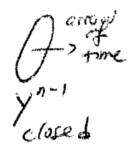
KITP/

Remarks on TQFT

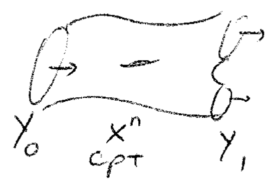
7/31/08

Def'n: An n-dim'l TQFT is a symmetric monoidal functor

$$F : (\mathrm{Bord}_n, \amalg) \longrightarrow (\mathrm{Vect}_{\mathbb{C}}, \otimes)$$

Explicate: objects: 

$\longmapsto F(Y)$ ~~C-vector space~~

morphisms: 

$\longmapsto \big(F(X) : F(Y_0) \longrightarrow F(Y_1)\big)$

$\left\{ \begin{array}{l} \text{disjoint union} \longmapsto \otimes \\ \text{gluing law.} \end{array} \right.$

Remarks: ① Extra structure — orientations, spin, framings, ...

② codomain could be $(Ab, \otimes)$, $(dg\text{-}Ab, \otimes)$, ...

③ Compare cohomology theory

$$h : (\mathrm{Top}, \amalg) \longrightarrow (\mathrm{Vect}_{\mathbb{C}}, \oplus)$$

which satisfies Mayer-Vietoris. $[(\mathrm{Top}_*, \amalg) \longrightarrow (\mathbb{Z}\text{-graded Vect}, \oplus)$ preserving shift.$]$

• Note $\oplus$ vs $\otimes$.

Extend to higher codimension:

crucial in geometric Langlands.

dim M	F(M)	category #
n	elt of $\mathbb{C}$	-1
$n-1$	$\mathbb{C}$-vector space	0
$n-2$	$\mathbb{C}$-linear category	1
$\vdots$	$\vdots$	$\vdots$

Remark: Familiar in ~~topology~~ cohomology theories as well. But there encoded in spaces (spectra): the TQFT is different. Better framework?

Today say two <u>concrete</u> things in particular example which involve categorical aspect: one very old, one very new. Both in Chern-Simons. L2

$$\left.\begin{array}{l} G \text{ cpt Lie group} \\ \ell \in H^4(BG) \end{array}\right\} \Rightarrow F_{(G,\ell)} \quad \text{a 3d TQFT}$$

Introduced by Witten ~~20~~ 20 years ago and a nice example in which to study <u>structure</u> of QFT in general. Lessons learned here apply elsewhere, e.g. in geometric Langlands. Two episodes:

(1) G finite group: $\boxed{\text{hep-th/9212115}}$

(2) G torus: w/ Teleman, Lurie, Hopkins (ongoing discussions)

Both involve taking theory down to a point; $F(pt)$, which has codim 3. Naively expect a 2-category, or more generally an object in a 3-category. But will find sth more concrete.

<u>G finite</u>: Nice feature is path $\int$ reduced to finite sum; can extend to "path $\int$" in dim 3, 2, 1, 0.

<u>classical fields</u>: $\begin{array}{c} P \\ \downarrow \\ M \end{array}$ principal G-bundles. Groupoid of fields.

$\in H^4(BG;\mathbb{Z}) \cong H^3(BG, \mathbb{R}/\mathbb{Z})$ gives, for $\begin{array}{c} P \\ \downarrow \\ X^3 \end{array}$ ~~or~~ oriented closed $\quad S_X(P) \in \mathbb{R}/\mathbb{Z}$ characteristic number.

$$F_{(G,\ell)}(X) = \sum_{\left\{ \begin{smallmatrix} P \\ \downarrow \\ X \end{smallmatrix} \right\}/\cong} e^{iS_X(P)} \cdot \frac{1}{\#\mathrm{Aut}(P)}.$$

The fields and action are both *local* — not only can factorize on surface, but can go down to points. Let's look at dim 1 (codimension 2), so $M = S^1$.

