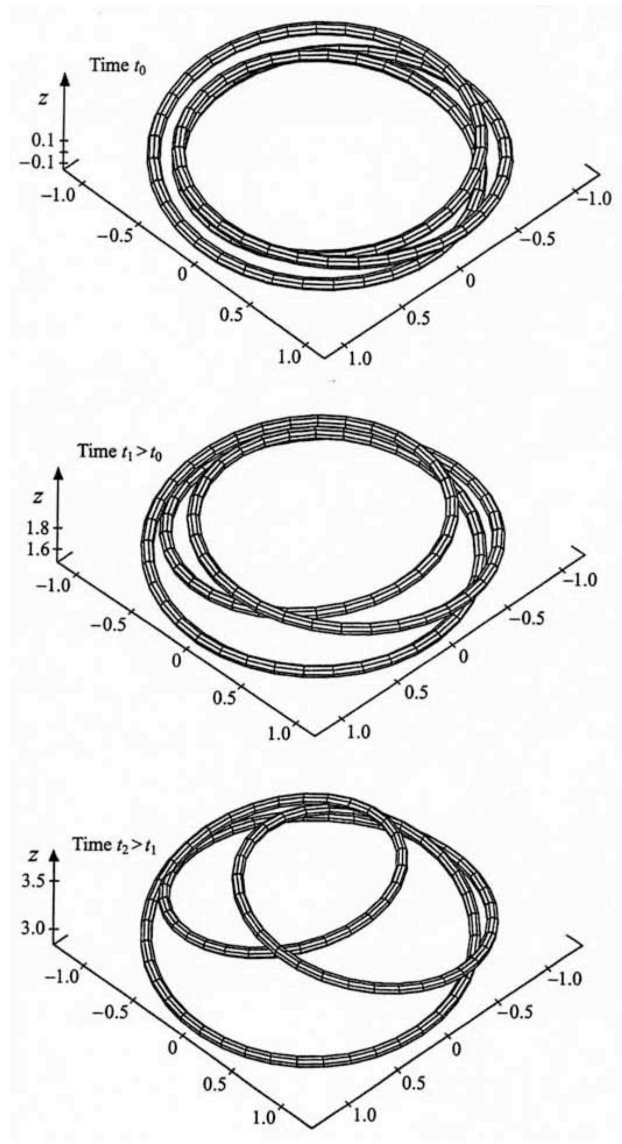
Numerical evolution of $\mathcal{T}_{p,q}$ under LIA

