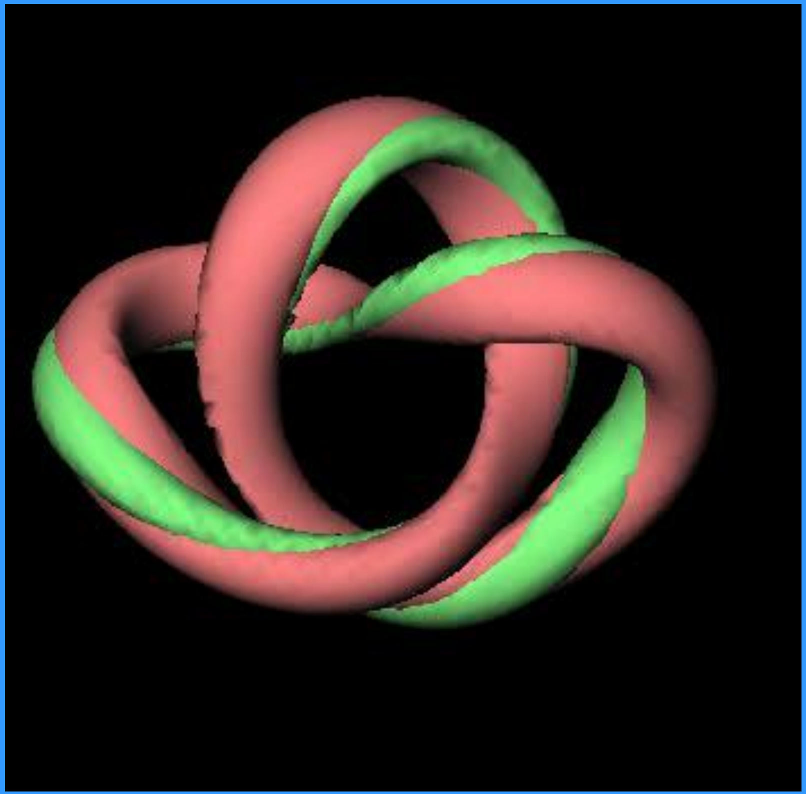
Solitonic Knots



Paul Sutcliffe
Durham University

Battye & Sutcliffe, Phys. Rev. Lett. 81, 4798 (1998)
Sutcliffe, Proc. R. Soc. A 463, 3001 (2007)
Harland, Speight & Sutcliffe, Phys. Rev. D. 83, 065008 (2011)

Outline

- Topology of Hopf solitons.
- The Skyrme-Faddeev model.
- Soliton solutions and knots.
- Conclusion.

Topology of maps from the plane to the sphere

$$\mathbf{n} : R^2 \mapsto S^2$$

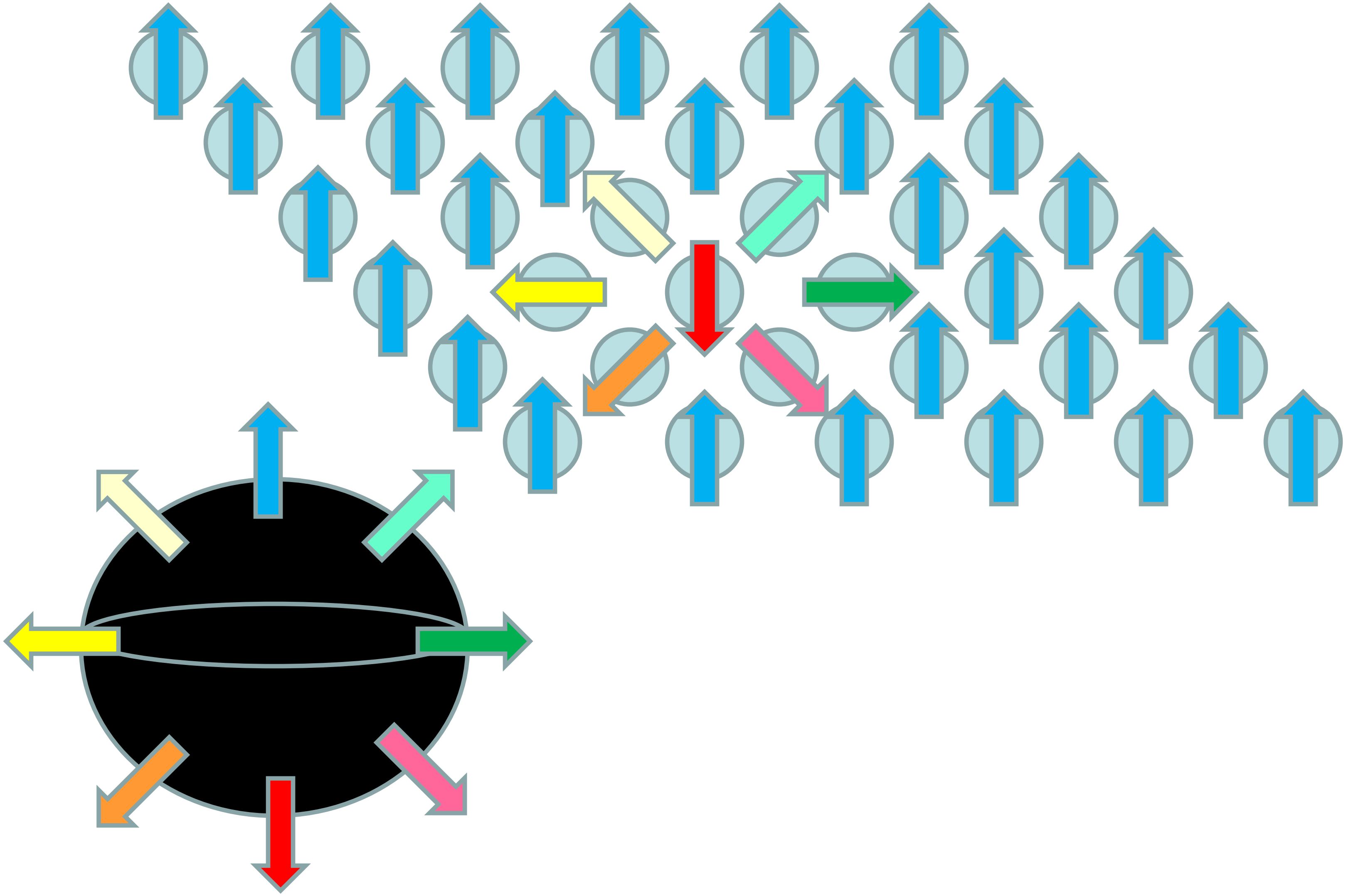
$$\mathbf{n}(x, y) = (n_1, n_2, n_3), \quad \text{with } \mathbf{n} \cdot \mathbf{n} = 1$$

Boundary condition $\mathbf{n}(\infty) = (0, 0, 1) = \mathbf{k}$ compactifies R^2 to S^2.

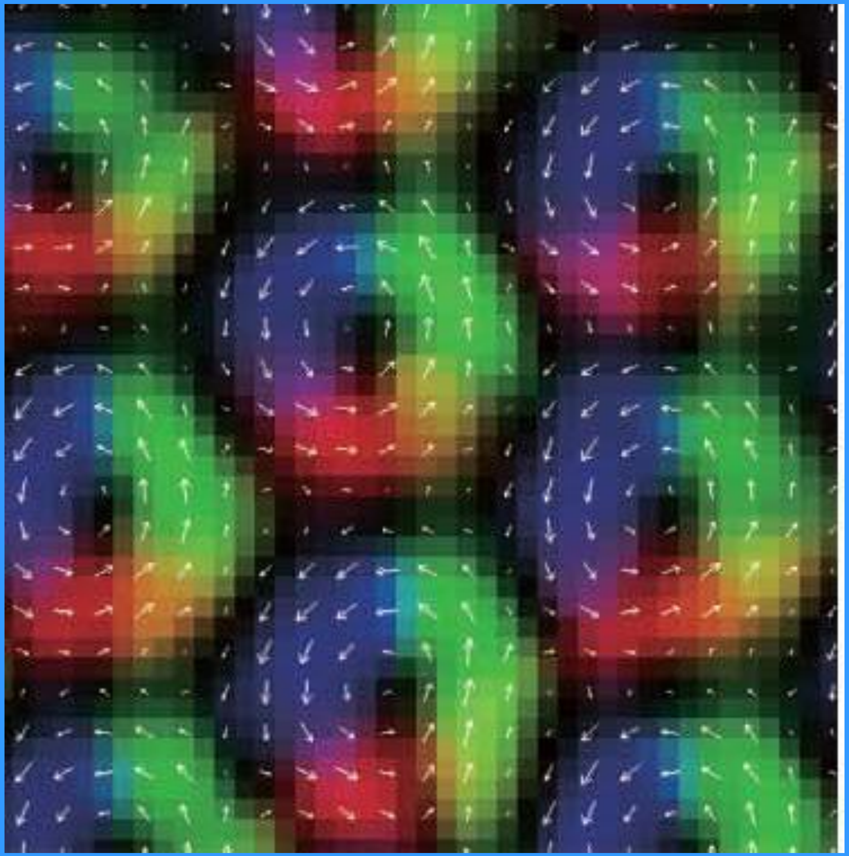
Such maps are classified by $q \in Z = \pi_2(S^2)$.

$$q = \frac{1}{4\pi} \int \mathbf{n} \cdot \left(\frac{\partial \mathbf{n}}{\partial x} \times \frac{\partial \mathbf{n}}{\partial y} \right) d^2x = \frac{1}{4\pi} \int_{S^2} \mathbf{n}^* \omega$$

$q = 1$, the vector $\mathbf{n}$ winds once around the sphere of directions



Baby skyrmions in chiral magnets



skyrmion size ~ 100nm

Experiments on Fe-Co-Si alloy, imaged using transmission electron microscopy (TEM)

Yu et al, Nature 465, 901 (2010)

Topology of maps from R^3 to the sphere

$$\mathbf{n} : R^3 \mapsto S^2$$

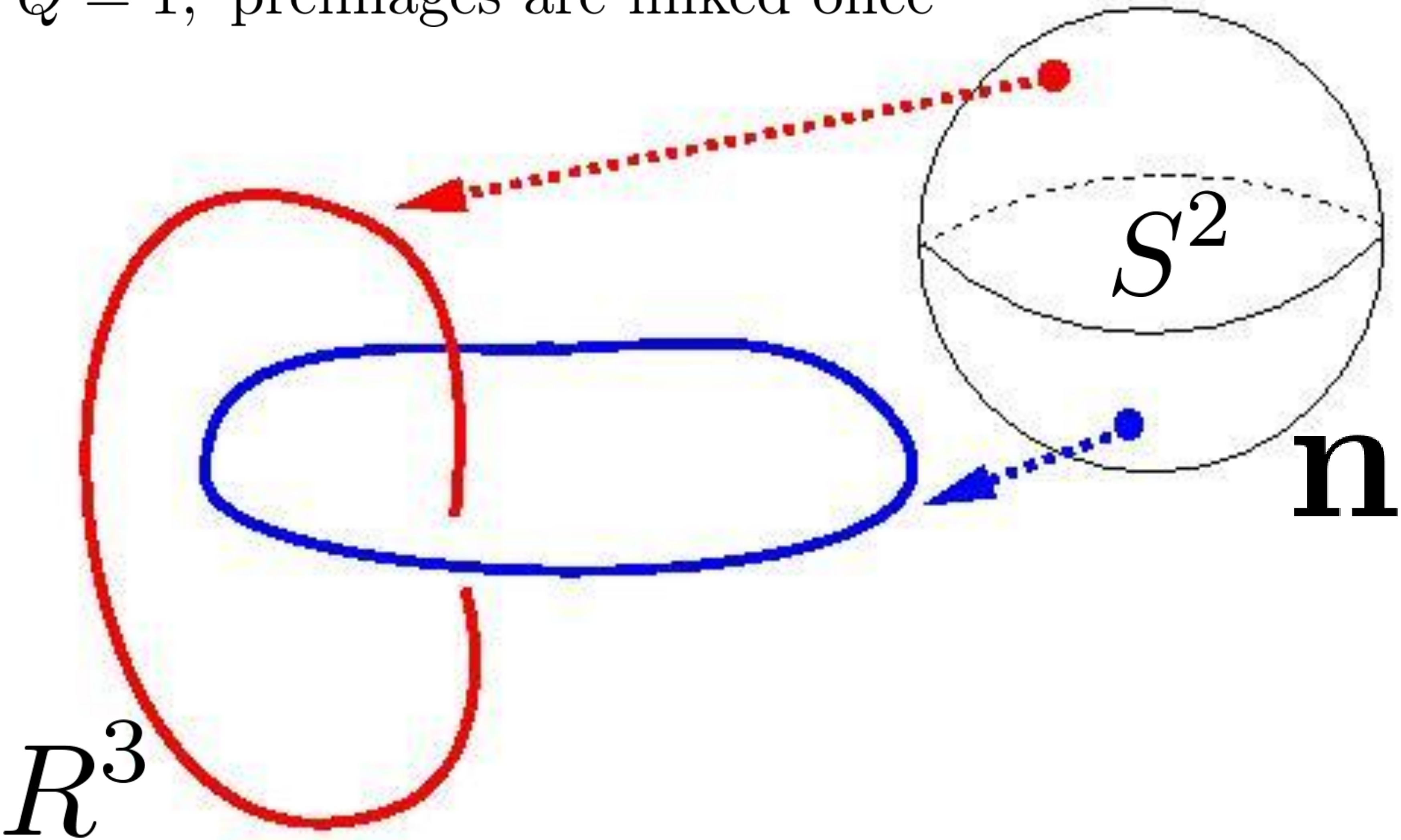
$$\mathbf{n}(x, y, z) = (n_1, n_2, n_3), \quad \text{with } \mathbf{n} \cdot \mathbf{n} = 1$$

