D. Ben Zvi

TFT & Representation Thy

Geometric Langlands Program

$\Uparrow$ [Kapustin - Witten]

Topological Field Thy

TFT in codim 1 & 2

- toy models

$\rightsquigarrow$ geometry of derived categories

$\boxed{\text{TFT}}$ algebraic structures axiomatizing coarse features of QFT

M = spacetime smooth oriented n-mfld

n-dim TFT (w/ 1 tier)

is a multiplicative invariant of Ms

"partition fn" $Z: M \mapsto Z(M) \in \mathbb{C}$

$$\amalg \mapsto \cdot$$
$$\phi \mapsto 1$$

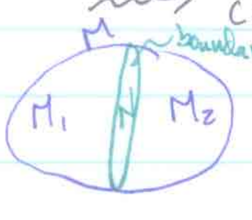
$$Z(M) = \int \mathcal{D}\varphi \, e^{-S(\varphi)}$$

$\{\varphi\}$ fields on M: local expressions

(fns, sections, bundles, connections, maps to target $M \to X$)

Locality of Z in M

$\rightsquigarrow$ cut & paste formula



$$Z(M) = \langle Z(M_1), Z(M_2) \rangle_{Z(N)}$$

2-tier TFT

- $Z : (n-1)\text{ mflds} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$

$$N \longmapsto Z(N)$$

functorial & multiplicative

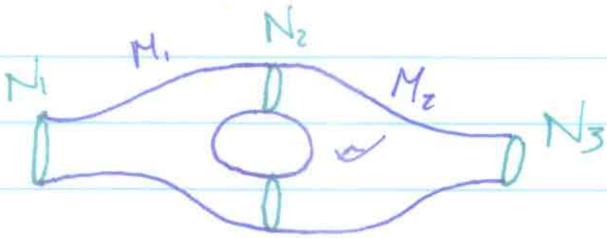
$$\amalg \longmapsto \otimes$$

$$\emptyset \longmapsto \mathbb{C}$$

$$(\ )^{op} \longmapsto (\ )^{*}$$

- $Z :$ M, n-mflds w/ boundary $\longmapsto Z(M) \in Z(\partial M)$

M s.t. above gluing rule holds

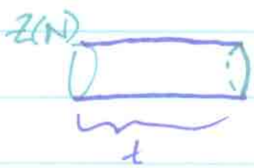


$$Z(M) \in Z(\partial M) = Z(N_3) \otimes Z(N_1)^{*} = \mathrm{Hom}(Z(N_1), Z(N_3))$$

$$Z(N_1) \xrightarrow{Z(M_1)} Z(N_2) \xrightarrow{Z(M_2)} Z(N_3)$$

If think of N as a fixed-time slice of M

$Z(N)$ is Hilbert space



$$Z(N \times I) = id \in \mathrm{Hom}(Z(N), Z(N))$$

Construction $\quad Z(N)$ = linearization/quantization (i.e. some kind of fn on) space of boundary values of fields $\varphi|_{\partial M = N}$

$$Z(M)(\varphi_\partial) = \int_{\varphi|_N = \varphi_\partial} e^{-S(\varphi)} \mathcal{D}\varphi$$

<u>Evolution</u>) push-pull operators

$$\begin{tikzcd} & \text{Fields}(M) \arrow[dl] \arrow[dr] & \\ \text{Fields}(N_1) & & \text{Fields}(N_2) \end{tikzcd}$$

$$v \in Z(N_1) \longrightarrow Z(N_2)$$

$$Z(M)(v)(\varphi_2) = \int_{\varphi|_{N_2} = \varphi_2} \mathcal{D}\varphi \; v(\varphi|_{N_1})\, e^{-S(\varphi)}$$

<u>Fn thy 101</u>

X finite set $\rightsquigarrow$ Fun(X) unambiguous

- carries comm. mult.
- $\pi : X \longrightarrow Y$

$$\pi_* : \mathrm{Fun}(X) \underset{\longrightarrow}{\longleftarrow} \mathrm{Fun}(Y) : \pi^*$$

All linear opers $\mathrm{Fun}(X) \longrightarrow \mathrm{Fun}(Y)$ are integral transforms w/ kernel $K = K(x,y) \in \mathrm{Fun}(X \times Y)$

$$\begin{tikzcd} & K \\ & X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd}$$

$$K * f(y) = \int f(x)\, K(x,y)\, dx = \pi_{2*}(\pi_1^* f \cdot K)$$

$$\mathrm{Fun}(X) \simeq \mathbb{C}^n$$
$$\mathrm{Fun}(Y) \simeq \mathbb{C}^m$$
$$\mathrm{Fun}(X \times Y) = \mathrm{Mat}_{m \times n}$$

convolution

$$\begin{tikzcd}
 & X \times_Y X \times_Y X \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & \\
\underset{A}{X \times_Y X} & \underset{A \cdot B}{X \times_Y X} & \underset{B}{X \times_Y X}
\end{tikzcd}$$

$$A \cdot B = \pi_{13*}(\pi_{12}^* A \ \pi_{23}^* B)$$

$$\mathrm{Fun}(X \times_Y X) = \begin{array}{c c} & \\ X & \begin{pmatrix} \pi_0 = Y_1 & & 0 \\ & Y_2 & \\ 0 & & Y_3 \end{pmatrix} \\ & X \end{array}$$

