23 July 2008
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Topological Field Theory and Representation Theory, II

2-tier TFT

$Z : \{M^n\} \to \mathbb{C}$

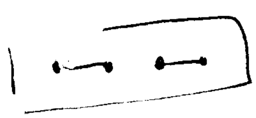
$Z : \{N^{n-1}\} \to \mathrm{Vect}$

Local operators

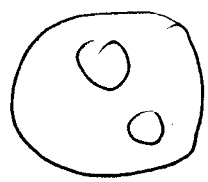
In any TFT (n-dim)

$Z(S^{n-1})$ is an algebra given by "pair of pants"

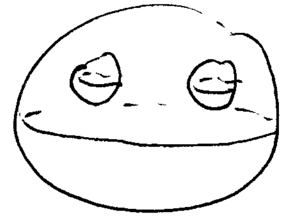
$n=1$



$n=2$



$n=3$



$Z(S^2) \otimes Z(S^2) \longrightarrow Z(S^2)$ or $Z(S^1)$ or $Z(S^0)$

$Z(S^0)$ associative
$Z(S^1)$ commutative
$Z(S^2)$ more commutative

Ob: n-1 manifolds $\longrightarrow$ Vect
Mor: n-manifold cobordism/diffeos   nums.

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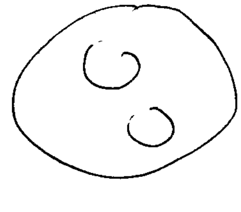
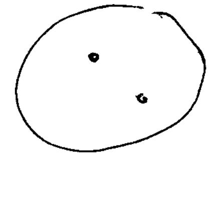
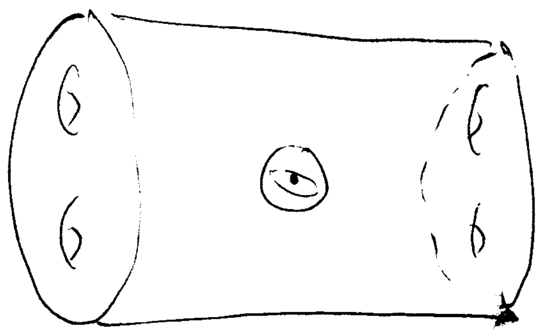
topological OPE





$Z(S^{n-1}) \circlearrowright Z(N)$ any N^{n-1}

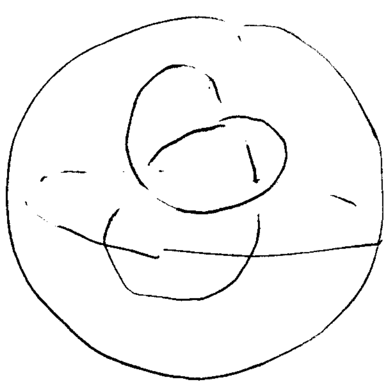


$N \times I$ $\qquad x \in N^{n-1}$

$Z(N) \otimes Z(S^{n-1}) \to Z(N)$

<u>Other labelings</u>

3-manifold $\supset L$



$\partial(\text{tub}) \simeq S^1 \times S^{n-2}$

$v \in Z(S^1 \times S^{n-2})$

<u>order & disorder</u>

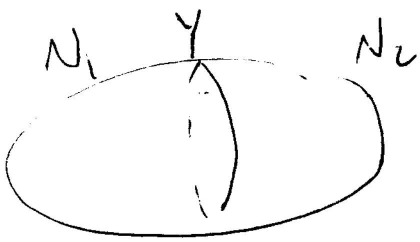
operators can be measurements $\mathcal{O}_x(\varphi)$

or can be functionals on ∂ values of fields, supported on fields with a given singularity

3 tiers (extended TFT)

express locality of the vector space $Z(N)$ wrt cut & paste

$$N = N_1 \coprod_{Y^{n-2}} N_2$$



$$\Rightarrow Z(N) = \langle Z(N_1), Z(N_2) \rangle_{Z(Y)}$$

$Z(Y) \rightsquigarrow$ a $\mathbb{C}$ linear category

$$Z(N_1) \in \mathrm{Ob}\, Z(\partial N_1)$$

needed to fix boundary values on ∂M.

$\Rightarrow$ $\mathbb{C}$-valued function on bdry values of fields

$\Rightarrow$ vector bundle (sheaf) on space of bdry values Fields(Y)

$Z(Y) =$ sheaves on Fields(Y)

$\mathbb{C}$-linear category

2-category

Objects: $n-2$ manifolds Y

1-morphisms: $n-1$ manifold with bdry (cobordisms) 

2-morphisms: n manifold with corner 

metacobordisms / diffeo

TFT: multiplicative functor to some target

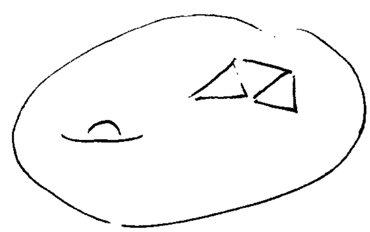
Ob: linear categories

1-morphism: functors

2-morphism: natural transformations 

Philosophy

points are boring



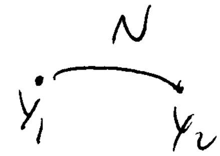
2d gauge theory, G finite

$Z(\cdot)$ = vector bundles on $\underbrace{\text{Fields on } Y = \cdot}_{= G\text{-bundles}}$