Boundary condition $\mathbf{n}(\infty) = (0, 0, 1) = \mathbf{k}$ compactifies R^3 to S^3.

Such maps are classified by $Q \in Z = \pi_3(S^2)$.

$$f = \mathbf{n}^* \omega = da, \qquad Q = \frac{1}{4\pi^2} \int_{S^3} f \wedge a$$

$Q = 1$, preimages are linked once



The Skyrme-Faddeev model

Faddeev (1975); Faddeev & Niemi, Nature (1997).

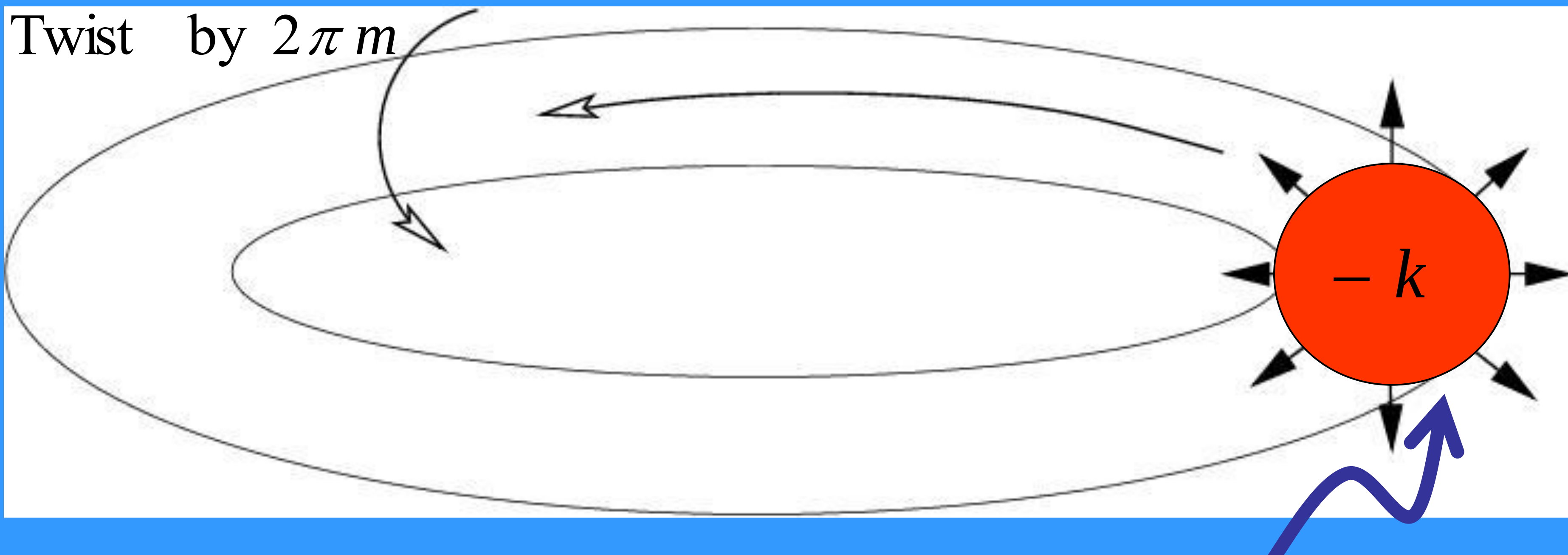
$$E = \frac{1}{32\pi^2\sqrt{2}} \int \left(|\nabla \mathbf{n}|^2 + \left| \frac{\partial \mathbf{n}}{\partial x} \times \frac{\partial \mathbf{n}}{\partial y} \right|^2 + \text{ cyclic} \right) d^3x$$

Energy bound, $E \geq c|Q|^{3/4}.$

Proved with $c = \left(\frac{3}{16}\right)^{3/8} \approx 0.5338$. Conjectured with $c = 1$.

Vakulenko & Kapitanski (1979). Ward (1999).

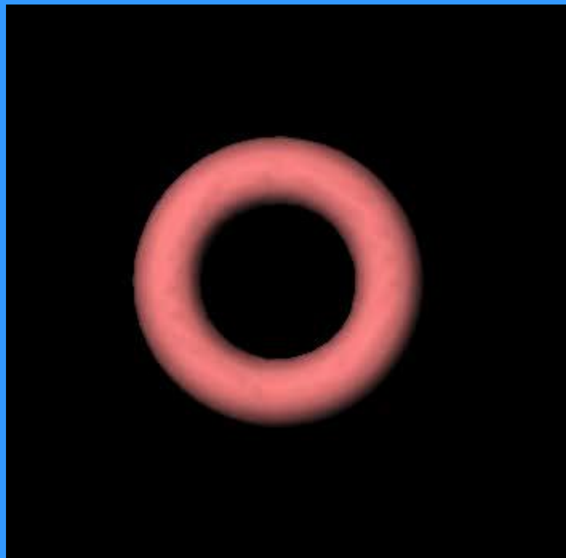
Axial Hopf solitons $\mathbb{A}_{m,q}$

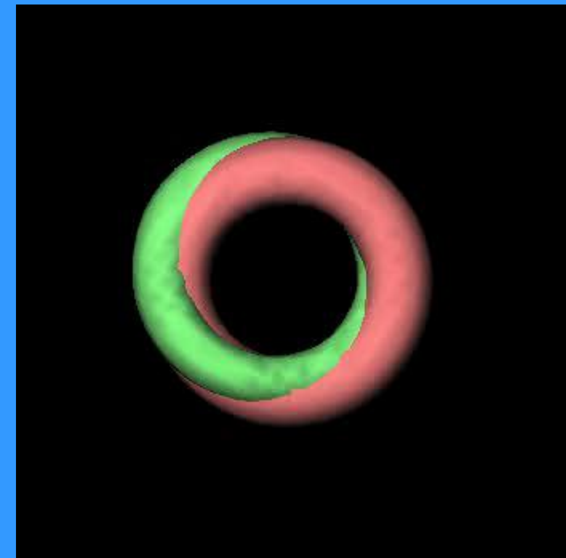


2D soliton with $q \in \mathbb{Z} = \pi_2(S^2)$.

$$Q = m\,q$$

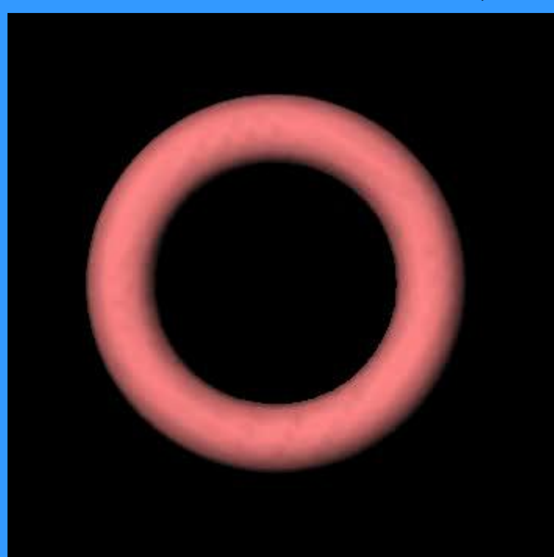
$Q = 1; \quad \mathbb{A}_{1,1}; \quad E/Q^{3/4} = 1.204$

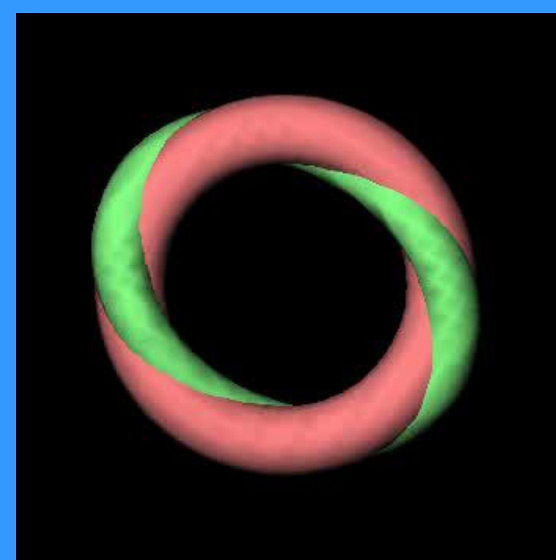




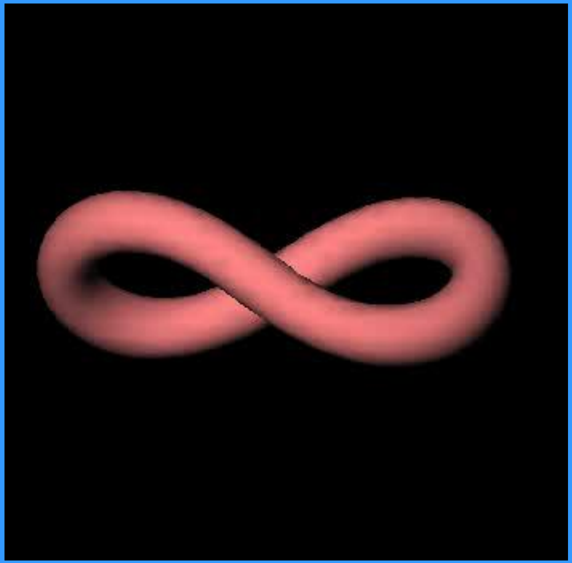
Soliton position (preimage of $-k$). Linking (preimage of 2 points).

