24 July 2008
D. Ben-Zvi

Representation Theory and Topological Field Theory, I

$\mathcal{Z}$ 4d TFT

$$\mathcal{Z} : M^4 \longmapsto \mathcal{Z}(M^4) \in \mathbb{C}$$
$$N^3 \longmapsto \text{Vect}$$
$$\Sigma (=C) \longmapsto \text{Category}$$

$\Sigma \longmapsto$ 2d TFT

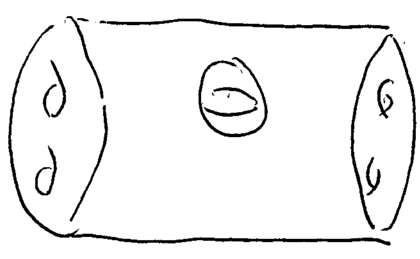
$$\mathcal{Z}_\Sigma(Y) = \mathcal{Z}(Y \times \Sigma)$$
$$\mathcal{Z}(\Sigma) = \mathcal{Z}_\Sigma(\cdot) \qquad \text{CY category}$$

Fields $(\Sigma \times Y)$ some kind of $Y \to$ Fields(Σ)

$\mathcal{Z}_\Sigma$ is a σ-model on Fields(Σ)

- $\mathcal{Z}(S^2)$ is a tensor category (symmetric monoidal)

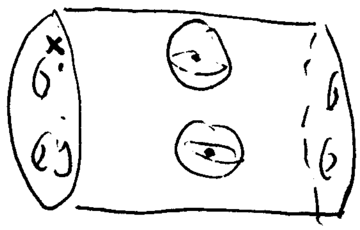
- $\forall x \in \Sigma$ get action of $\mathcal{Z}(S^2)$ on $\mathcal{Z}(\Sigma)$



all commute

$Z(\Sigma) \circlearrowleft$ action of $\{Z(S^2)\}_{x \in X}$

$x, y \in \Sigma$ compare $Z(S^2)_x$, $Z(S^2)_y$



$x \overset{\sigma}{\rightsquigarrow} y$

get isom of the actions

$\rightsquigarrow$ <u>flat connection</u> on the actions

$$Z(\Sigma) \circlearrowleft \{Z(S^2)\}_{x \in \Sigma}$$

"Hilbert space"

$\rightsquigarrow$ CY category

4d gauge theory, G finite.

$$\mathcal{F}\text{ields}(\Sigma) = \mathcal{M}_G(\Sigma) = \{G\text{-bundles}\}$$

$$= \coprod_{P_i \text{ bundle}} \bullet / \mathrm{Aut}\, P_i$$

$$Z(\Sigma) = \mathrm{Vect}(\mathcal{F}(\Sigma)) = \bigoplus_i \mathrm{Rep}\, P_i$$

$$Z(S^2) = \mathrm{Vect}(\mathcal{F}(S^2)) = \mathrm{Vect}(\cdot / G) = \mathrm{Rep}\, G, \otimes$$

$x \in \Sigma$, $V \in \mathrm{Rep}\, G \Rightarrow$ operator on $Z(\Sigma)$

— tensor product with a vector bundle on $\mathcal{F}(\Sigma)$

$W_{x,V}$

$P \in \mathcal{F}(\Sigma)$, $x, V \rightsquigarrow W_{x,V}|_P$ = Fiber at x of vector bundle on Σ assoc. to P in rep V.

Fibers at x, y isomorphic:

but isom depends on path & monodromy = monodromy of P.

Function-sheaf

$$\Sigma \rightsquigarrow \Sigma \times S^1$$

$$Z(\Sigma \times S^1) = K(Z(\Sigma))$$
= Grothendieck grp

$$\mathcal{F}(\Sigma \times S^1) = G\text{-bundles on } \Sigma \times S^1$$

$$= \coprod_{P_i \text{ bundles}} \frac{\operatorname{Aut} P_i}{\operatorname{Aut} P_i} \leftarrow \text{holonomy}$$

$$= \coprod \frac{H_i}{H_i}$$

$$Z(\Sigma \times S^1) = \bigoplus \operatorname{Fun}\left(\frac{H_i}{H_i}\right) = \bigoplus \mathbb{C} H_i^{H_i}$$
$$= \bigoplus \text{Rep ring of } H_i$$

$W_{x,V} \circlearrowright Z(\Sigma \times S^1)$, multiplication by a function on $\mathcal{F}(\Sigma \times S^1)$

$W_{x,V}(P)$ = trace of holonomy around loop l_x (S^1) of P in rep V.

Dim reduction $\longleftrightarrow$ decat.

