22 July 2008
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D-branes, II

B-branes on a CY X

$D(X)$ means $D^b(X)$

e.g. Non compact CY (toric CY)

$\mathbb{C}^3/\mathbb{Z}_3$ is blown up to produce $X = \mathcal{O}_{\mathbb{P}^2}(-3)$

Let $\mathcal{O}(n)$ be a line bundle over X which restricts to $\mathcal{O}_{\mathbb{P}^2}(n)$ on $\mathbb{P}^2$.

X has homogeneous coords p, x, y, z

Let $S = k[p, x, y, z]$

remove $x=y=z=0$ and quotient by k^*

$(p, x, y, z) \simeq (\lambda^{-3}p, \lambda x, \lambda y, \lambda z)$

Coherent sheaves on X are given by graded S-modules

"modulo sheaves supported at $x=y=z=0$"

- S-modules annihilated by a power of (x, y, z)

Note that $\mathrm{Hom}(\mathcal{O}, \mathcal{O}(1))$ given by all functions of degree 1 in S

e.g. x, y, z, px^4, px^2yz, etc.

infinite dim'l

Fact: $D(X)$ is generated by $\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)$
(i.e., $S, S(1), S(2)$)

$$0 \to S \to S(1)^{\oplus 3} \to S(2)^{\oplus 3} \to S(3) \to \frac{S}{(x,y,z)} \to 0$$

$\Rightarrow$ quasi-iso. between $S(3)$ and

$$\left(S \to S(1)^{\oplus 3} \to \underline{S(2)^{\oplus 3}} \right)$$

↑ 0^{th} position

Similarly, any $S(n)$ can be written as a complex involving $S, S(1), S(2)$.

Any finitely generated module has a finite free resolution in terms of $S(n)$.

Put $M = \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$

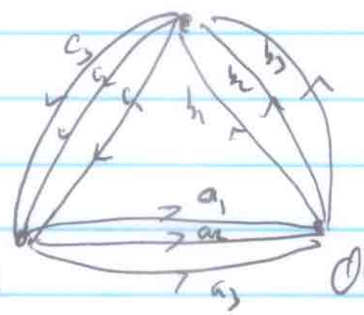
Define $A = \mathrm{End}(M) = \mathrm{Hom}(M, M)$

$$= \begin{pmatrix} \mathrm{Hom}(\mathcal{O},\mathcal{O}) & \mathrm{Hom}(\mathcal{O}(1),\mathcal{O}) & \\ \mathrm{Hom}(\mathcal{O},\mathcal{O}(1)) & & \\ & & \end{pmatrix}$$

A is a non-commutative algebra

- infinite dimensional

Path Algebra of a quiver



$a_1 = b_1 = \times x$

$a_2 = b_2 = \times y$

$a_3 = b_3 = \times z$

$c_1 = \times px$

$c_2 = \times py$

$c_3 = \times pz$

Relations $a_i b_j = a_j b_i \quad \ldots$

Let $D(\text{mod-}A)$ be the bounded derived category of finitely generated right A-modules

An A-module is a quiver representation

Associate a vector space $\mathbb{C}^{N_i}$ to each node v_i and a $N_j \times N_i$ matrix to each arrow $v_i \to v_j$ satisfying the relations.

Consider the functor

$$\mathrm{Hom}_X(M, -) : \text{Sheaves} \to \text{mod-}A$$

If $f: M \to F$ is a sheaf hom, then fa acts as

$$M \xrightarrow{a} M \xrightarrow{f} F$$

We also have a functor

$$- \otimes_A M : \text{mod-}A \to \text{Sheaves}$$

These are adjoint functors

Theorem (Baer, Bondal) + Rickard ...

These adjoint functors give an equivalence if

1) Summands of M generate $D(X)$

2) $\mathrm{Hom}(M, M[n]) = 0$ for any $n \neq 0$

If M satisfies 1) and 2), it is called a "tilting object".