Assertion: $\mathrm{Fields}(S^1) \approx G/\!/G$. Add basepoint and take holonomy.

Remark: This for any gp G where fields are G-connections.

Consider first $\ell = 0$. Then

passage from classical to qtm given

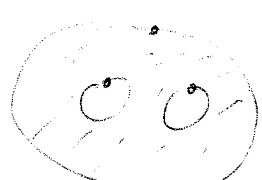
$$F_{G,0}(S^1) = \mathrm{Vect}_G(G), \quad \text{category of } G\text{-equiv. v.b. over } G.$$

Observe: Let $\mathcal{G}$ denote this groupoid and $\mathbb{C}[\mathcal{G}]$ its path algebra.

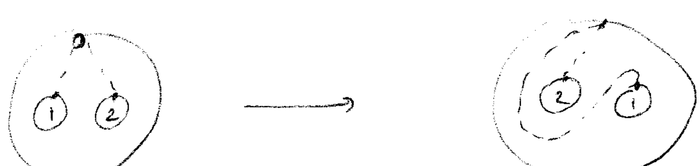
Then
$$\mathrm{Vect}_G(G) \approx \mathbb{C}[\mathcal{G}]\text{-mod}.$$

$\mathbb{C}[\mathcal{G}]$ is a (quasi-triangular quasi-) *Hopf algebra*, the Drinfeld double of $\mathbb{C}[G]$ the group algebra of G.

Already see concrete computations: e.g. the coproduct on $\mathbb{C}[\mathcal{G}]$ is derived from the monoidal structure on $F(S^1)$ which, in turn, is derived from studying "path integral"



on pair of pants. The R-matrix is computed from a diffeomorphism



In fact, $\mathrm{Vect}_G(G)$ has structure of a <u>modular tensor category</u>.

what about $\ell \neq 0$? what ℓ gives is a <u>central extension</u> of G by $\mathbb{T}$

$$1 \to \mathbb{T} \to \tilde{G}_\ell \to G \to 1$$

So a ~~$\mathbb{T}$...~~ hermitian line $\xrightarrow{a}$ L_a for each arrow, and

coherent isos $\xrightarrow{a}\xrightarrow{b}$ $\quad L_{ba} \xrightarrow{\cong} L_b \otimes L_a$.

Now

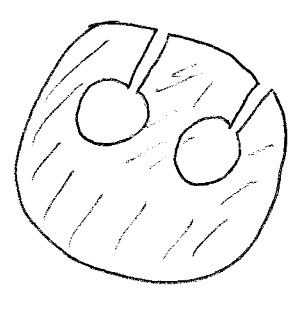
$$F_{(G,\ell)}(S^1) = \mathrm{Vect}^{(\ell)}_G(G) = \left\{ \begin{matrix} W \\ \downarrow \\ G \end{matrix}, \quad L_a \otimes W_x \longrightarrow W_y \right\} \qquad x \xrightarrow{a} y$$

<u>Remark</u>: In retrospect see twisted K-theory.

Now
$$\mathrm{Vect}^{(\ell)}_G(G) \approx \mathbb{C}[\hat{G}]\text{-mod}_1 \quad : \quad \text{modules where central } \mathbb{T} \text{ acts as scalar multiplication.}$$
$$\approx \mathcal{A}_\ell\text{-mod}$$

for $\mathcal{A}_\ell$ a "twisted Drinfeld double".

Remark: This was written down by Dijkgraaf - Pasquier - Roche (orbifold models)
To recover their formulas needed certain coherent trivializations —
used higher gluing law to do so: disk $\to$ pair of pants.



$\underline{F(pt)}$: Fields $(pt) \simeq pt/\!\!/G$. So just the group G. Now

$$\ell \in H^4(BG;\mathbb{Z}) \cong H^2(BG;\mathrm{Pic})$$

$\mathrm{Pic} = K(\mathbb{Z},2)$ = lines.