30-35 radii



$\mathcal{T}_{2,3}$ $(w > 1)$

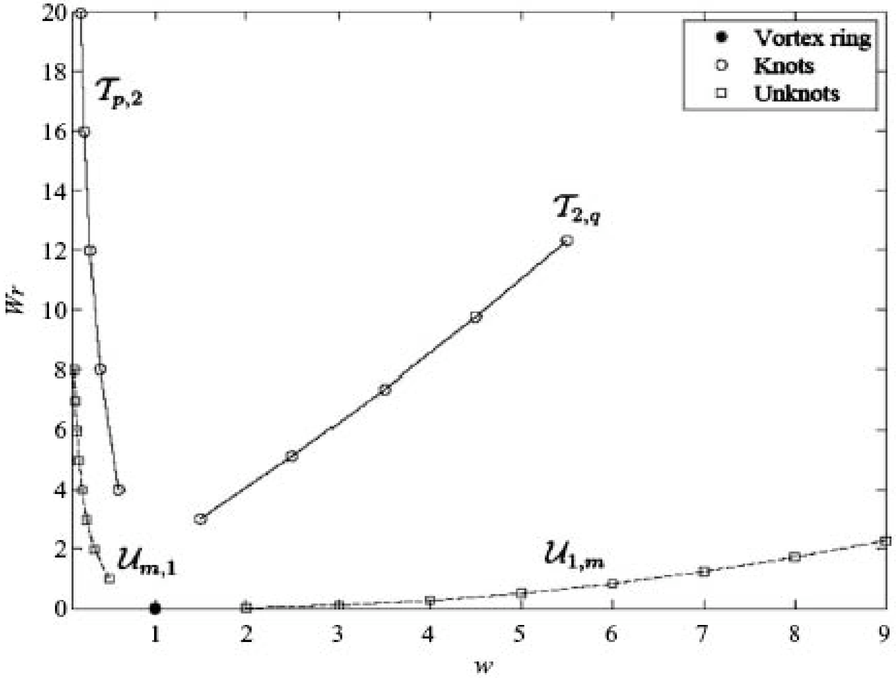
3-5 radii

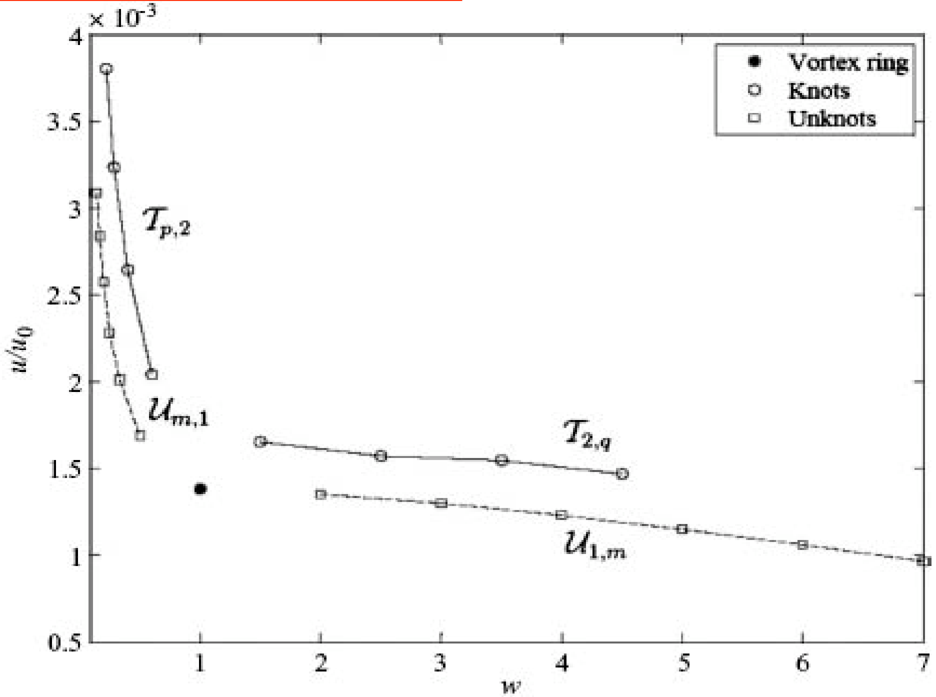


$\mathcal{T}_{3,2}$ $(w < 1)$

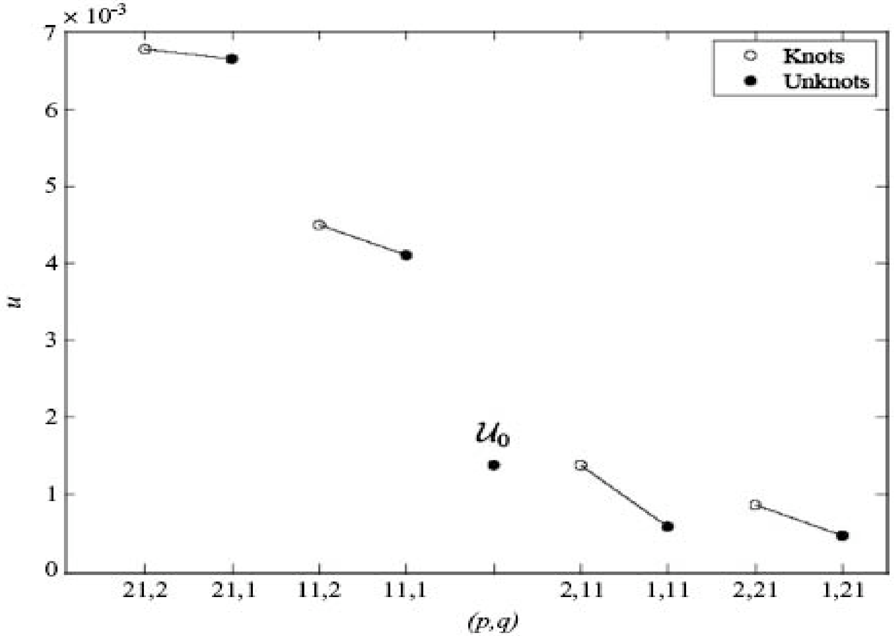
(Ricca et al., JFM 391, 1999)