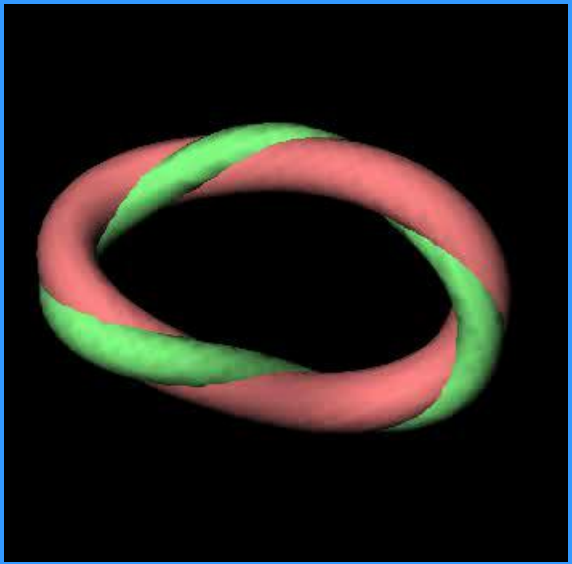
$Q = 2; \quad \mathbb{A}_{2,1}; \quad E/Q^{3/4} = 1.170$



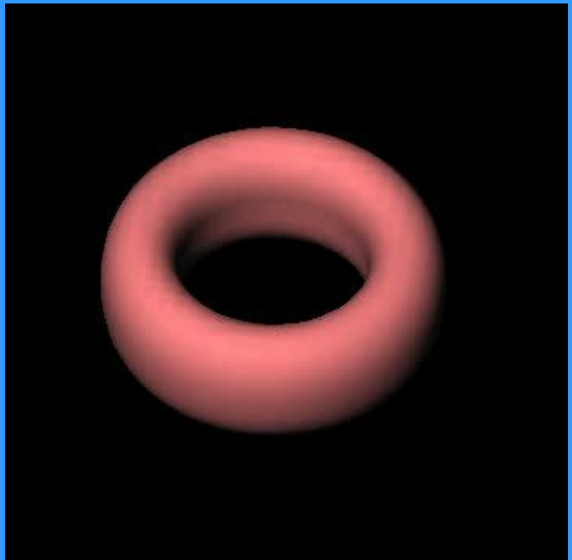


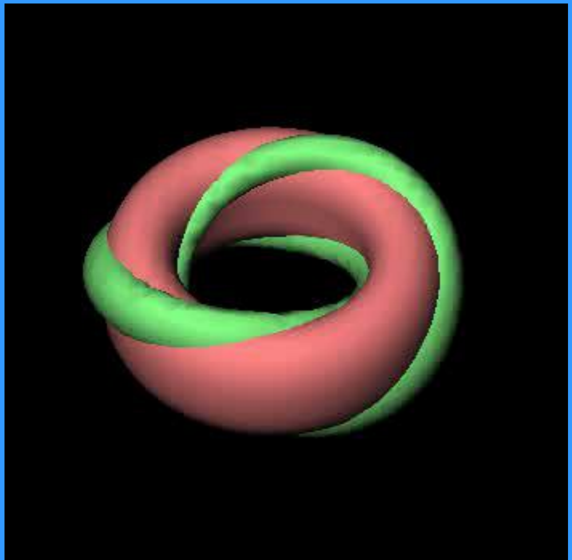
$Q = 3; \quad \tilde{\mathbb{A}}_{3,1}; \quad E/Q^{3/4} = 1.208$



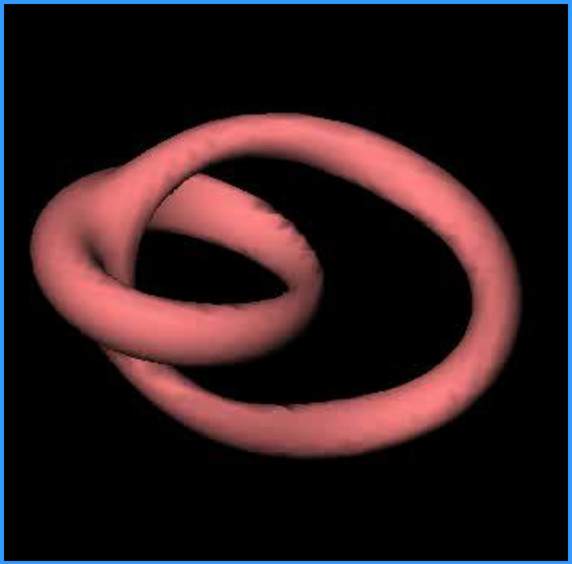


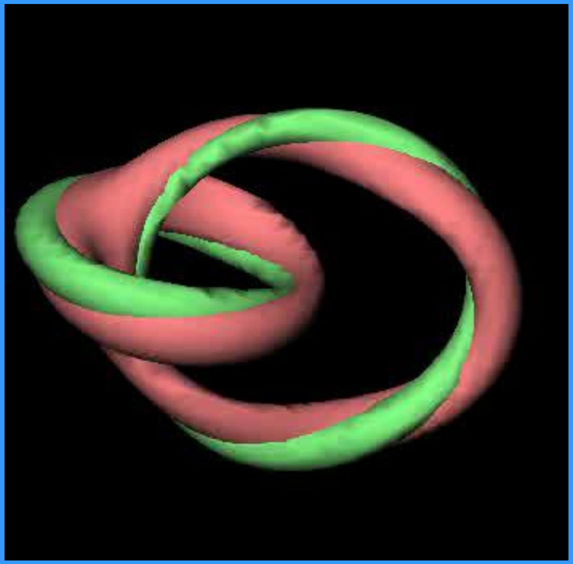
$Q = 4; \quad \tilde{\mathbb{A}}_{2,2}; \quad E/Q^{3/4} = 1.218$



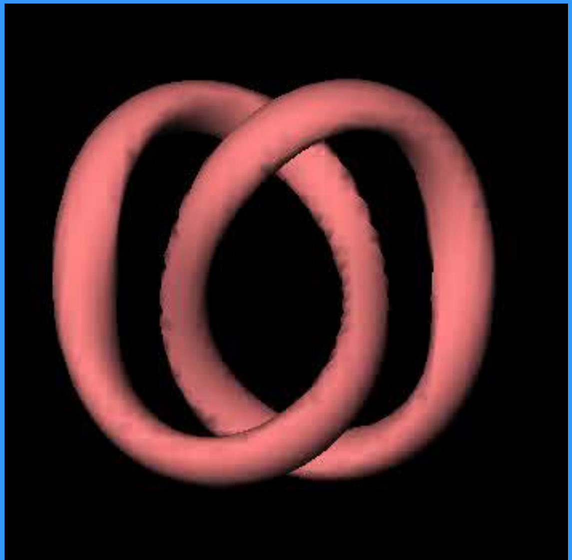


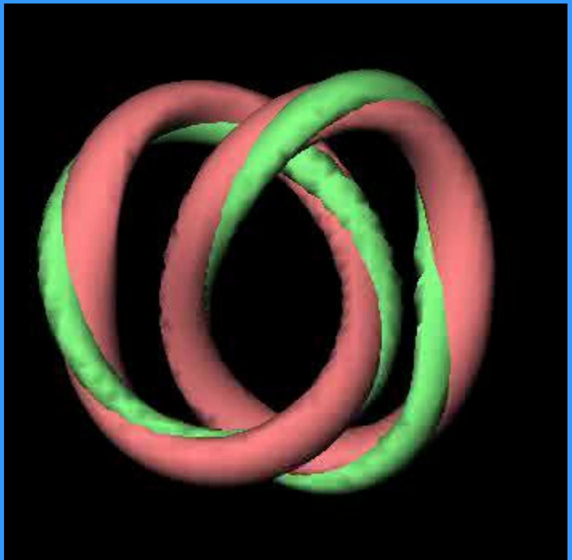
$Q = 5; \quad \mathbb{L}^{1,1}_{1,2}; \quad E/Q^{3/4} = 1.225$



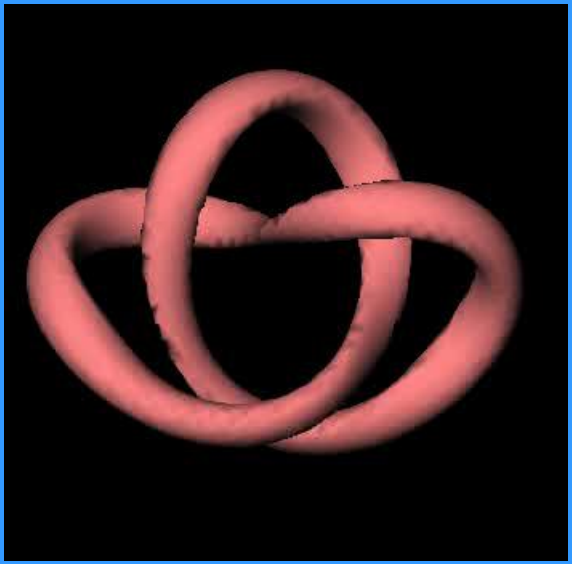


$Q = 6; \quad \mathbb{L}^{1,1}_{2,2}; \quad E/Q^{3/4} = 1.213$





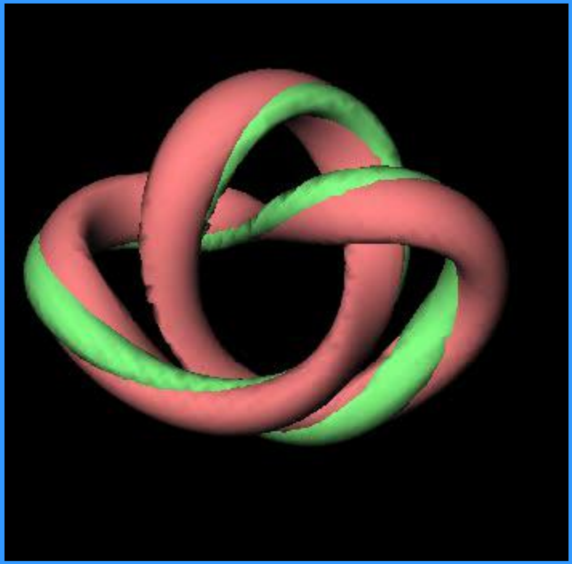
$Q = 7; \quad \mathbb{K}_{3,2}$ (trefoil knot) ; $\quad E/Q^{3/4} = 1.218$



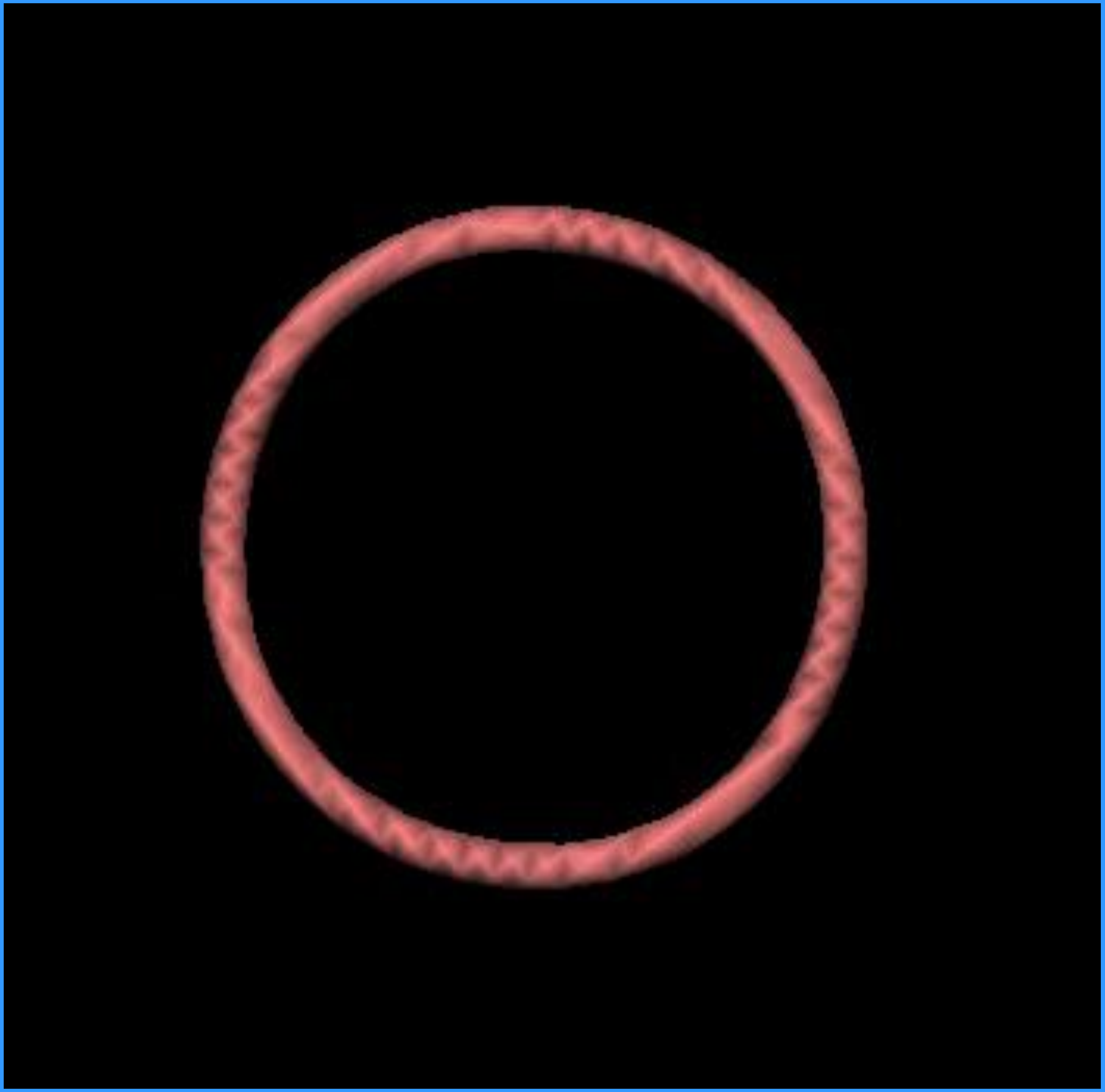
$$Q = 3 + 4 = 7$$

self-linking (crossing)

