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Introduction to the Langlands Program, II

3/4 tracks

Number Theory   Algebraic Curves / finite fields   Complex Algebraic Curves   Physics

$\downarrow$

Langlands Correspondence

Galois (motivic) Data $\longleftrightarrow$ Automorphic (Rep. Th.) Data

$$\left\{\begin{array}{c}\text{irreducible} \\ n\text{-dimensional} \\ \text{motivic reps} \\ \text{of } Gal(\overline{\mathbb{Q}}/\mathbb{Q})\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{(complex)} \\ \text{irreducible cuspidal} \\ \text{automorphic reps of} \\ GL_n(\mathbb{A}_{\mathbb{Q}})\end{array}\right\}$$

[e.g. $Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_n$]

(condition at ∞)

$\mathbb{A}_{\mathbb{Q}}$ — ring of adèles of $\mathbb{Q}$.

"Motivic" means that it is an irr. constituent in $H^i_{et}(Z \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$ $\circlearrowleft$ $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$

~~Example~~

$$\left\{\begin{array}{c}\text{Frobenius eigenvalues} \\ Fr_p\end{array}\right\} \longleftrightarrow \left\{\text{Hecke eigenvalues, } H_p\right\}$$

all but finitely many primes p

Example Elliptic curve / $\mathbb{Q}$

$$y^2 = x^3 + ax + b. \qquad a, b \in \cancel{\mathbb{Q}}\ \mathbb{Z}$$

Smooth: $4a^3 + 27b^2 \neq 0$.

$\cancel{H^1_{et}(E, \mathbb{Q}_\ell)}$

$H^1_{et}(E \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$ – 2-dimensional rep of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$

Frob. eigenvalues: $\# E(\mathbb{F}_p) = \# \{x, y \mid y^2 = x^3 + ax + b \bmod p\} + 1$

$\forall\, p$ which do not divide $4a^3 + 27b^2$.

$\sigma_E \longrightarrow \pi_E$, aut. rep. of $Gl_2(\mathbb{A}_{\mathbb{Q}})$

$\# E(\mathbb{F}_p) \longrightarrow$ Hecke eigenvalues attached to π_E

$\mathbb{Q} \hookrightarrow \mathbb{A}_{\mathbb{Q}} = \prod$(all completions of $\mathbb{Q}$)

$$= \mathbb{R} \times \prod_{p \text{ prime}}' \mathbb{Q}_p = \{ (\underset{\mathbb{R}}{\underset{\cap}{x_\infty}}, \underset{\mathbb{Q}_p}{\underset{\cap}{(x_p)}}) \mid x_p \in \underset{\{\sum_{n \geq 0} a_n p^n\}}{\underset{\parallel}{\mathbb{Z}_p}} \text{ for all but finitely many } p \}$$

$$Gl_2(\mathbb{Q}) \hookrightarrow Gl_2(\mathbb{A}_{\mathbb{Q}}) = Gl_2(\mathbb{R}) \times \prod_{p \text{ prime}}' Gl_2(\mathbb{Q}_p)$$

rep. of $gl_2(\mathbb{R})$

Irr. Rep. of $Gl_2(\mathbb{A}_{\mathbb{Q}})$ $\quad \pi = \pi_\infty \otimes \bigotimes_p' \pi_p$

Choose a compact subgroup $K \subseteq Gl_2(\mathbb{A}_{\mathbb{Q}})$ so that π^K is 1-dimensional.

<u>Ref</u> "Lectures on Langlands Program and CFT"
hep-th/0512172

Aut. rep. of $GL_n(\mathbb{A}_\mathbb{Q})$ is an irred. const. in $L^2\left(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}_\mathbb{Q})\right)$ $\circlearrowleft$ $GL_n(\mathbb{A}_\mathbb{Q})$

$\pi \subset$

<u>1-dim'l</u> $\pi^K \subset \text{Function}\left(GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}_\mathbb{Q})/K\right)$

$K = SO(2) \times \prod_p K_p$ $\qquad K_p = GL_2(\mathbb{Z}_p)$ for almost all p.

For some p, $K_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{matrix} a,b,c,d \in \mathbb{Z}_p \\ c \equiv 0 \bmod p^{N_p} \end{matrix} \right\}$

$$GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}_\mathbb{Q})/K \simeq \Gamma_0(N)\backslash GL_2^+(\mathbb{R})/SO_2$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| c \equiv 0 \bmod N \right\}$$

$$N = \prod N_p$$

π_E^K gives a ~~function~~ on $\Gamma_0(N)\backslash GL_2^+(\mathbb{R})/SO_2$

gives a modular form f_E on $\quad \mathbb{H}_+ = \{\underset{\tau}{\alpha + \beta i} \mid \alpha \in \mathbb{R}, \beta \in \mathbb{R}_{>0}\}$

$\quad \shortparallel$

$\quad SL_2(\mathbb{R})/SO_2$

$$f_E\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 f_E(\tau)$$

What are the Hecke eigenvalues?

$$f_E(\tau) = \sum_{n=0}^{\infty} a_n q^n \qquad q = e^{2\pi i \tau}$$

($a_0 = 0$ because it is cuspidal)

$$f_E(\tau) = \sum_{n=1}^{\infty} a_n q^n$$

normalize: $a_1 = 1$.

Now the Hecke eigenvalues $\Longleftrightarrow \{a_p\}_{p \text{ prime not dividing } N}$

$$a_{nm} = a_n a_m \text{ if } (n,m) = 1.$$

Matching of Frob. eigenvalues + Hecke eigenvalues amounts to:

$$\boxed{a_p = p+1 - \#(E(\mathbb{F}_p))}$$

Taniyama-Shimura-Weil conjecture, proved by Wiles et al.

$$\left| \# E(\mathbb{F}_p) - 1 \right| \le 2\sqrt{p}$$

$$\shortparallel$$

$$2\sqrt{p} \cos\theta_p.$$

B-branes

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to {}^L G$$

A-brane

$$\cancel{GL_n(\mathbb{A}_\mathbb{Q})} \quad G(\mathbb{A}_\mathbb{Q})$$