G finite group

group alg.

$$\mathrm{Fun}(G) = \mathbb{C}G$$
$$\mu : G \times G \to G$$
$$f * g = \mu_*(\pi_1^* f \cdot \pi_2^* g)$$

eg | 2d TFT : G-gauge thy , G finite

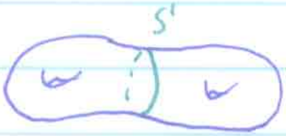
fields: $\mathrm{Loc}_G(M) = \mathcal{M}_G(M)$ $\left\{ \begin{array}{l} \text{principal bundles on } M \iff G\text{-covering spaces} \\ \iff \text{(flat) } G\text{-connections on } M \\ \iff \{\pi_1(M) \to G\}/\mathrm{conj.} \end{array} \right.$

Σ  surface

$Z(\Sigma) = \#\text{ fields} = \#\{\pi_1(\Sigma)\to G\}/_{\text{conj}}$

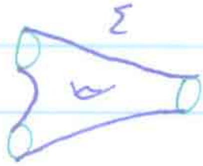
$= \#\left\{ \begin{matrix} A_1,\dots,A_g \\ B_1,\dots,B_g\end{matrix} \in G \quad \prod [A_i, B_i] = 1 \right\}$

S^1

 $Z(S^1)$ = linearization on spc of fields on S^1

$= \mathrm{Fun}\left(\frac{G}{G}\right) = \mathbb{C}G^G$ = class fns

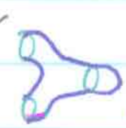
Σ

 $Z(\Sigma): Z(S^1)^{\otimes 2} \to Z(S^1)$

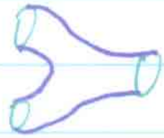
"folk"
Thm | 2d TFT (2 tiers)
$\Longleftrightarrow$
commutative Frobenius algebra
(unital com. alg. w/ non-deg. trace)

$Z \mapsto \mathcal{H} = Z(S^1)$

 $\mathbb{C}\to\mathcal{H}$  $\mathcal{H}\to\mathbb{C}$ $\mathcal{H}\otimes\mathcal{H}$  $\mathcal{H}$

$Z($  $) = Z($  $)$

$\mathcal{H}\otimes\mathcal{H}\to\mathbb{C}$ nondeg.

 =  unique (up to diffeo)

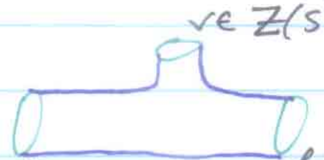
eg/ $\mathbb{C}G$ is an assoc. Frobenius alg

$tr(f) = f(1)$ (dim of a char)

$Z(S^1) = \mathbb{C}G^G = \underbrace{Z}_{\text{center}}(\mathbb{C}G)$, comm. Frob. alg.

geometric basis: conjugacy classes in G

to solve our thy



simultaneously diagonalize $\mathcal{H} = \mathbb{C}G^G$

$\longleftrightarrow$ spectral decomposition

$\operatorname{Spec} \mathbb{C}G^G = \{ \text{homomorphisms } \mathbb{C}G^G \to \mathbb{C} \}$

$=$ joint eigenvalues

$= \{\text{irred. } \cancel{\text{chars}} \text{ reps of } G\} = \widehat{G}$

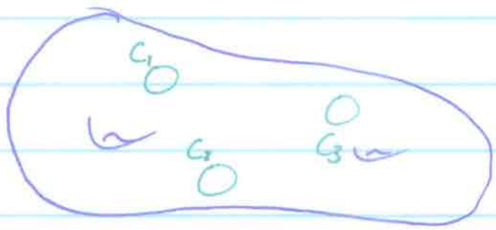
$\mathbb{C}G^G = \operatorname{Fun}(\widehat{G})$ ptwise

$\operatorname{char} V(g) \longleftrightarrow V$

$= tr_V(g)$

"QM on $\mathcal{H}$ is completely integrable"

$$Z(\Sigma) = \sum_{\chi \in \text{irreps of } G} \left(\frac{|G|}{\dim \chi} \right)^{2\underset{\text{genus}}{g}-2}$$



C_i conjugacy classes in G

$$\downarrow$$

$$1_{C_i} \in Z(S^1)$$

$Z(S^1) \rightsquigarrow$ algebra of local opers