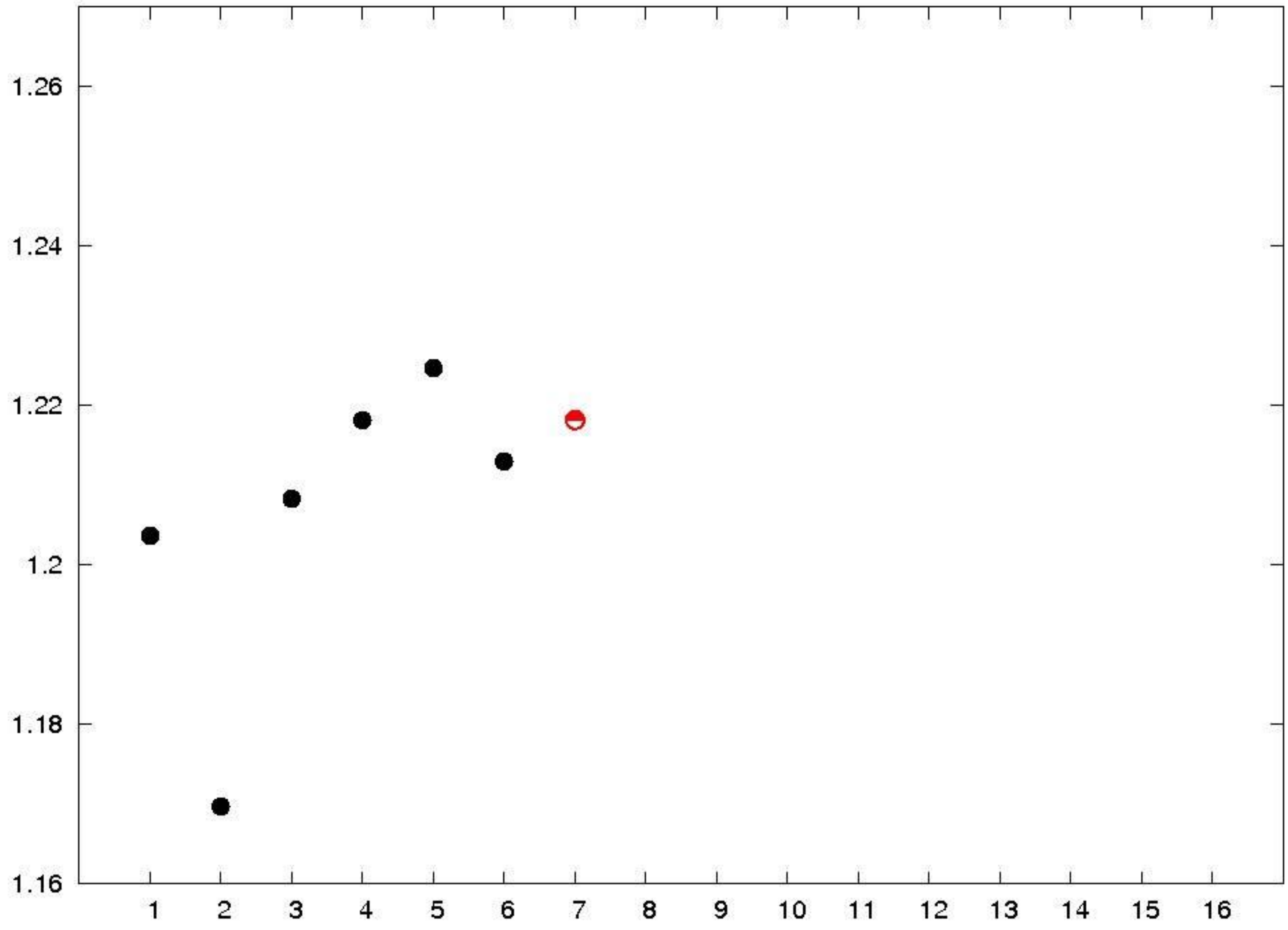
twist



$Q = 7; \quad \mathbb{A}_{7,1} \to \mathbb{K}_{3,2}$ (trefoil knot); energy minimizati on.



$E / Q^{3/4}$

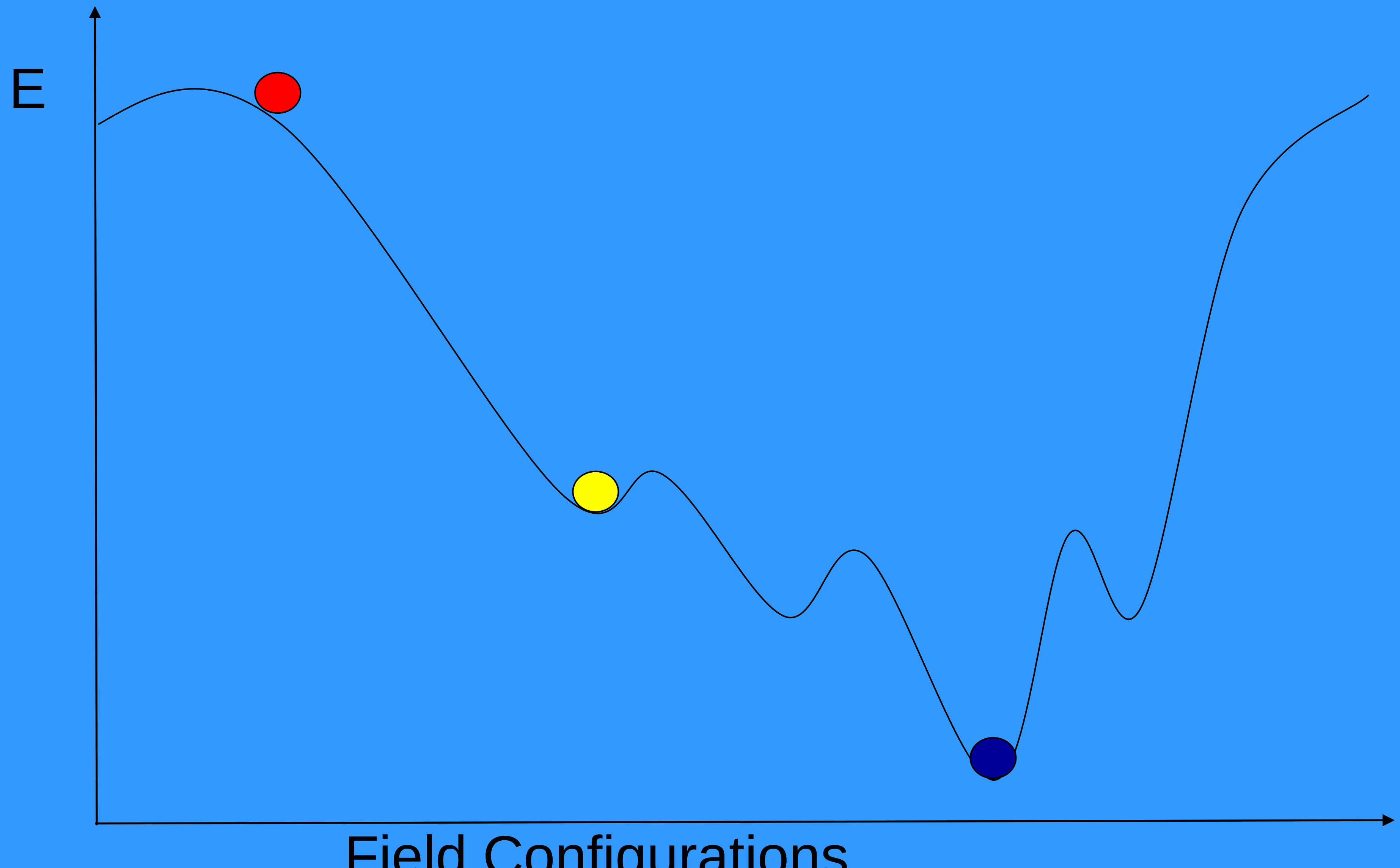


Q

unknot/link

trefoil knot

Local Minima & Initial Conditions



Torus Knots

$\mathbb{K}_{a,b}$ Torus knot, where $a > b$ are coprime integers.

$\mathbb{K}_{3,2}$ Trefoil knot (3_1), $C = 3$.

$\mathbb{K}_{5,2}$ Solomon's seal knot (5_1), $C = 5$.

$\mathbb{K}_{4,3}$ (8_{19}), $C = 8$.

$\mathbb{K}_{a,b}$, $C = a(b-1)$.

Torus Knots & Rational Maps

$\mathbb{K}_{a,b}$ is $S^3 \cap \{Z_1^a + Z_0^b = 0\}.$

$$(Z_1, Z_0) \in \mathbb{C}^2 \supset S^3 = \{|Z_0|^2 + |Z_1|^2 = 1\}.$$

$$\begin{array}{ccc} n: \mathbb{R}^3 & \longrightarrow & S^2 \\ \Big\downarrow (Z_1, Z_0) & & \Big\downarrow W \\ W: S^3 & \longrightarrow & \mathbb{CP}^1 \end{array}$$