$= \mathrm{Vect}(\cdot_G \cong BG = \cdot/G)$

$= \mathrm{Rep}_{\mathbb{C}}\, G = \mathbb{C}G$-modules

$$\begin{tikzcd} & \mathrm{Fields}(N) \arrow[dl] \arrow[dr] & \\ \mathrm{Fields}(Y_1) & & \mathrm{Fields}(Y_2) \end{tikzcd}$$



$$\mathrm{Vect}\,F(Y_1) \xrightarrow{\pi_1^*} \mathrm{Vect}(F(N)) \xrightarrow{\pi_2^*} \mathrm{Vect}\,F(Y_2))$$

Moore-Segal
open-closed 2d TFT

$$V, W \in \mathrm{Rep}\,G \implies \mathrm{Hom}_G(V, W)$$

$\underline{\mathrm{Vect}} = Z(\emptyset)$

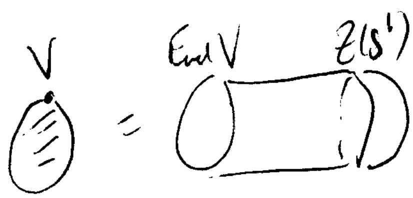
$$\mathrm{Hom}(U,V) \otimes \mathrm{Hom}(V,W) \to \mathrm{Hom}(U,W)$$

composition

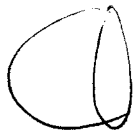
$\mathbb{C}G$ is a Frobenius algebra

Modules over a NC Frobenius algebra form a Calabi-Yau category

End V is a Frobenius algebra: nondegenerate trace



$$\mathrm{End}\,V \to Z(S^1) \to \mathbb{C}$$



$$\mathbb{C} \to Z(S^1) = (\mathbb{C}G)^G = \text{center of } \mathbb{C}G$$
$$\xrightarrow{\text{act}} \text{End}(V) \quad V = \text{rep of } \mathbb{C}G$$

$$V = \bigoplus_{V_i \text{ irred}} V_i \otimes \mathbb{C}^{n_i}$$

$$\text{End}\, V = \bigoplus \text{Mat}_{n_i \times n_i}$$
$$\text{Id}_{n_i} \longrightarrow \text{char}(V_i) \in \mathbb{C}G^G$$

Moore-Segal, Costello

2d TFT, $Z(\cdot)$ is a CY category

$Z(S^1)$ is the center ($\text{End}(\text{Id})$) or Hochschild cohomology

Example σ-model on a finite orbifold

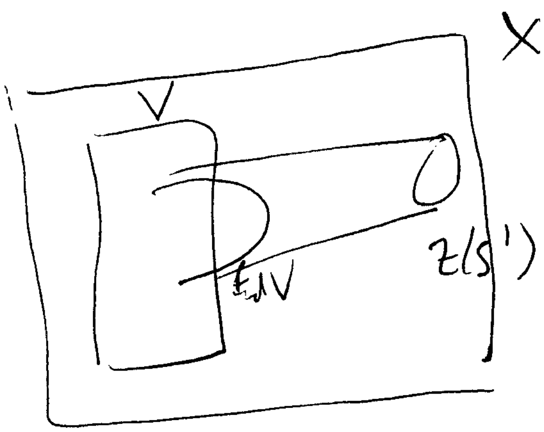
$$X = \coprod \bullet / H_i$$

$$F(Y) = \text{Maps}(Y \to X)$$
$$Z(\Sigma) = \#\, \text{Map}(\Sigma \to X)$$
$$= \sum_i \#\, H_i\text{-bundles on } \Sigma / \sim$$

$$Z(S^1) = \mathrm{Fun}(\underbrace{\mathrm{Map}(S^1 \to X) = LX}_{= \coprod \frac{H_i}{H_i}})$$

$$= \bigoplus_i \mathbb{C}\left[\frac{H_i}{H_i}\right] = \bigoplus_i (\mathbb{C}H_i)^{H_i}$$

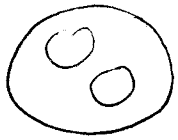
$$Z(\cdot) = \mathrm{Vect}(\mathrm{Map}(\cdot \to X)) = \bigoplus_i \mathrm{Rep}\, H_i$$



In any extended TFT,

$Z(S^{n-2})$ is an "algebra" (monoidal category)

$n=2$ associative



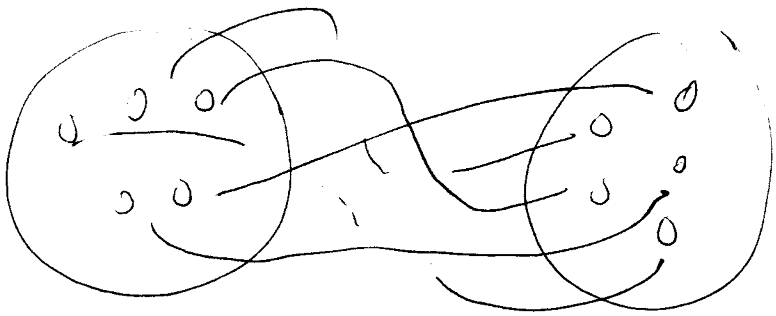
$Z(S^1)$

~~commutative~~ braided $\otimes$ category

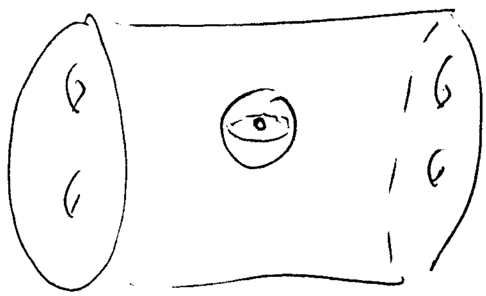
$V\otimes W \xrightarrow{\sim} W \otimes V$

$Z(S^2)$

symmetric monoidal ~~c~~ tensor



$Z(Y^{n-2}) \circlearrowright Z(S^{n-2})$



4d TFT

$Z(S^1)$ surface operator

End(S) line ops on S