If X is a CY smooth compact variety of dim d

$$\mathrm{Hom}(F, F) = \mathrm{Ext}^d(F, F) = \mathrm{Hom}(F, F[d])$$

What quiver rep corresponds to $\mathcal{O}$?

Hom$_X(M, \mathcal{O})$ is the space of paths ending at node V_0

Call this quiver rep P_0.

If e_0 is defined as the zero-length path beg + ending at node V_0, then $P_0 = e_0 A$.

Similarly, $\mathcal{O}(j)$ maps to $P_j = e_j A$ for $j = 1, 2$.

Let L_j be the one-diml quiver rep. concentrated at node j, $j = 0..2$.

We have an exact sequence of A-modules

$$0 \to P_0 \xrightarrow{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} P_1^{\oplus 3} \xrightarrow{\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}} P_2^{\oplus 3} \xrightarrow{(px\ py\ pz)} P_0 \to L_0 \to 0$$

So L_0 is q-iso to

$$\mathcal{O} \to \mathcal{O}(1)^{\oplus 3} \to \mathcal{O}(2)^{\oplus 3} \xrightarrow[(px\ py\ pz)]{} \mathcal{O}$$

But we know

$$\mathcal{O} \to \mathcal{O}(1)^{\oplus 3} \to \mathcal{O}(2)^{\oplus 3} \to \mathcal{O}(3)$$

is trivial, so L_0 is

$$\mathcal{O}(3) \xrightarrow{p} \mathcal{O}$$

But we have a short exact sequence

$$0 \to \mathcal{O}(3) \xrightarrow{p} \mathcal{O} \to \mathcal{O}_D \to 0$$

where D is $\mathbb{P}^1$.

$\Rightarrow$ L_0 is $\mathcal{O}_D$.

Homework: L_1 is $\Omega_D(1)[1]$, L_2 is $\mathcal{O}_D(-1)[2]$.

What is a point? $\mathcal{O}_x$, $x \in X$

- This is a quiver representation with dimension vector $(1,1,1)$

If all arrows are 0, get $L_0 \oplus L_1 \oplus L_2$

~~In general~~

For a general quiver we have P_i's, L_i's associated to each node.

$$\cdots \to \bigoplus_{\substack{a \\ h(a)=i \\ t(a)=j}} P_j \to P_i \to L_i \to 0$$

$\Rightarrow \dim \mathrm{Ext}^1(L_i, L_j) = \#$ arrows $j \to i$

"Quiver Gauge Theories"

General methods in toric geometry

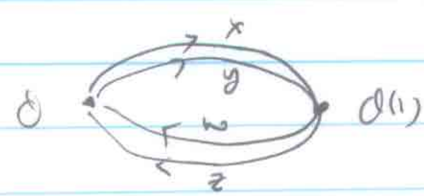
- always appear to be able to ~~detect~~ find an M as a sum of line bundles

Conifold: x, y, z, w

$+1 \quad +1 \quad -1 \quad -1$

remove $x = y = 0$

$M = \mathcal{O} \oplus \mathcal{O}(1)$



$xzy = yzx$ (Quartic Superpotential)

Phases We had $\mathcal{O}_{\mathbb{P}^2}(-3) = X$ given by $S = \mathbb{C}[p, x, y, z]$

$$(p, x, y, z) \sim (\lambda^{-3} p, \lambda x, \lambda y, \lambda z)$$

we removed $(p, 0, 0, 0)$

Instead remove $(0, x, y, z)$.

$p \neq 0 \Rightarrow$ use $\mathbb{C}^*$ action to fix $p = 1$

leaves $\mathbb{Z}_3$ acting on x, y, z.

$\mathbb{C}^3 / \mathbb{Z}_3$.

Now S, $S(1)$, $S(2)$ generate $D(X)$ again.

$S(3) \cong S(0)$

$\Rightarrow$ Same quiver