$$W = \frac{Z_1^{\alpha} Z_0^{\beta}}{Z_1^a + Z_0^b}$$

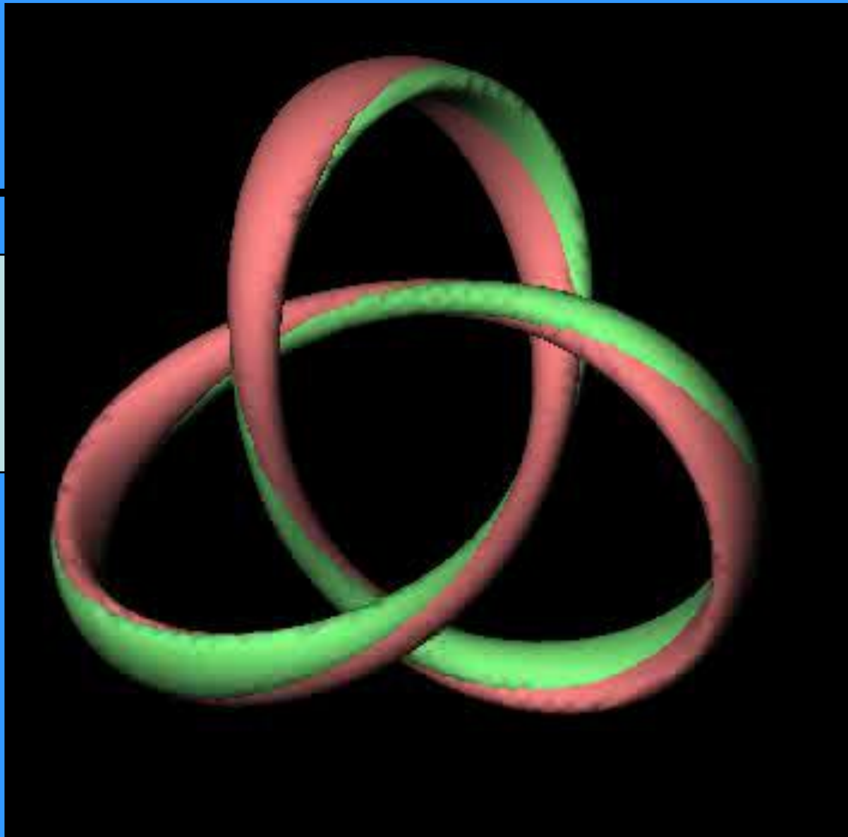
Hopf charge & rational maps

$$W = \frac{Z_1^{\alpha} Z_0^{\beta}}{Z_1^{a} + Z_0^{b}}$$

$\longrightarrow$

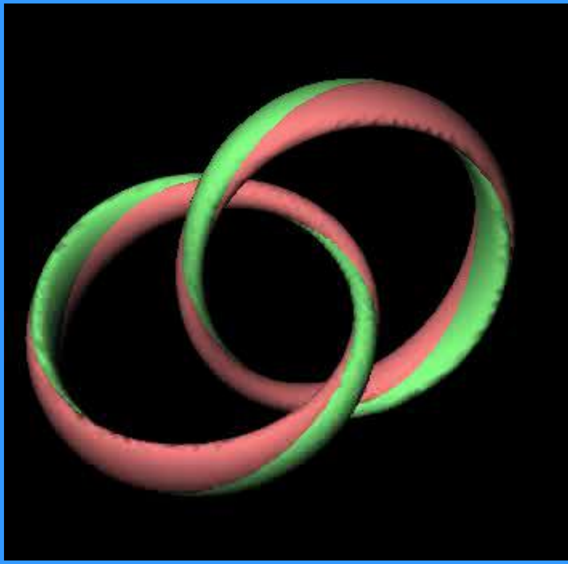
$$Q = \alpha b + \beta a$$

Eg. $Q = 7, \; \mathbb{K}_{3,2} :$ $W = \dfrac{Z_1^{2} Z_0}{Z_1^{3} + Z_0^{2}}$



Links & Rational Maps

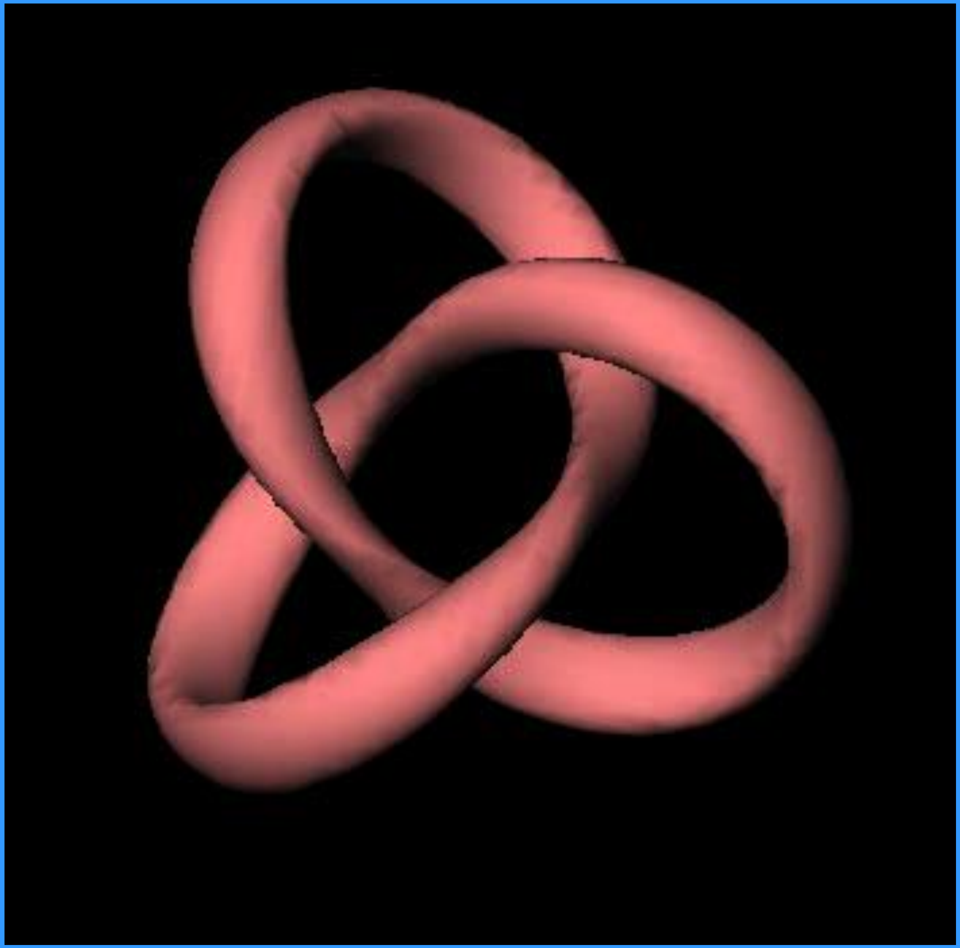
Eg. $Q = 6, \quad \mathbb{L}^{1,1}_{2,2}: \quad W = \dfrac{Z_1^2 Z_0}{Z_1^2 - Z_0^2}$



$Q = mq, \quad$ Axial $\quad \mathbb{A}_{m,q}: \quad W = \dfrac{Z_1^m}{Z_0^q}$

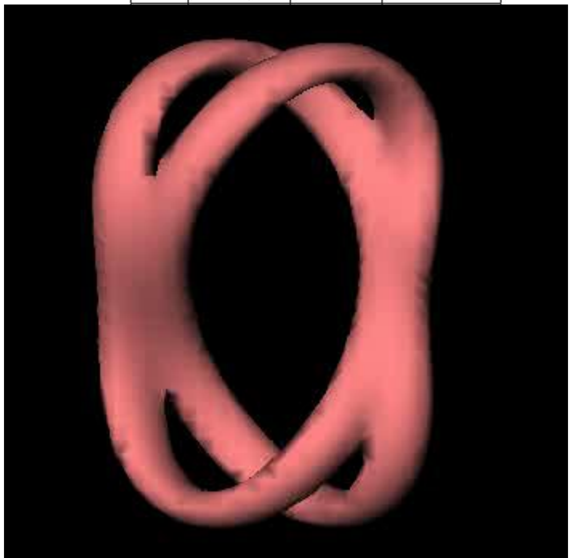
No trefoils with Q<7

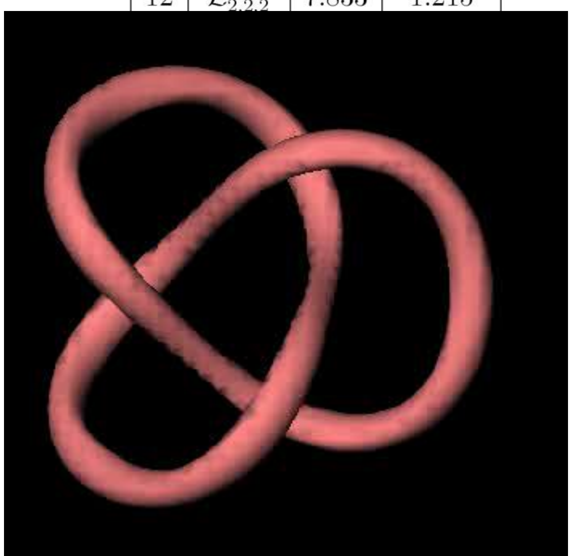
Eg. $Q = 5, \quad \mathbb{K}_{3,2} \rightarrow \mathbb{L}^{1,1}_{1,2}$



$Q = 8, \ \mathbb{L}^{1,1}_{3,3}$

Q	type	E	$E/Q^{3/4}$
1	$\mathcal{A}_{1,1}$	1.204	1.204
11	$\mathcal{L}_{3,4}$	7.555	1.247
11	$\mathcal{K}_{3,2}$	7.614	1.261
12	$\mathcal{L}^{2,2,2}_{2,2,2}$	7.833	1.215