$\text{Cat} \xrightarrow{K} \text{Vect}$

$\text{Sheaves} \xrightarrow{tr} \text{functions}$

"Solving" the TFT

$\longrightarrow$ simultaneously diagonalize action on $Z(\Sigma)$

of $\mathcal{H}(\Sigma) = \{Z(S^2)\}_{x \in \Sigma}$

R comm ring : $\operatorname{Spec} R = \operatorname{Hom}(R, \mathbb{C})$

$\operatorname{Spec} R(k) = \operatorname{Hom}(R, k)$

$\mathbb{C}$-algebra

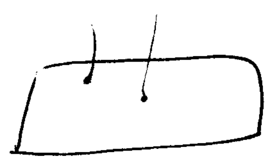
$R \circlearrowright V \Rightarrow V$ localizes as a sheaf on $\operatorname{Spec} R$

with fiber at λ = λ-eigenspace

$L \circlearrowright H$

$\circlearrowright$

$\mathbb{C}[L] \cong \mathbb{C}[x]$



$\mathbb{C} = \operatorname{Spec} \mathbb{C}[x]$

$\mathcal{C}$ tensor category $1, \otimes$

"Spec" $\mathcal{C}$ = $\mathrm{Fun}_{\otimes}(\mathcal{C}, \mathrm{Vect}_{\mathbb{C}})$

"Spec" $\mathcal{C}(k) = \mathrm{Fun}_{\otimes}(\mathcal{C}, \mathrm{Vect}_k)$, k $\mathbb{C}$-algebra

[R = alg. fns on Spec R]

$\mathcal{C}$ = coh. sheaves (span (vect. bundles)) on "Spec $\mathcal{C}$" [usually]

& $\mathcal{C}$-modules localize / spectrally decompose over "Spec" $\mathcal{C}$

<u>Tannakian story</u>

e.g. $\mathcal{C} = \mathrm{Rep}\, G^\vee$

"Spec" $\mathcal{C}$ = { fiber functors }

$= \cdot/G^\vee = BG^\vee$

$\mathcal{C} = \mathrm{Vect}(\text{"Spec"}\,\mathcal{C})$

$\quad\quad\quad\quad\quad\quad\quad\quad \shortparallel$

$\quad\quad\quad\quad\quad\quad\quad\quad \cdot/G^\vee$

e.g. $\mathcal{C}$ = { graded vect }, $\otimes$

"Spec" $\mathcal{C} = \cdot / \boxed{\begin{matrix}\mathrm{Galois}(\mathcal{C}) \\ \mathrm{Tannaka}(\mathcal{C})\end{matrix}} = \mathbb{C}^*$

$\mathcal{C} = \mathbb{C}^*$-rep.

$\mathcal{C} = \{$ flat vector bundles on $X \}$, $\otimes$
reasonable

"Spec" $\mathcal{C} = \cdot / \text{Galois}(\mathcal{C}) = \pi_1(X)$

$\mathcal{C} = \text{Rep}\, \pi_1(X)$

$Z(\Sigma) \circlearrowleft \mathcal{H}(\Sigma) = \{ Z(S^2) \}_{x \in \Sigma}$ Hecke algebra

$Z(S^2)$ is Tannakian

$= \text{Rep}\, G^\vee$ $\quad$ ["Spec" $Z(S^2) = \cdot / G^\vee$]

$G^\vee$-bundles on a pt

"Spec" $\mathcal{H}(\Sigma) = \text{Loc}_{G^\vee}(\Sigma)$, flat $G^\vee$ bundles on Σ

Theory Z wants to be $G^\vee$ gauge theory

$Z(S^2)$ want to be Wilson operators

$\text{Vect}(\text{Loc}_{G^\vee} \Sigma) = \mathcal{C}$

Hope: $Z(\Sigma) \simeq \mathcal{H}(\Sigma) = \text{Coh}(\text{Loc}_{G^\vee} \Sigma)$

multiplicity one

[module $\simeq$ ring]

Geometric Langlands

Z_G 4d TFT [$\mathcal{N}=4$ SYM in GL twist with $\Psi=0$]

$Z_G(\Sigma) = $ A-branes on $T^*\mathrm{Bun}_G\Sigma$ (e.g. Lagrangian)

$\xleftrightarrow[\text{Nadler-Zaslow}]{\text{Kapustin-Witten}}$ $\mathcal{D}$-modules on $\mathrm{Bun}_G\Sigma$ (e.g. holonomic)

GLC: $Z_G(\Sigma)$ "multiplicity one"

$$\simeq \mathrm{Coh}(\mathrm{Loc}_{G^\vee}\Sigma)$$

$$Z(S^2) = \mathrm{Shv}(\mathrm{Bun}_G S^2) \simeq \mathrm{Rep}\, G^\vee.$$

Geometric Satake theorem (Mirkovic - Vilonen, Lusztig - Ginzburg, Drinfeld)

$$Z(\Sigma) \supset \{\mathrm{Rep}\, G^\vee\}_{x\in\Sigma}$$

$\circlearrowleft$ $\mathrm{Vect}(\mathrm{Loc}_{G^\vee}\Sigma)$ 't Hooft operators

$$Z(\Sigma) \simeq D^b_{coh}(\mathrm{Loc}_{G^\vee}\Sigma)$$

$$Z_\Sigma \simeq \text{B-model on } \mathrm{Loc}_{G^\vee}\Sigma$$