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D. Auroux

<u>Special Lagrangian fibrations and mirror symmetry</u>

<u>SYZ</u> 0706.3207
+ in progress with M. Abouzaid, L. Katzarkov

Calabi-Yaus

(X, J, ω, Ω) Kähler mfld, $K_X \cong \mathcal{O}_X$
$\Omega \in \Omega^{n,0}(X)$

$$\updownarrow$$

$(X^\vee, J^\vee, \omega^\vee, \Omega^\vee)$

Strominger-Yau-Zaslow

$X, X^\vee$ carry dual fibrations by special Lagrangian tori *.

($\to$ Joyce, Fukaya, Kontsevich-Soibelman, <u>Gross-Siebert</u>)

point $p \in X^\vee \longleftrightarrow \mathcal{O}_p \in D^b Coh(X^\vee) \leftrightarrow \mathcal{L}_p \in D^b \mathcal{F}(X)$

$\mathcal{L}_p$ = Lagrangian $T^n \subset X$, flat line bundle $(L, \nabla) \to T^n$; $\mathcal{F}(X)$ = Fukaya category

$H^*(T^n, \mathbb{C}) \simeq Ext^*(\mathcal{O}_p, \mathcal{O}_p)$

$X^\vee$ as a moduli space of

$$\left\{ \begin{array}{l} L \subset X \ \text{(special) Lagrangian tori} \\ \nabla \ \text{flat } U(1) \text{ conn. on } \mathbb{C} \to L \end{array} \right.$$

Landau-Ginzburg models.

X Kähler manifold, $c_1(X) \neq 0$.

$\cup$

D anticanonical divisor, $[D] = c_1(X)$.

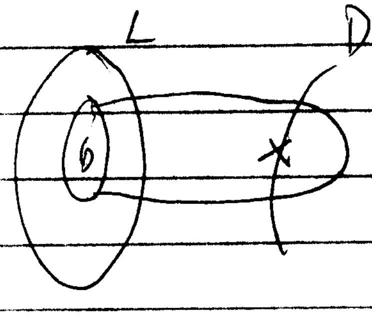
$X \setminus D$ open Calabi-Yau

$D = \sigma^{-1}(0)$, $\sigma \in H^0(X, K_X^{-1}) = \Omega^{0,n}$

$\Omega = \sigma^{-1}$ holom vol. form with poles along D.

$\rightsquigarrow M$ open CY mirror to $X \setminus D$ (via SYZ)

$L \subset X \setminus D$ Lagrangian $\to HF_{X \setminus D}(L, L)$ ✓



$HF_X(L, L) \to$ often zero

L ~~zero~~ object in $F(X)$

$HF(L, L')$ typically not defined — obstruction

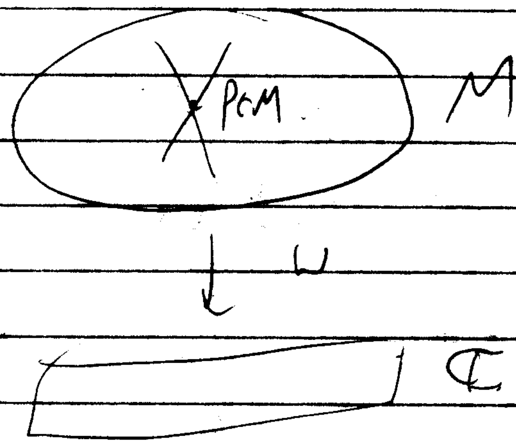
$\partial^2 = m_0$

Fukaya-Oh-Ohta-Ono

mirror: Landau-Ginzburg model
M Kähler manifold, $W: M \to \mathbb{C}$ holomorphic

$\cancel{D^b(Coh(M))}$

$\rightsquigarrow D^b_{sing}(M,W) \cong MF(W)$ (Kapustin-Orlov ...)



$D^b(Coh(W=\lambda))/Perf$ singularities of W.

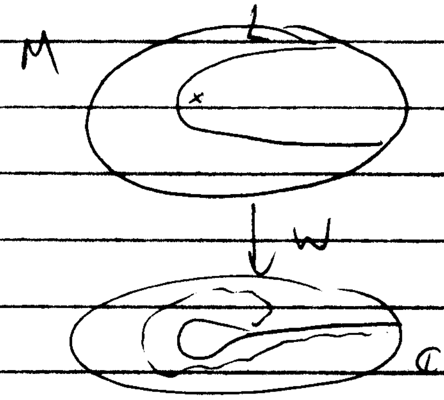
$\mathcal{O}_p = 0$ in D^b_{sing} unless $dW(p)=0$.