So a central extension of G by lines. With a choice

This is given by a 2-cocycle on G w/ values in lines:

$\{K_{xy},\ x,y \in G\}$ and isos

$$(*) \qquad \omega_{x,y,z}: K_{x,y} \otimes K_{xy,z} \xrightarrow{\cong} K_{x,yz} \otimes K_{y,z}$$

for all triple of gp. elts. Relation for quartets. ~~...~~ To see coh class, choose $k_{x,y} \overset{\neq 0}{\in} K_{x,y}$ and then $\omega_{x,y,z} \in \mathbb{C}^\times$ is cocycle.

Use K to ~~...~~ form a twisted group algebra of G w/ coefficients in $\mathrm{Vect}_{\mathbb{C}}$.

So define a monoidal category

$R_{(G,\ell)}$: objects $W \to G$ vector bundles

monoidal: ~~...~~

$$(W_1 * W_2)_x = \bigoplus_{x_1 x_2 = x} K_{x_1,x_2} \otimes W_{x_1} \otimes W_{x_2}.$$

$(*)$ gives proper associativity. Twisted "Categorification" of group algebra $\mathbb{C}[G]$.

$$F_{(G,\ell)}(pt) = R_{(G,\ell)}\text{-mod}, \quad \text{a 2-category}.$$

Or, as won't have time to explain, best to just think of $R_{(G,\ell)}$ as $F_{(G,\ell)}(pt)$.

$\underline{\text{Recall } F(S^1)}$: Categorical version of Drinfeld double — see Ben-Zvi - Francis - Nadler.

$\underline{\text{Def}^n}$: Let R be a monoidal category. Its center $Z(R)$ has as objects pairs (x,β_x) of $x \in R$ and β_x a functor

$$\beta_x(y): x \otimes y \longrightarrow y \otimes x, \qquad y \in R$$

If $(W,\beta) \in Z(\mathcal{R}_{(G,\ell)})$, then for $\mathbb{C}_y$ skyscraper at some pt y,

$$\beta_W(\mathbb{C}_y) : \quad W * \mathbb{C}_y \longrightarrow \mathbb{C}_y * W$$

at $x \in G$: $\quad \bigoplus_{zy=x} K_{z,y} \otimes W_z \longrightarrow \bigoplus_{yz'=x} K_{y,z'} \otimes W_{z'}$

so $\quad K_{xy^{-1},y} \otimes W_{xy^{-1}} \longrightarrow K_{y,y^{-1}x} \otimes W_{y^{-1}x}$

$x' = xy^{-1}$

$y^{-1}x = y^{-1}x'y$.

or

$$\frac{K_{x',y}}{K_{y,y^{-1}x'y}} \otimes W_{x'} \longrightarrow W_{y^{-1}x'y}$$

So if $\; x' \xrightarrow{\;a\;} y^{-1}x'y \;$ set $L_a = \dfrac{K_{x',y}}{K_{y,y^{-1}x'y}}$, Verify composition law for L_a and find

$$Z(\mathcal{R}_{(G,\ell)}) \simeq F(S^1) = \mathrm{Vect}_G^{(\ell)}(G).$$

Justifies $\quad F(pt) = \mathcal{R}_{(G,\ell)}$.

~~Finite Group~~ Torus Groups

Somewhat analogous picture, but new ingredient. Propose $F(pt)$.

Notation: $\quad \mathbb{T}$ torus

$\quad \mathfrak{t}$ Lie algebra

$$\mathfrak{t} \supset \Pi = \mathrm{Hom}(\mathbb{T}, T) \cong \pi_1 T \cong H_1 T \cong H_2 BT$$

$$\mathfrak{t}^* \supset \Lambda = \mathrm{Hom}(T, \mathbb{T}) \cong \qquad H^1 T \cong H^2 BT$$