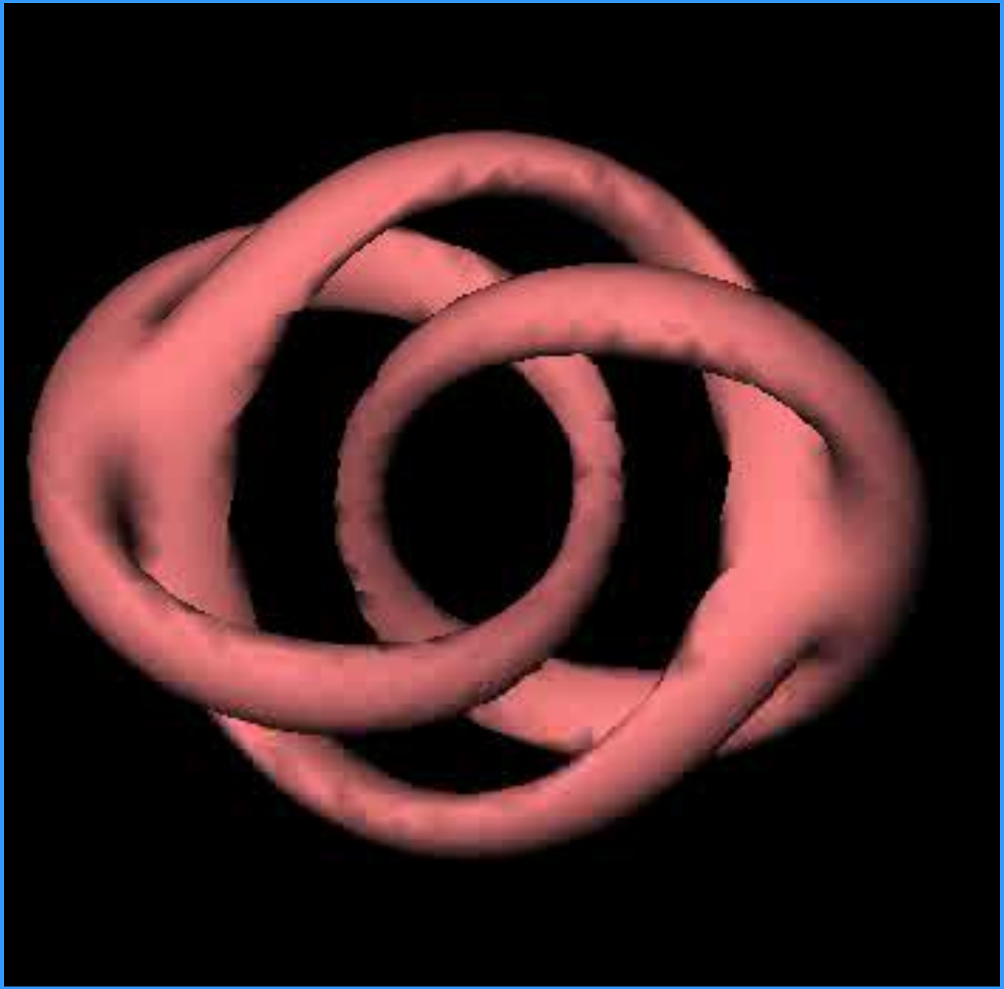




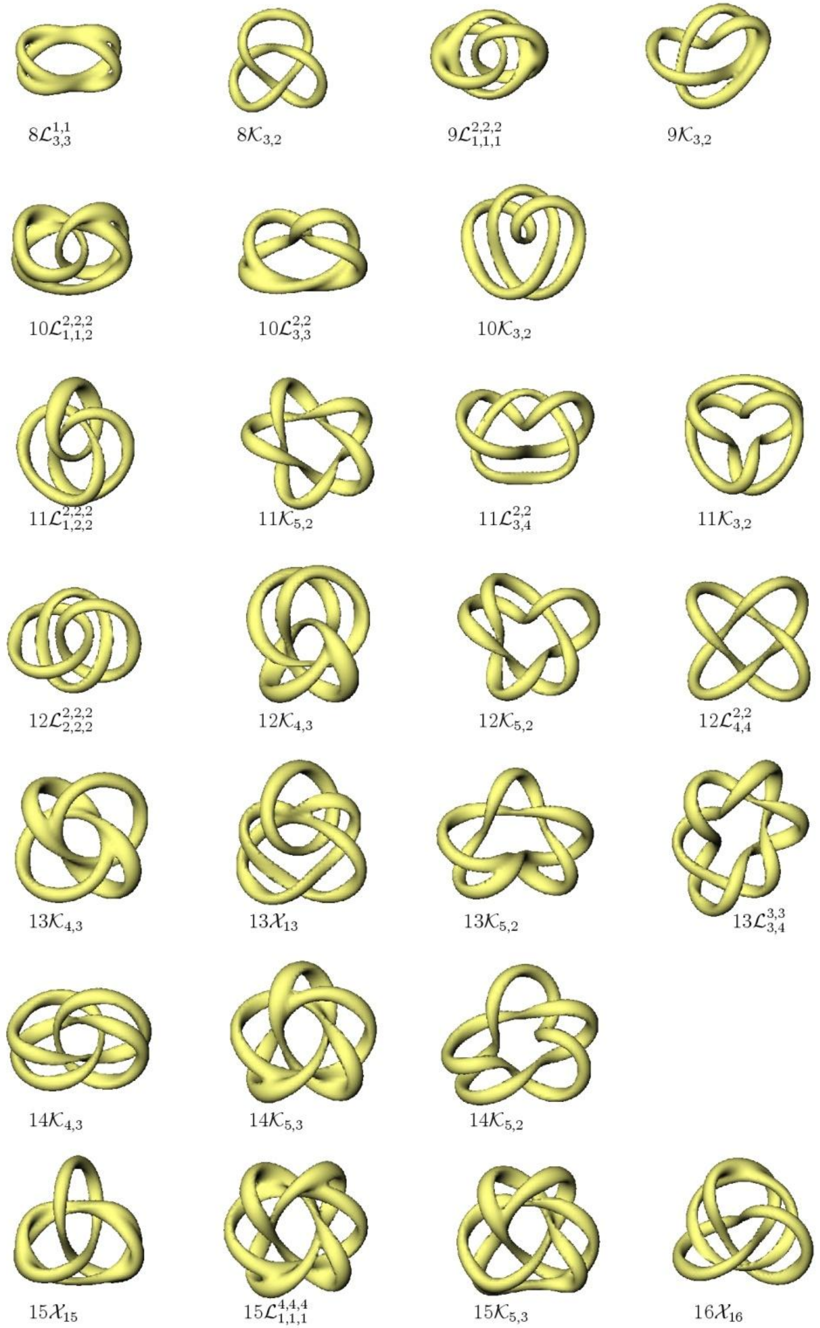
$Q = 8, \ \mathbb{K}_{3,2}$

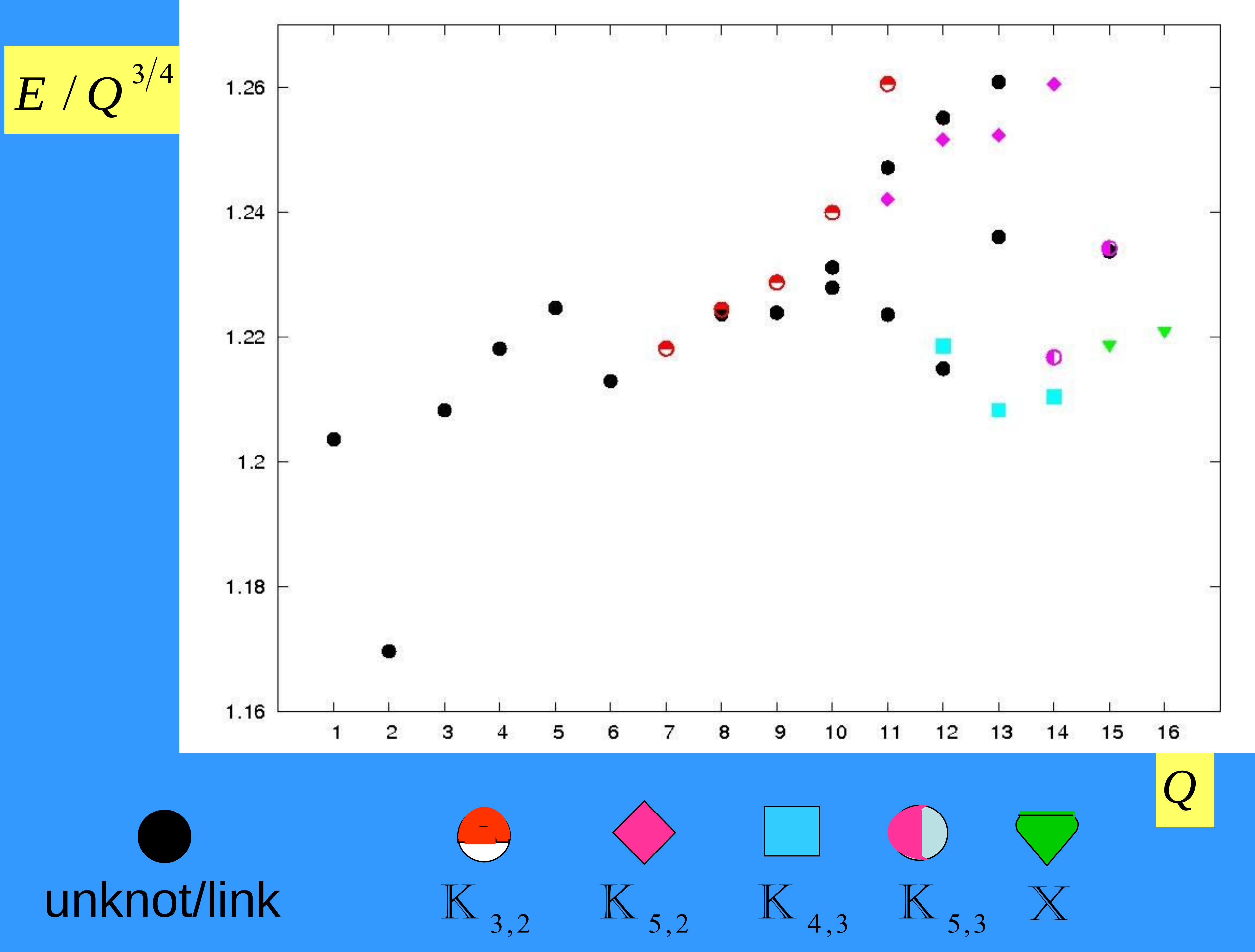
Q	initial $\downarrow$ final					
5	$\mathcal{K}_{3,2} \downarrow \mathcal{L}^{1,1}_{1,2}$					
6	$\mathcal{K}_{3,2} \downarrow \mathcal{L}^{1,1}_{2,2}$					
7	$\mathcal{K}_{3,2} \downarrow \mathcal{K}_{3,2}$	$\mathcal{L}^{1,1}_{2,3} \downarrow \mathcal{K}_{3,2}$				
8	$\mathcal{L}^{1,1}_{3,3} \downarrow \mathcal{L}^{1,1}_{3,3}$	$\mathcal{L}^{2,2}_{2,2} \downarrow \mathcal{L}^{1,1}_{3,3}$	$\mathcal{L}^{1,1}_{4,2} \downarrow \mathcal{L}^{1,1}_{3,3}$	$\mathcal{K}_{5,2} \downarrow \mathcal{L}^{1,1}_{3,3}$	$\mathcal{K}_{3,2} \downarrow \mathcal{K}_{3,2}$	$\mathcal{A}_{8,1} \downarrow \mathcal{K}_{3,2}$
9	$\mathcal{L}^{2,2,2}_{1,1,1} \downarrow \mathcal{L}^{2,2,2}_{1,1,1}$	$\mathcal{K}_{4,3} \downarrow \mathcal{L}^{2,2,2}_{1,1,1}$	$\mathcal{K}_{3,2} \downarrow \mathcal{K}_{3,2}$	$\mathcal{K}_{5,2} \downarrow \mathcal{K}_{3,2}$	$\mathcal{L}^{2,2}_{2,3} \downarrow \mathcal{K}_{3,2}$	$\mathcal{L}^{1,1}_{3,4} \downarrow \mathcal{K}_{3,2}$
10	$\mathcal{L}^{2,2,2}_{1,1,2} \downarrow \mathcal{L}^{2,2,2}_{1,1,2}$	$\mathcal{K}_{4,3} \downarrow \mathcal{L}^{2,2,2}_{1,1,2}$	$\mathcal{L}^{2,2}_{3,3} \downarrow \mathcal{L}^{2,2}_{3,3}$	$\mathcal{L}^{1,1}_{5,3} \downarrow \mathcal{L}^{2,2}_{3,3}$	$\mathcal{K}_{5,2} \downarrow \mathcal{L}^{2,2}_{3,3}$	$\mathcal{K}_{3,2} \downarrow \mathcal{K}_{3,2}$
11	$\mathcal{K}_{4,3} \downarrow \mathcal{L}^{2,2,2}_{1,2,2}$	$\mathcal{K}_{5,2} \downarrow \mathcal{K}_{5,2}$	$\mathcal{K}_{7,2} \downarrow \mathcal{L}^{2,2}_{3,4}$	$\mathcal{K}_{3,2} \downarrow \mathcal{K}_{3,2}$		
12	$\mathcal{K}_{5,3} \downarrow \mathcal{L}^{2,2,2}_{2,2,2}$	$\mathcal{K}_{3,2} \downarrow \mathcal{K}_{4,3}$	$\mathcal{K}_{5,2} \downarrow \mathcal{K}_{5,2}$	$\mathcal{K}_{7,2} \downarrow \mathcal{L}^{2,2}_{4,4}$		
13	$\mathcal{K}_{3,2} \downarrow {}^*\mathcal{K}_{4,3}$	$\mathcal{K}_{5,3} \downarrow {}^*\mathcal{L}^{2,3,3}_{1,2,2}$	$\mathcal{K}_{5,2} \downarrow {}^*\mathcal{K}_{5,2}$	$\mathcal{K}_{7,2} \downarrow \mathcal{L}^{3,3}_{3,4}$		
14	$\mathcal{K}_{4,3} \downarrow \mathcal{K}_{4,3}$	$\mathcal{K}_{5,3} \downarrow \mathcal{K}_{5,3}$	$\mathcal{K}_{3,2} \downarrow \mathcal{K}_{5,2}$			
15	$\mathcal{K}_{5,3} \downarrow \mathcal{X}_{15}$	$\mathcal{K}_{4,3} \downarrow \mathcal{L}^{4,4,4}_{1,1,1}$	$\mathcal{K}_{3,2} \downarrow \mathcal{K}_{5,3}$			
16	$\mathcal{K}_{4,3} \downarrow \mathcal{X}_{16}$	$\mathcal{L}^{3,3,3,3}_{1,1,1,1} \downarrow \mathcal{X}_{16}$	$\mathcal{K}_{3,2} \downarrow \mathcal{X}_{16}$			