$\mathcal{E}_0 \underset{Q}{\overset{P}{\rightleftarrows}} \mathcal{E}_1$

$PQ = (W-\lambda)\, Id_{\mathcal{E}_1}$

$QP = (W-\lambda)\, Id_{\mathcal{E}_0}$

HMS: $D^b_{sing}(M,W) \cong D^b F(X)$



$D^b Fu(M,W)$

$\to$ Serre functor $S \longleftrightarrow - \otimes K_X[n]$

$\to$ canonical natural transformation $S \Rightarrow \mathrm{id}$

$\updownarrow$

section of K_X^{-1}

$X, D \rightsquigarrow M, W$

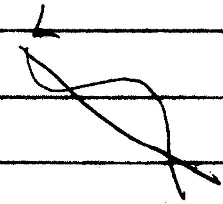
symplectic geom ⇝ complex, holom. f^n

$L \subset X \setminus D$ special Lagrangian if $\omega|_L = 0$

$\mathrm{Im}\,\Omega|_L = 0.$

Deform^{ns} of special Lagrangian $\longleftrightarrow H^1(L,\mathbb{R})$

$v \in C^\infty(NL)$



$\longleftrightarrow \alpha = -\iota_v \omega$

$\cap$

$\Omega^1(L,\mathbb{R})$

$d\alpha = 0$

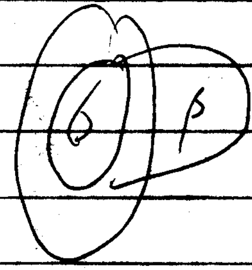
$d^*(\psi\alpha) = 0$

$\uparrow |\Omega|$

$$M = \left\{ (L,\nabla) \;\middle|\; \begin{array}{l} L \subset X \setminus D, \text{ SLag } T^n \\ \nabla \text{ flat } U(1) \text{ conn. on } \underline{\mathbb{C}} \to L \\ \qquad\qquad \text{up to gauge} \end{array} \right\}$$

$$T_{(L,\nabla)} M = \left\{ (v, i\alpha) \in C^\infty(NL) \oplus i\,\Omega^1(L,\mathbb{R}) \;\middle|\; -\iota_v\omega + i\alpha \in H^1(L)\otimes\mathbb{C} \right\}$$

Complex structure



$\beta \in \pi_2(X,L)$

$$z_\beta = \exp\left(-\int_\beta \omega\right) \mathrm{hol}_{\partial\beta}(\nabla) : M \to \mathbb{C}^*$$

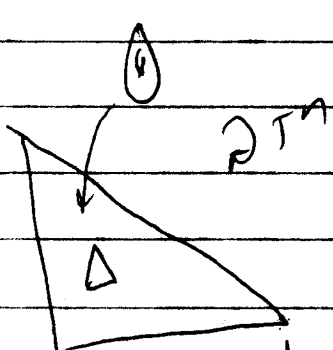
holom. function

<u>Toric manifolds</u> (Hori, Cho-Oh ...)

$X = \mathbb{CP}^2 \supset (\mathbb{C}^*)^2 = X \setminus D$

(or any toric)

D = toric divisor = 3 lines

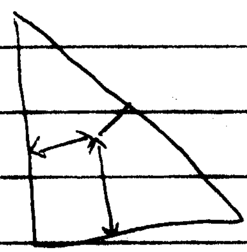


moment polytope

ω = toric Kähler form

$$\Omega = \frac{dz_1 \wedge dz_2}{z_1 z_2}$$

T^n-orbits are special Lagrangian.



$S^1(r_1) \times S^1(r_2)$ bounds $D(r_1) \times \{x_2\} \leftarrow \beta_1$

$\{x_1\} \times D(r_2) \leftarrow \beta_2$

$\cdots \leftarrow \beta_3$

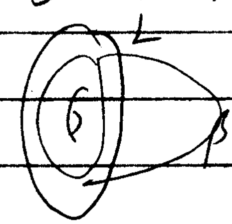
z_{β_1}, z_{β_2} coords on $\mathcal{M} \subset (\mathbb{C}^*)^2$

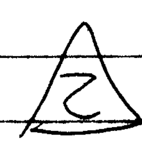
(bounded subset?)

$$W(L, \nabla) = \sum_{\substack{\beta \in \pi_2(X, L) \\ \mu(\beta) = 2}} n_\beta z_\beta$$

$\mu(\beta) = 2\beta \cdot D$

n_β = # holom. discs in class β through a generic pt $p \in L$





$W = z_{\beta_1} + z_{\beta_2} + z_{\beta_3}$ $\qquad [\beta_1] + [\beta_2] + [\beta_3] = [\mathbb{CP}^1]$

$$W = z_{\beta_1} + z_{\beta_2} + z_{\beta_3}$$

$$z_{\beta_1} z_{\beta_2} z_{\beta_3} = \exp\left(-\int_{\mathbb{CP}^1} \omega\right)$$

X = toric variety $\qquad$ D = toric anticanonical divisor

$\cup$
H = one facet

$\cup$
Y hypersurface in H, smooth

$\hat{X}_Y$ blowup of X along Y

$\hat{\omega}$ Kähler form

$\hat{\Omega} = \pi^* \Omega$.

$$\begin{array}{c} \hat{X}_Y \\ \downarrow \pi \\ X \end{array}$$

polarity

$\hat{D}$ = proper transform of D.