LIA knots and unknots: writhing and velocity vs. $w = q/p$

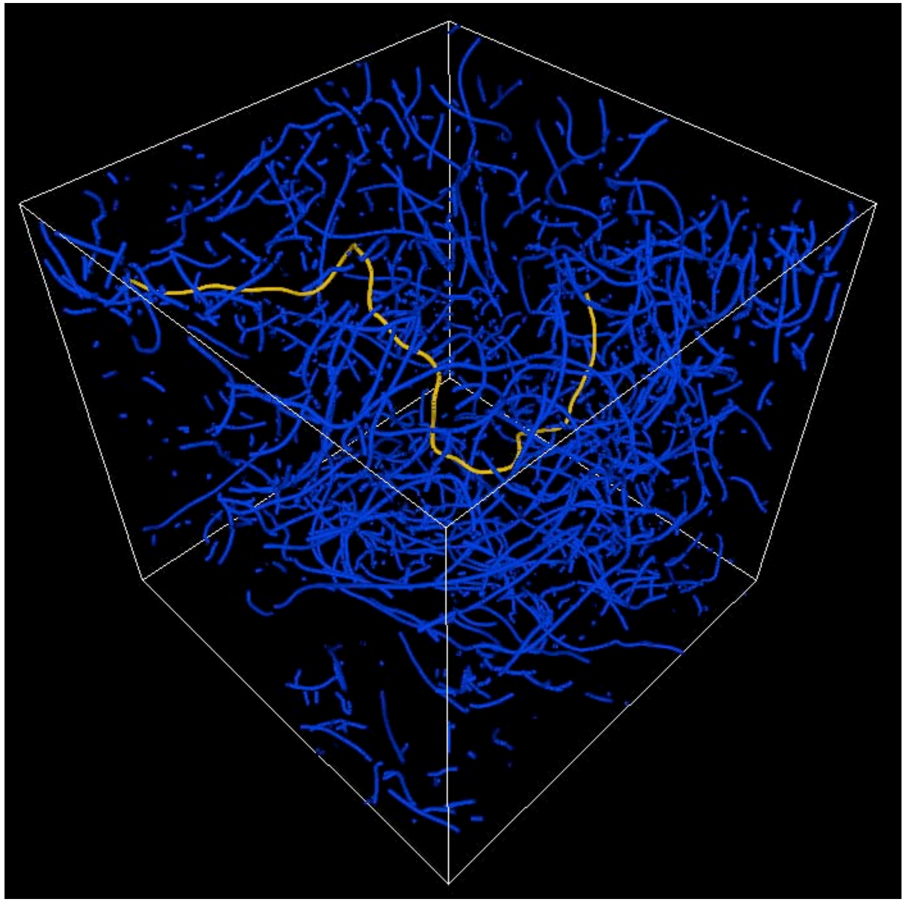




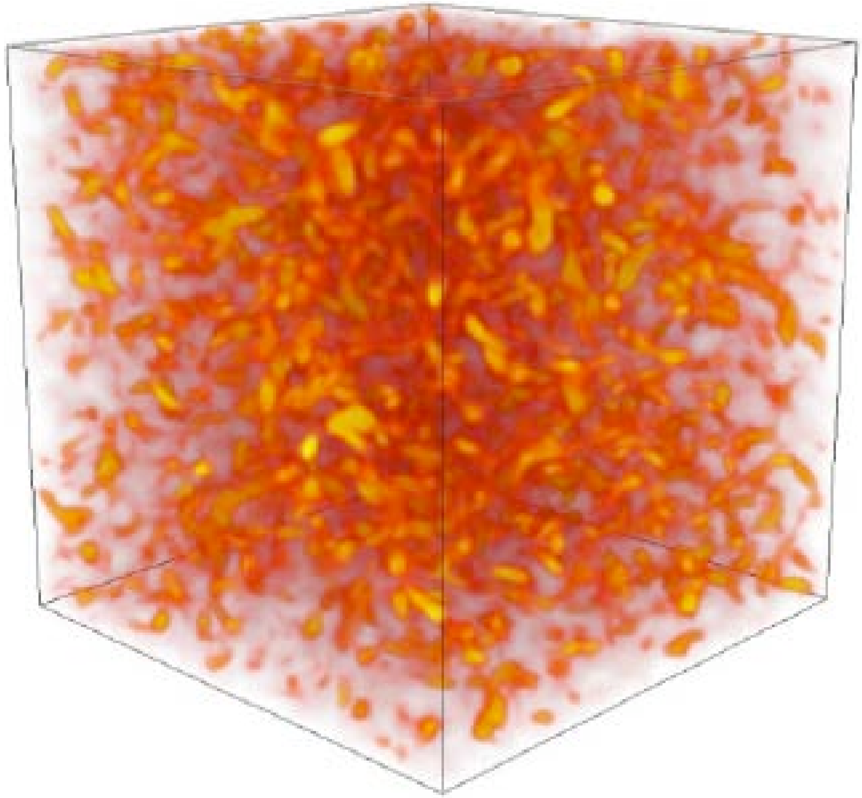
Maggioni et al.,
Phys Rev. E 82, 2010



Vorticity localization in classical and quantum fluids



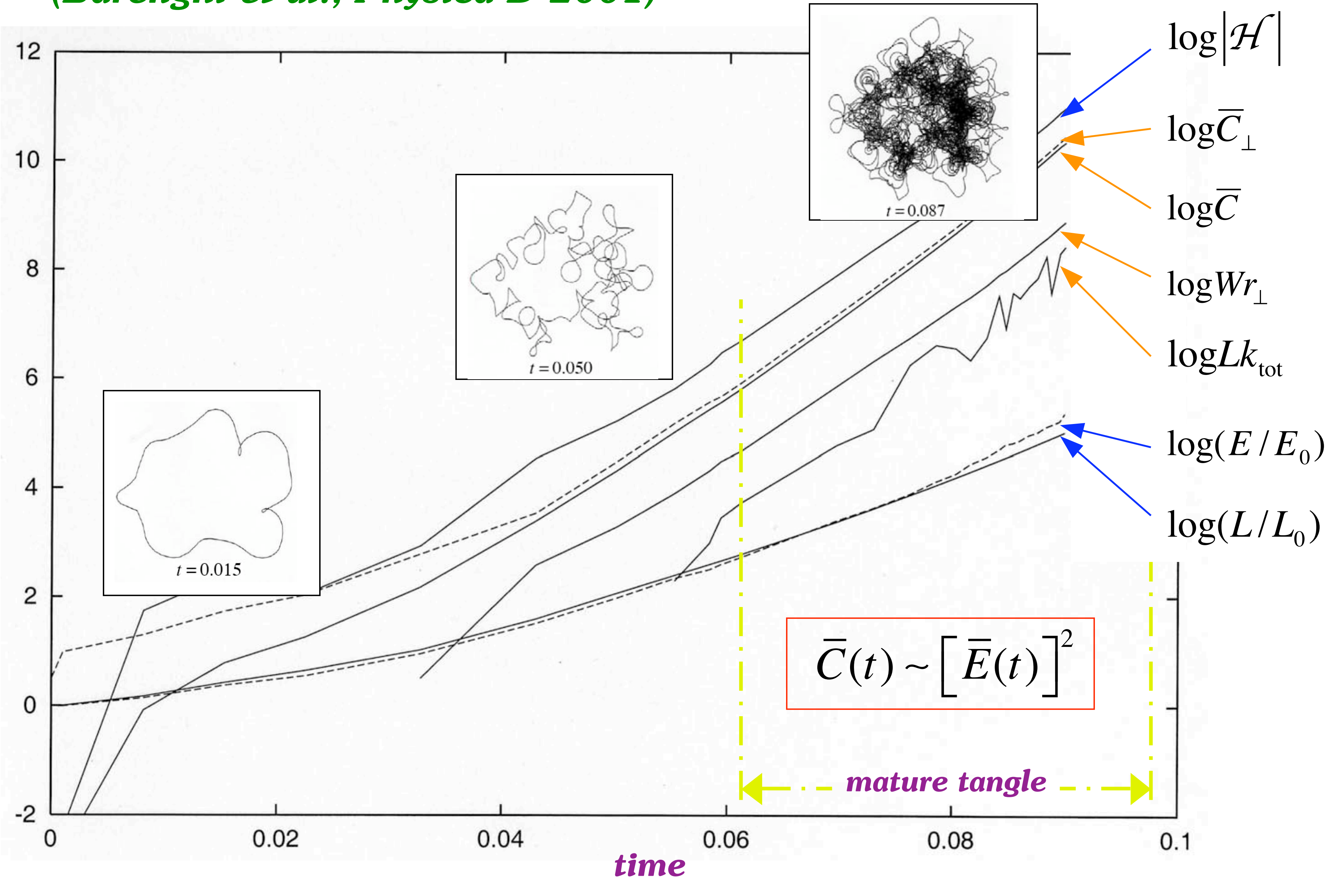
Kida et al.
(Toki-Kyoto 2002)



Baggaley et al.
(EPL 2012)

Energy-complexity relation for vortex tangles

(Barenghi et al., Physica D 2001)



Jones polynomial of fluid knots from helicity

Let $H \equiv \int_{V(\mathcal{K})} \mathbf{u}\cdot\boldsymbol{\omega}\; d^3X = \sum_i \kappa_i \oint_{\mathcal{K}_i} \mathbf{u}_v \cdot d\mathbf{l}$ and set $\kappa_i = 1 \;\; \forall i$. Then

- **Theorem (Liu & Ricca, 2012)**. Let $\mathcal{K}$ denote a fluid knot. If the helicity of $\mathcal{K}$ is $H = H(\mathcal{K})$, then

$$e^{H(\mathcal{K})} = e^{\oint_{\mathcal{K}} \mathbf{u}_v \cdot d\mathbf{l}}$$

appropriately re-scaled, satisfies the skein relations of the Jones polynomial $V = V(\tau)$.

- In general: $\tau = e^{-4\lambda H(\gamma_+)}$, $(0 \le \lambda \le 1)$.