Table 2:



$$Q = 9, \quad \mathbb{L}^{2,2,2}_{1,1,1}$$

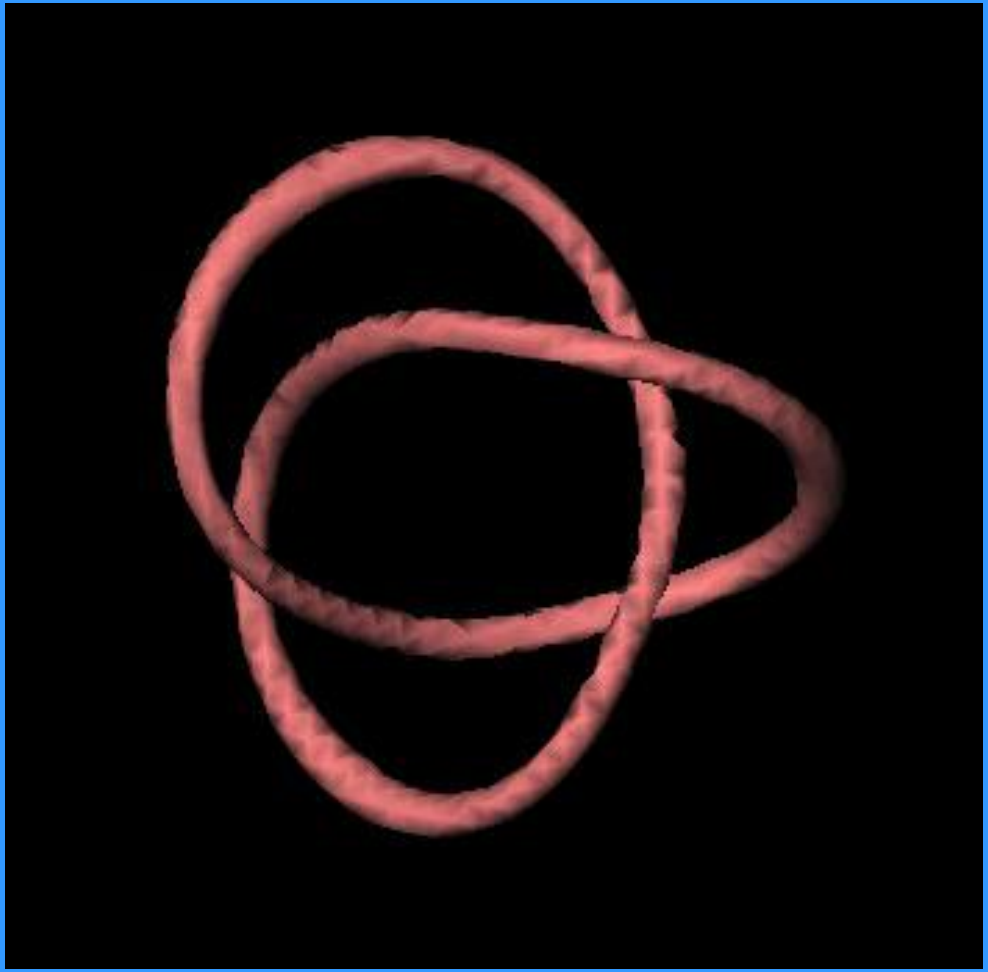

$8\mathcal{L}_{3,3}^{1,1}$
$8\mathcal{K}_{3,2}$
$9\mathcal{L}_{1,1,1}^{2,2,2}$
$9\mathcal{K}_{3,2}$
$10\mathcal{L}_{1,1,2}^{2,2,2}$
$10\mathcal{L}_{3,3}^{2,2}$
$10\mathcal{K}_{3,2}$
$11\mathcal{L}_{1,2,2}^{2,2,2}$
$11\mathcal{K}_{5,2}$
$11\mathcal{L}_{3,4}^{2,2}$
$11\mathcal{K}_{3,2}$
$12\mathcal{L}_{2,2,2}^{2,2,2}$
$12\mathcal{K}_{4,3}$
$12\mathcal{K}_{5,2}$
$12\mathcal{L}_{4,4}^{2,2}$
$13\mathcal{K}_{4,3}$
$13\mathcal{X}_{13}$
$13\mathcal{K}_{5,2}$
$13\mathcal{L}_{3,4}^{3,3}$
$14\mathcal{K}_{4,3}$
$14\mathcal{K}_{5,3}$
$14\mathcal{K}_{5,2}$
$15\mathcal{X}_{15}$
$15\mathcal{L}_{1,1,1}^{4,4,4}$
$15\mathcal{K}_{5,3}$
$16\mathcal{X}_{16}$

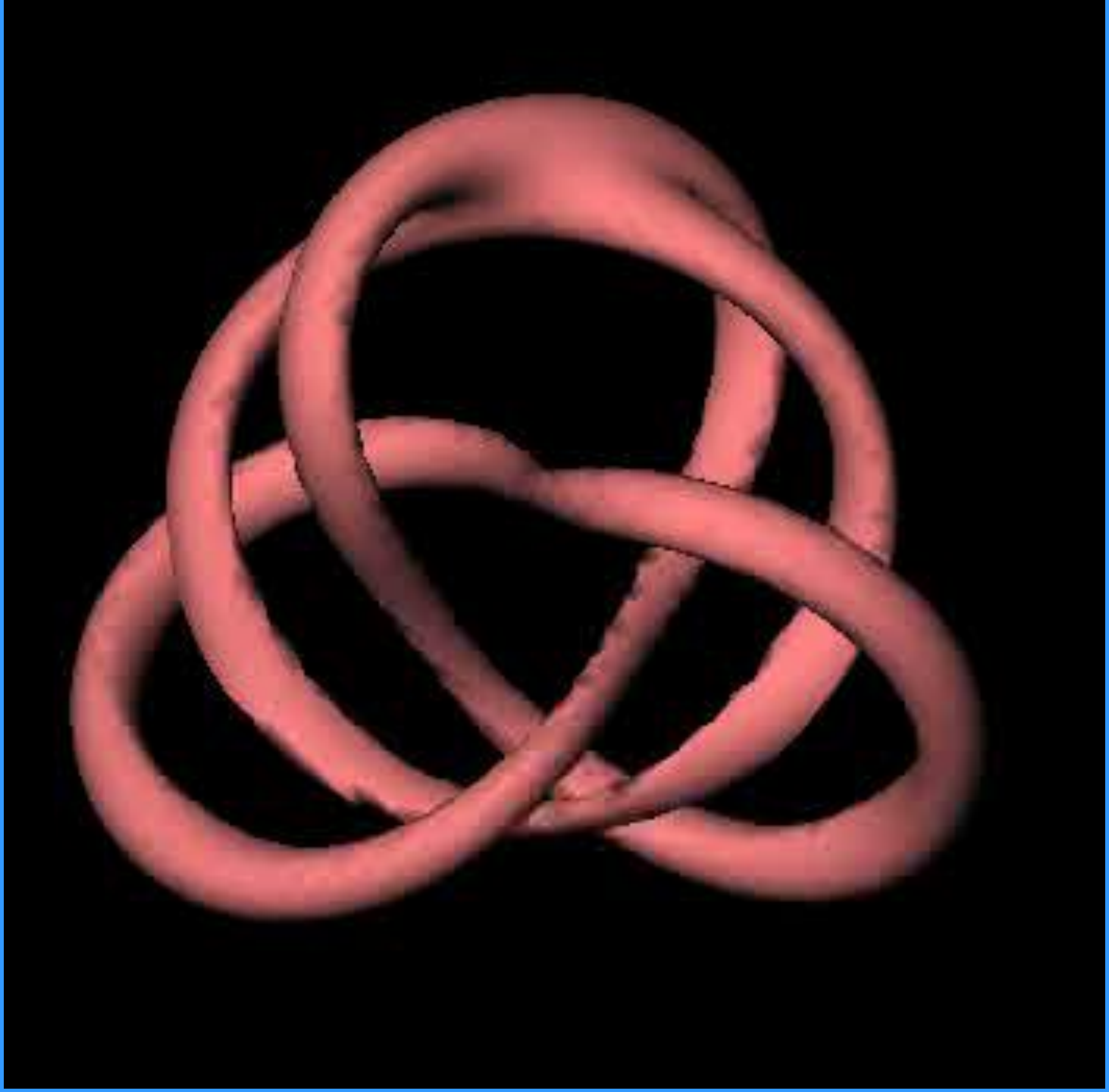


E / Q 3/4
1.26
1.24
1.22
1.2
1.18
1.16
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
Q
unknot/link
K 3,2
K 5,2
K 4,3
K 5,3
X

Knot transmutation

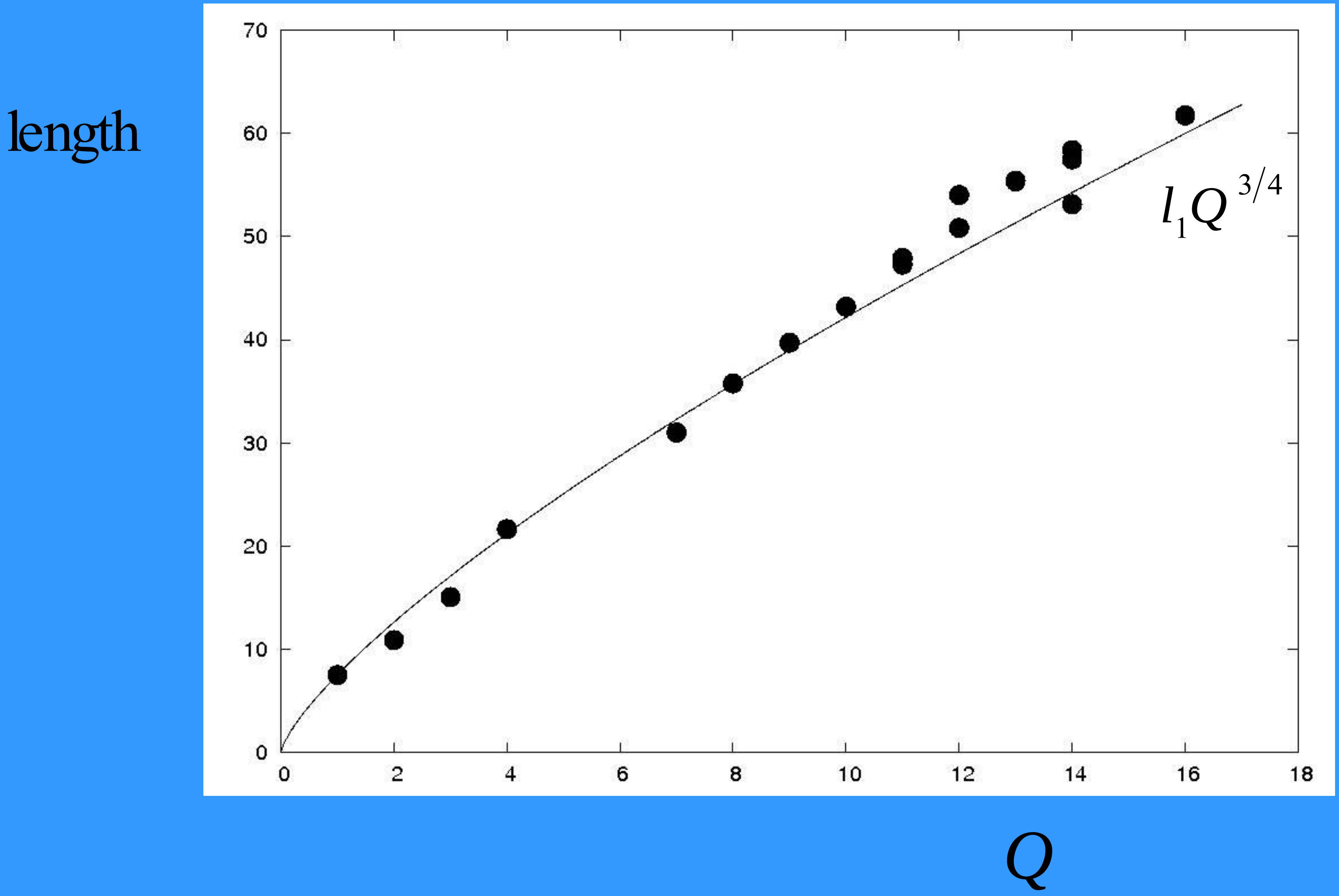
$$Q = 12, \quad \mathbb{K}_{3,2} \to \mathbb{K}_{4,3}$$