Program (in progress)

- build SYZ fibration on $\hat{X} \setminus \hat{D}$
- build $(M, W) \leftarrow$ discontinuous, piecewise hol. $\boxed{n_\beta \text{ jump! Wall-crossing}}$
- analyse wall-crossing

$\Rightarrow$ find instanton corrections to mirror

$(\check{M}, W)$

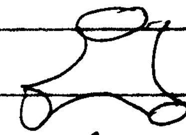
Laurent holo. function

<u>Why?</u> $D^b Coh(\hat{X}_Y) = \langle D^b Coh(Y), D^b Coh(X) \rangle$

~~Bondal-Orlov~~

Can think of (part of) mirror of $\hat{X}_Y$ as substitute for ~~mirror of Y~~

<u>Example</u> mirror of pair of pants $\mathbb{CP}^1 \setminus 3$ pts



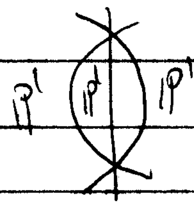
$\updownarrow$

LG-model

$W = xyz$ on $\mathbb{C}^3$.

genus 2 curve $\underset{\substack{\| \\ Y}}{C} \underset{(2,3)}{\subset} \mathbb{P}^1 \times \mathbb{P}^1 \subset \underset{\substack{\| \\ X}}{(\mathbb{P}^1)^3}$

$\rightsquigarrow$ 3-fold $\tilde{M}$, $\tilde{W}$



<u>Toy example</u> $\mathbb{C}^* \times \mathbb{C}$, $\Omega = \dfrac{dx \wedge dy}{xy}$, ω std

blow up $(1,0)$ $\quad D = \mathbb{C}^* \times \{0\}$.

$\rightsquigarrow$

$\tilde{X}$, $\tilde{D}$ proper transform, $\tilde{\omega} = \tilde{\omega}_\varepsilon$ $\quad \int_E \tilde{\omega} = \varepsilon$

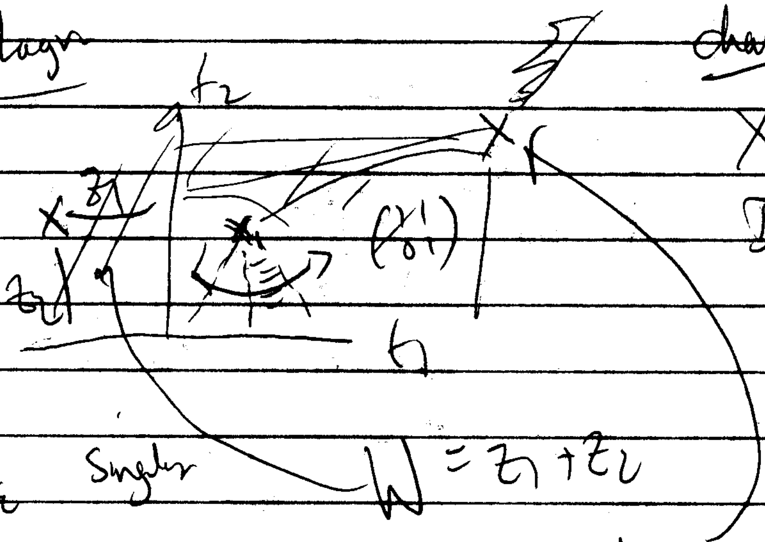
S^1 acts by y-rotation

fixed pts $= \tilde{D} \cup \{p_0\}$

$\mu =$ moment map for S^1-action

$$L_{t_1,t_2} = \left\{ \begin{array}{l} \log|x| = t_1 \\ \mu = t_2 \end{array} \right\}$$

Special Lagn



change

$X = \mathbb{C} \times \mathbb{C}$

$D = \mathbb{C} \times 0 \cup 0 \times \mathbb{C}$

$L_{0,\varepsilon}$ singular

$W = z_1 + z_2$

(z_1, z_2)
$(\tilde{z}_1, z_2)$

$W =$ Smoothg also $= \tilde{z}_1 + z_2 + \tilde{z}_1 z_2 e^{\varepsilon}$

glue $(z_1, z_2) \updownarrow (\tilde{z}_1, z_2)$ $z_1 = \tilde{z}_1(1+e^{\varepsilon} z_2)$

$\rightsquigarrow \{(u, v, z_2) \mid uv = 1 + e^{\varepsilon} z\} \subset \mathbb{C}^2 \times \mathbb{C}^*$

$W = u + z_2$