- For a homogeneous tangle of knots and links of same circulation κ, we have

$$\bar{\lambda} = \langle \lambda \rangle = \frac{1}{2}, \qquad \langle H(\gamma_+) \rangle = \frac{\kappa^2}{2},$$

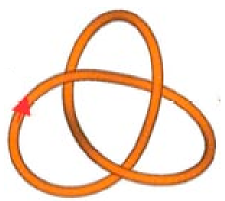
hence

$$\tau = e^{-\kappa^2}$$

Tackling structural complexity by knot polynomials

$$V(\mathcal{K}(\tau)) \to V(\mathcal{K}(\kappa)) = f(\mathcal{K};\kappa)$$

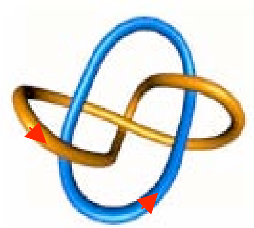
- **Examples: (i) trefoil knots**



(a) $\boldsymbol{T}^L$ (left-handed): $V(\boldsymbol{T}^L) = e^{\kappa^2} + e^{3\kappa^2} - e^{4\kappa^2}$

(b) $\boldsymbol{T}^R$ (right-handed): $V(\boldsymbol{T}^R) = e^{-\kappa^2} + e^{-3\kappa^2} - e^{-4\kappa^2}$

(ii) Whitehead link:



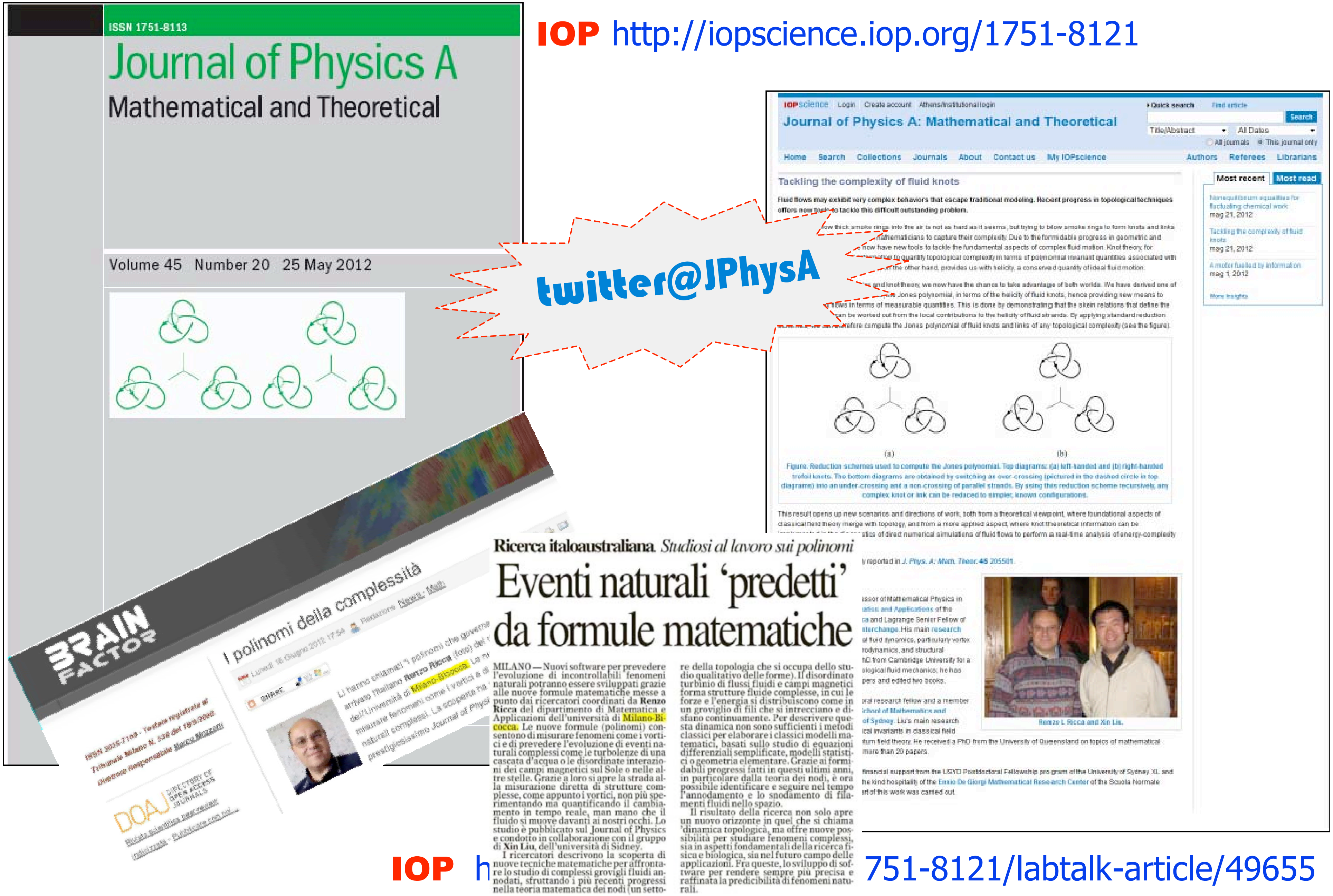
(a) $\boldsymbol{W}_+$ = (b) $\boldsymbol{W}_-$ = $\boldsymbol{W}$:

$$V(\boldsymbol{W}) = e^{-\frac{3}{2}\kappa^2}\left(-1 + e^{\kappa^2} - 2e^{2\kappa^2} + e^{3\kappa^2} - 2e^{4\kappa^2} + e^{5\kappa^2}\right)$$

- **Future work:**
  - extension to Vassiliev finite type invariants, Kovhanov homology and Heegaard Floer theory;
  - implementation of topological analysis in advanced visiometrics for predictive diagnostics of vortical flows.

Fluid knots in the press

IOP http://iopscience.iop.org/1751-8121



MILANO — Nuovi software per prevedere l'evoluzione di incontrollabili fenomeni naturali potranno essere sviluppati grazie alle nuove formule matematiche messe a punto dai ricercatori coordinati da **Renzo Ricca** del dipartimento di Matematica e Applicazioni dell'università di Milano-Bicocca. Le nuove formule (polinomi) consentono di misurare fenomeni come i vortici e di prevedere l'evoluzione di eventi naturali complessi come le turbolenze di una cascata d'acqua o le disordinate interazioni dei campi magnetici sul Sole o nelle altre stelle. Grazie a loro si apre la strada alla misurazione diretta di strutture complesse, come appunto i vortici, non più sperimentando ma quantificando il cambiamento in tempo reale, man mano che il fluido si muove davanti ai nostri occhi. Lo studio è pubblicato sul Journal of Physics e condotto in collaborazione con il gruppo di **Xin Liu**, dell'università di Sidney.

I ricercatori descrivono la scoperta di nuove tecniche matematiche per affrontare lo studio di complessi grovigli fluidi annodati, sfruttando i più recenti progressi nella teoria matematica dei nodi (un settore della topologia che si occupa dello studio qualitativo delle forme). Il disordinato turbinio di flussi fluidi e campi magnetici forma strutture fluide complesse, in cui le forze e l'energia si distribuiscono come in un groviglio di fili che si intrecciano e disfano continuamente. Per descrivere questa dinamica non sono sufficienti i metodi classici per elaborare i classici modelli matematici, basati sullo studio di equazioni differenziali semplificate, modelli statistici o geometria elementare. Grazie ai formidabili progressi fatti in questi ultimi anni, in particolare dalla teoria dei nodi, è ora possibile identificare e seguire nel tempo l'annodamento e lo snodamento di filamenti fluidi nello spazio.

Il risultato della ricerca non solo apre un nuovo orizzonte in quel che si chiama 'dinamica topologica, ma offre nuove possibilità per studiare fenomeni complessi, sia in aspetti fondamentali della ricerca fisica e biologica, sia nel futuro campo delle applicazioni. Fra queste, lo sviluppo di software per rendere sempre più precisa e raffinata la predicibilità di fenomeni naturali.

IOP h… 751-8121/labtalk-article/49655