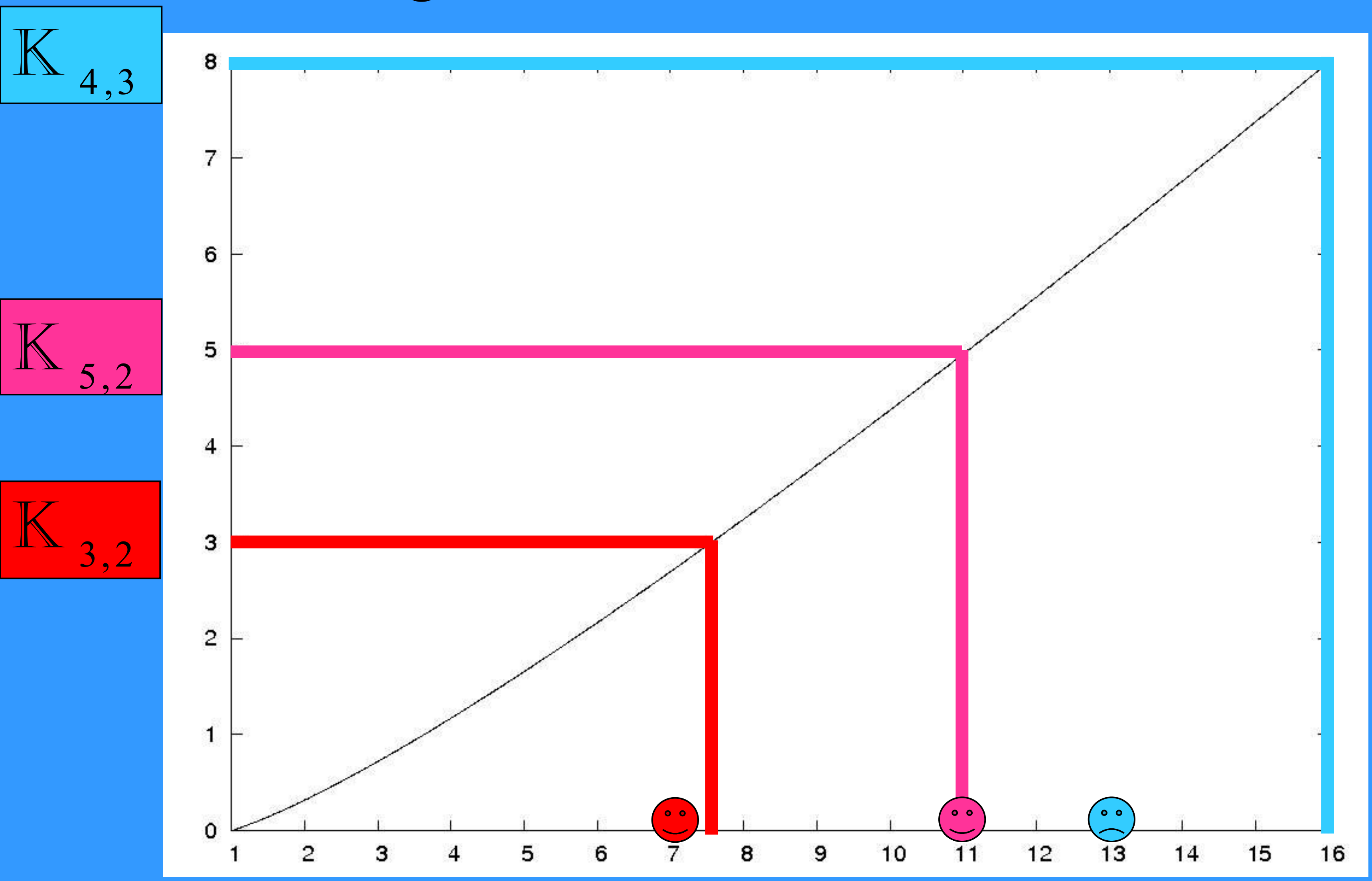
$$Q = 16, \quad \mathbb{X}_{16}$$

String Length

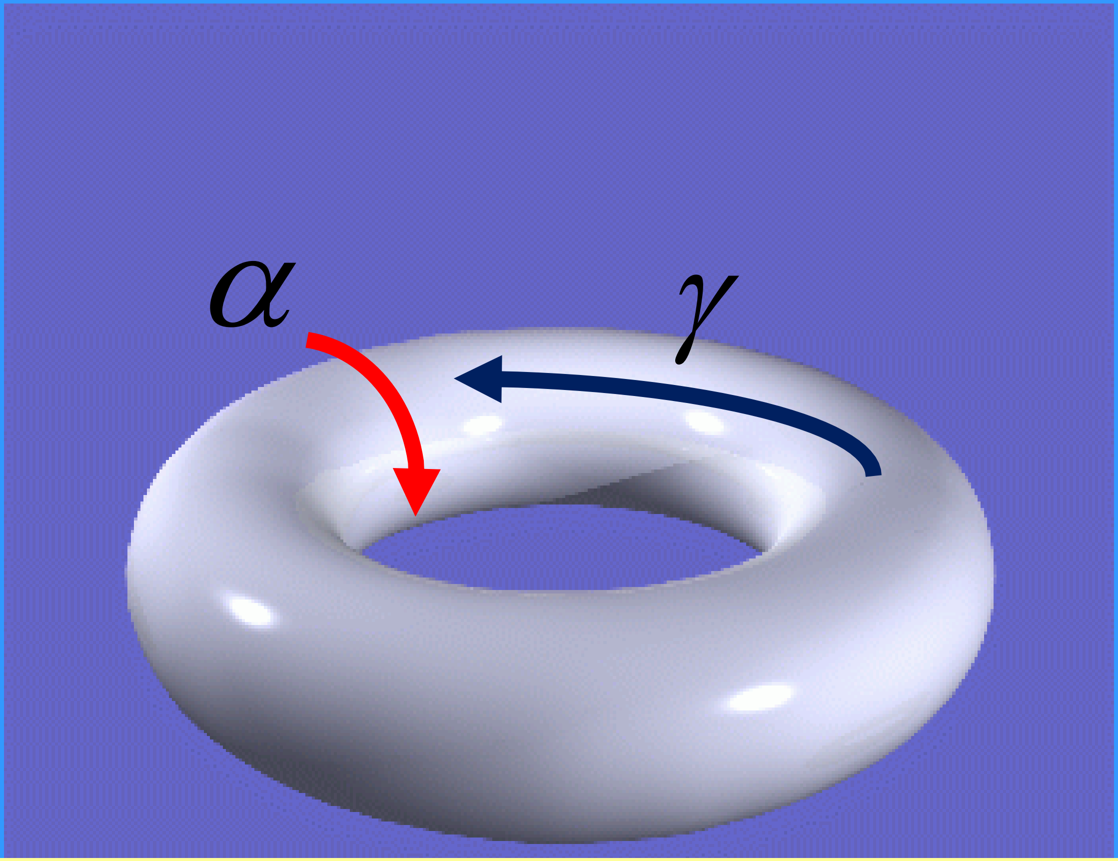


Knot Crossing Number

$$\text{Crossing} = Q - \text{Twist} \approx Q - Q^{3/4}$$



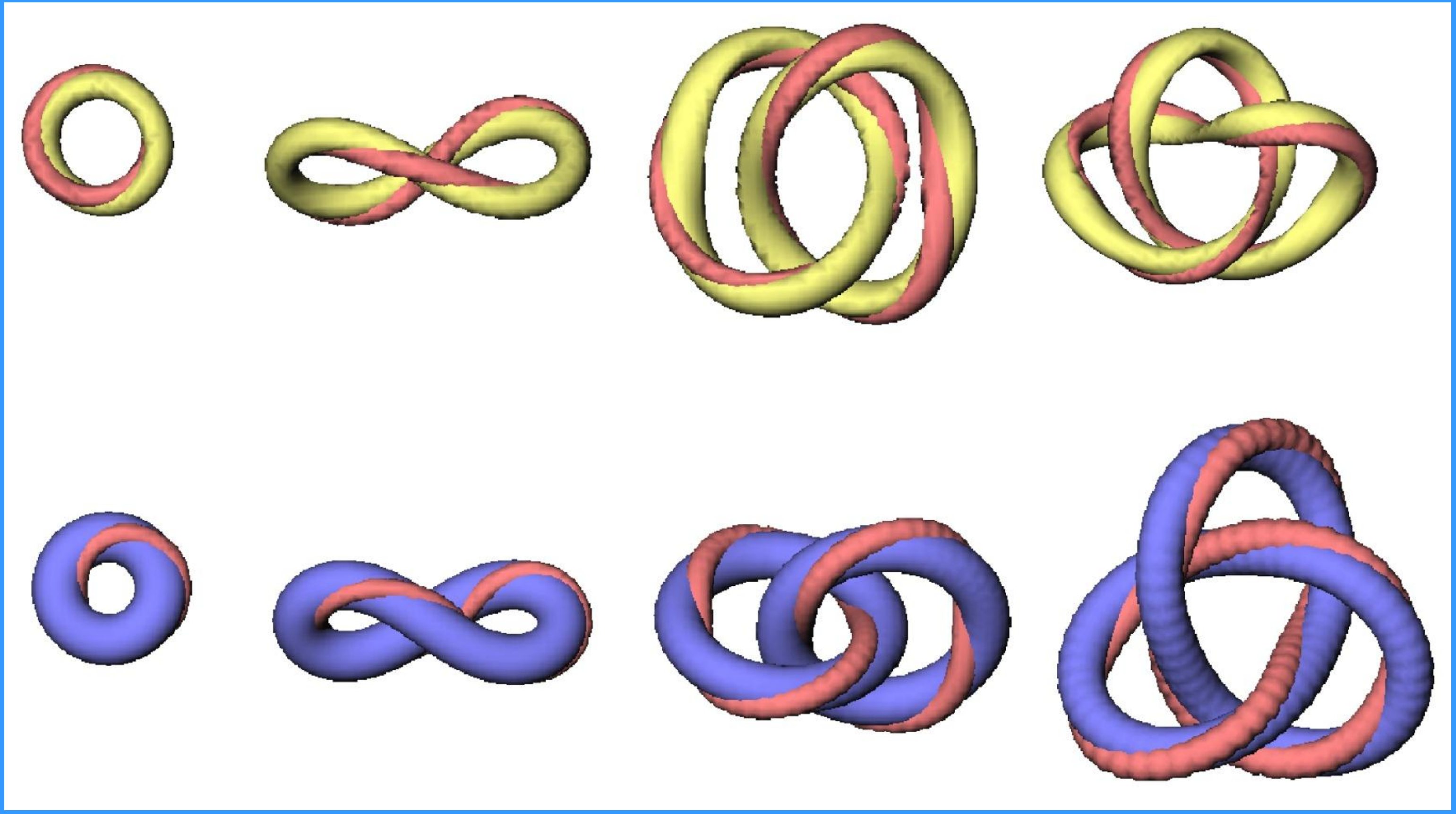
Hopf solitons as elastic rods



$$E = \int_0^L 1 + \kappa^2 + C(\alpha' - \tau)^2 \, ds$$

Hopf solitons as elastic rods

$$E = \int_0^L 1 + \kappa^2 + C(\alpha' - \tau)^2 \, ds$$



Conclusion

- Knots form as minimal energy solitons
- Various torus knots appear at different charges
- Qualitative description in terms of rational maps
- Elastic rod model captures main features
- Other models with solitonic knots
- Physical applications?

Topological Solitons

NICOLAS MANTON
AND PAUL SUTCLIFFE

CAMBRIDGE MONOGRAPHS
ON MATHEMATICAL PHYSICS

THE END