$H^4(BT) \cong \mathrm{Sym}^2 \Lambda$. Identify with quadratic functions $g : \Pi \to \mathbb{Z}$ which satisfy $g(n\pi) = n^2 g(\pi)$, $\pi \in \Pi$. Associate even symmetric bilinear

$$\langle \ , \ \rangle : \Pi \times \Pi \longrightarrow \mathbb{Z}$$

$$\langle \pi_1, \pi_2 \rangle = g(\pi_1 + \pi_2) - g(\pi_1) - g(\pi_2).$$

N.B. If $l \in \mathrm{Sym}^2 \Lambda$, lift to $\hat{l} \in \Lambda^{\otimes 2} \twoheadrightarrow \mathrm{Sym}^2 \Lambda$ and set

$$g(\pi) = \hat{l}(\pi \otimes \pi).$$

Check indep. of lift.

Assumption: $\langle \ , \ \rangle$ is nondegenerate on $\Pi \otimes \mathbb{R}$. So define

$$\tau : \Pi \longrightarrow \Lambda \qquad \text{injection}$$

$$\tau : \mathfrak{t} \longrightarrow \mathfrak{t}^* \qquad \text{isomorphism}.$$

As before, construct two line bundles $K, L \longrightarrow T \times T$. The bundle K depends on choices and ~~[illegible]~~ comes with isos $\omega_{x,y,z}$ as before. Here

$$L_{x,y} = \frac{K_{x,y}}{K_{y,x}}$$

is a bihomomorphism into Pic:

$$L_{x,yz} \xrightarrow{\ \cong\ } L_{x,y} \otimes L_{x,z}.$$

So for $x \in T$ this gives a central extension $\mathbb{T} \rightleftarrows \tilde{T}_x \to T$. Let Λ_x be the Λ-torsor of splittings. The union of Λ_x is a principal Λ-bundle

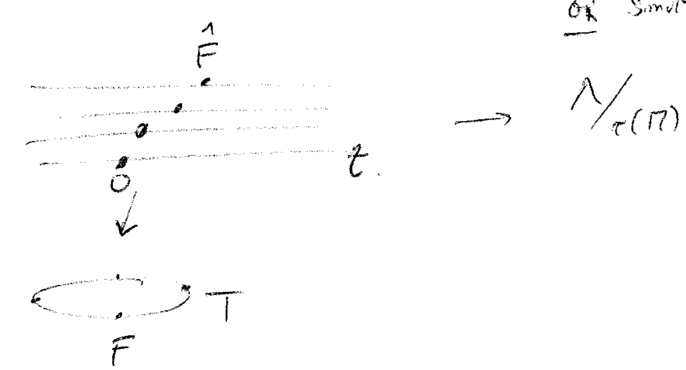
$$\begin{tikzcd}
 & C \cong \mathfrak{t} \times_{\Pi} \Lambda \arrow[dl, "(\Lambda)"'] \arrow[dr, shift left, "(\mathfrak{t})"'] & \\
T & & \Lambda / \tau(\Pi) \arrow[ul, shift left, "s"']
\end{tikzcd}$$

[illegible] $\Pi \to \Lambda$. So iso as above + projection section is

$$s(\lambda) = \left(\tau^{-1}(\lambda), \lambda\right) \in \mathfrak{t} \times \Lambda .$$

Image of s is a finite set $\hat{F} \subset \mathcal{B}$ which projects to a finite group $F \subset T$ which is, in fact, the kernel of $\tau \otimes \mathbb{T} : T \longrightarrow T^*$.
OR simultaneous kernel of all of $\tau(\Pi) \subset \mathrm{Hom}(T, \mathbb{T})$

Picture: $\mathcal{B}$:



Given $(\xi, \lambda) \in \mathfrak{t} \times \Lambda$ want to describe corresponding character of $\hat{T}_{x = \exp \xi}$.
In fact choice of ξ trivializes central extension, so can describe as characters of T, and do so by infinitesimal character $\xi' \in \mathfrak{t} \longrightarrow \mathbb{R}$:

$$\boxed{(\xi, \lambda) \cdot \xi' = \left\langle \frac{\xi}{2} - \tau^{-1}(\lambda), \xi' \right\rangle}$$

