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Introduction to the Langlands Program, I

Number Theory

$\mathbb{Q}$ - field of rational numbers

(more generally, finite extension of $\mathbb{Q}$)

Algebraic curves defined over a finite field $\mathbb{F}_q$

F = Field of rational functions on a curve $X/\mathbb{F}_q$

Riemann surface Σ

Complex geometry of various moduli spaces associated to R.S.

[= complex algebraic curve

$y^2 = x^3 + ax + b$

$a, b \in \mathbb{C}$

$(a, b \in \mathbb{F}_q)$

defines an elliptic curve (torus)]

Field of meromorphic [= rational] functions on Σ

$\mathbb{F}_p$, p prime $= \{0, 1, \ldots, p-1\}$ mod p arithmetic

More generally, $\mathbb{F}_q$, $q = p^n$ p-prime, degree n ext. of $\mathbb{F}_p$.

$\mathbb{Q} = \left\{\frac{p}{q}\right\}$
$\cup$
$\mathbb{Z}$

e.g. $X = \mathbb{P}^1$.

$F = \left\{ \frac{P(t)}{Q(t)} \,\middle|\, P, Q \in \mathbb{F}_q[t] \text{ polynomials with coefficients in } \mathbb{F}_q \right\}$

$F \supset \mathcal{O}_F = \mathbb{F}_q[t]$, polynomials

e.g. $p = 3$, $\mathbb{F}_3 = \{0, 1, 2\}$, $\frac{1+t}{1+t^2} \in F$

André Weil

$\mathbb{Q}$	$F = \left\{ \frac{P(t)}{q(t)} \right\}$	
$\cup$	$\cup$	
$\mathbb{Z}$ - integers	$\mathcal{O}_F = \mathbb{F}_q[t]$	
$\cup$	prime ideals	
$p\mathbb{Z}$ prime ideals	e.g. $\{tP(t)\}$	
$\mathbb{Z}/p\mathbb{Z} \simeq \mathbb{F}_p$	$\mathcal{O}_F/(t) \simeq \mathbb{F}_q$	
	more general:	
	$\{p(t)P(t) \mid p(t) \text{ irred}\}$	
char 0	$\text{char}(\mathbb{F}_q) = p$	
Original Langlands Program		Geometric Langlands Program $\updownarrow$ S duality, Mirror symmetry

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$\mathbb{Q}$, adjoin roots of polynomial with rational coeff

$\implies F$, new field, extension of $\mathbb{Q}$.

$\text{Gal}(F/\mathbb{Q})$ - Galois group, $\left\{ \sigma : F \to F \;\middle|\; \begin{array}{l} \sigma \text{ preserves } +, \cdot, 0, 1 \\ \sigma(x) = x \;\; \forall x \in \mathbb{Q} \subseteq F \end{array} \right\}$

eg adjoin roots of x^2+1, $\pm i$.

$$F = \mathbb{Q}(i) = \{a + bi \mid a, b \in \mathbb{Q}\}.$$

$$\text{Gal}(F/\mathbb{Q}) = \{1, \underset{i \to -i}{\tau = \text{complex conj}}\} \simeq \mathbb{Z}_2$$

$\overline{\mathbb{Q}}$ = algebraic closure of $\mathbb{Q}$

adjoin all roots of all polynomials with rational coefficients

$Gal(\overline{\mathbb{Q}}/\mathbb{Q})$

Look at finite dimensional reps of this group.

Arise as cohomologies of algebraic varieties defined over $\mathbb{Q}$.

e.g. an elliptic curve/$\mathbb{Q}$

$$y^2 = x^3 + ax + b \quad , \quad a, b \in \mathbb{Q}$$

Étale cohomology, Z - algebraic variety/$\mathbb{Q}$

$$H^m_{\acute{e}t}(Z \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_\ell) \qquad H^m_{\acute{e}t}(Z \times_{\operatorname{Spec} \mathbb{Q}} \operatorname{Spec} \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$$

ℓ - prime

$\mathbb{Q}_\ell$ - field of ℓ-adic numbers

completion of $\mathbb{Q}$

$$\left\{ \sum_{n \geq -N} a_n \ell^n , \quad a_n = \{0, 1, \ldots, \ell-1\} \right\}$$

addition + mult. with carry

(field of char 0)

$Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $H^m_{\acute{e}t}(Z \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell^{\oplus 2}$

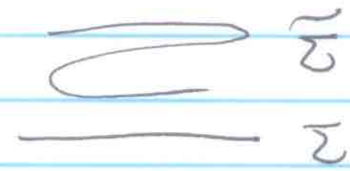
$Z = E$, elliptic curve

So we get a 2-dim'l rep of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. Such reps. are called <u>motivic</u>.

Analogy: Galois group $\mapsto$ fundamental group of $X/\mathbb{F}_q$ or $\Sigma/\mathbb{C}$

$Gal(\overline{\mathbb{Q}}/\mathbb{Q})$

$F = \mathbb{C}(\Sigma)$, rational function on Σ



$\mathbb{C}(\tilde{\Sigma}) \supseteq \mathbb{C}(\Sigma)$

auts are deck transformation

$\pi_1(\Sigma, \sigma)$

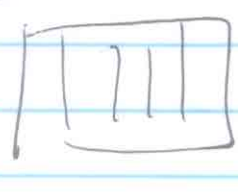
More precisely, $Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \varprojlim_{\text{punctures}} \pi_1(\Sigma_{\text{punctured}})$

$$\varprojlim_{n \to \infty} \pi_1(\Sigma \setminus \{x_1, \dots, x_n\})$$

$\downarrow$ (left) $Gal^{un}(\overline{\mathbb{Q}}/\mathbb{Q})$ $\cap$ maximal unramified extension

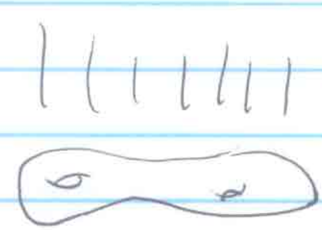
$\downarrow$ (right) $\pi_1(\Sigma)$

$\mathbb{Z}$ over $\mathbb{Q}$



x x x x Spec $\mathbb{Q}$

$\mathbb{Z}$ - variety (scheme) over X or Σ



$\ni p$

$H^n(\mathbb{Z}_p, \mathbb{Q}) \circlearrowleft \pi_1(\Sigma, p)$

These Galois reps contain important arithmetic info

{ Frobenius eigenvalues, Fr_p, defined for all but finitely many primes }

Ex E = elliptic curve $y^2 = x^3 + ax + b$ $a, b \in \mathbb{Z}$

For each prime p get # solution of the eqn mod p.

$\sim p+1 \; \pm ?$

Langlands Correspondence

$$\left\{ \begin{array}{c} \text{irreducible} \\ n\text{-dimensional} \\ \text{motivic Galois} \\ \text{representations} \\ \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{irreducible cuspidal} \\ \text{automorphic representation} \\ \text{of } GL_n(\mathbb{A}_{\mathbb{Q}}) \\ + \text{ extra conditions} \end{array} \right\}$$

$$\{ \text{Frobenius eigenvalues} \} \longleftrightarrow \{ \text{Hecke eigenvalues} \}$$

Fr_p, almost all p $\qquad$ H_p, almost all p

B-branes $\qquad$ A-branes

$\sum a_n q^n$