Now construction of $F_{(T,\ell)}(pt)$

$$\mathcal{R} = \text{monoidal cat. of finite sheaves on } T$$
$$= FSh^{(\ell)}(T)$$
$$= \{W_x\}_{x \in T} \quad \text{with } W_x = 0 \text{ for all but finitely many } x.$$

$$(W_1 * W_2)_x = \bigoplus_{x_1 x_2 = x} K_{x_1, x_2} \otimes (W_1)_{x_1} \otimes (W_2)_{x_2}$$

A model for twisted $K_0(T)$ à la Segal.

<u>Claim</u>: $Z(R) = FSh(G)$

Computation as above. This is <u>not</u> $F_{(T,\ell)}(S^1)$: a continuum of simple objects. Must cut down.

<u>Def'n</u>: Suppose $R, \mathcal{E}$ monoidal and $G : \mathcal{E} \longrightarrow Z(R)$ a monoidal functor. Then $Z^{\mathcal{E}}(R)$ is the subcategory of $Z(R)$ consisting of objects (x, β_x) such that

$$x \otimes G(e) \xrightarrow{\beta_x(G(e))} G(e) \otimes x \xrightarrow{\beta_{G(e)}(x)} x \otimes G(e)$$

is the identity for all $e \in \mathcal{E}$.

Can check $Z^{\mathcal{E}}(R)$ is a monoidal subcategory.

In our case set

$$\mathcal{E} = FSh(\mathbb{Z}) \xrightarrow{G} FSh(T)$$

Monoidal structure uses pullback of $K \to T \times T$ to $\mathbb{Z} \times \mathbb{Z}$ and addition in $\mathbb{Z}$: a twisted convolution.

<u>Claim</u>: $Z^{\mathcal{E}}(R) = Vect(\hat{F})$, so vector bundles on the finite set $\hat{F}$.

This has a monoidal structure: a twisted version of $Vect(F)$. This is the correct modular $\otimes$ category for Chern-Simons for the torus.

Computation. x = finite sheaf supported at $(\xi, \lambda) \in Z \times \Lambda \to G$.
e = – – – – – – – $(\xi', 0) \in Z$
} $\mathbb{C}$ at these points.

Then the braiding is $\exp(2\pi i)$ applied to

$$x \otimes G(e) \longrightarrow G(e) \otimes x \longrightarrow x \otimes G(e)$$

$$\langle \tfrac{\xi}{2} - \tau^{-1}(\lambda), \xi' \rangle \qquad \langle \tfrac{\xi'}{2}, \xi \rangle$$

Sum is $\langle \xi - \tau^{-1}(\lambda), \xi' \rangle$ which must vanish for all ξ'.

This implies $\xi = \tau^{-1}(\lambda)$ which puts us on $\hat{F}$.

Context: Not a 2-category, so more gen'l view of TQFT. Related to generators-and-relations conjectures of Baez-Dolan, now proved in $\le 2d$ (Hopkins-Lurie) and plan of proof in gen'l.

Conjecture: Let $\mathcal{C}$ be an symmetric monoidal infinity n-category. Then the space of symmetric monoidal functors

$$F : \mathrm{Bord}_n^{fr} \longrightarrow \mathcal{C}$$

is the space $\mathcal{C}^{fd}$ of "fully dualizable" objects in $\mathcal{C}$.

Won't explain terminology. But says it is enough to specify $F(pt)$ and the only constraint is the existence of enough duals.

Here would seek a suitable 3-category:

<u>objects</u>: pairs $(\mathcal{R}, \mathcal{E})$ of monoidal cats, $G : \mathcal{E} \longrightarrow Z(\mathcal{R})$

1-<u>morphisms</u>: certain bimodules

⋮

But more to do to complete this picture ...