4 August 2008
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Instantons beyond topological theory

(joint work with A. Losev & N. Nekrasov)

Part I hep-th/06...
Part II hep-th/08...
Part III to appear
Summary hep-th/07...

<u>TQFT</u> 2D twisted SUSY sigma model

$\Sigma \longrightarrow X$ Q - super charge

Look at closed observables $Q \cdot \mathcal{O}_i = 0$.

$\langle \mathcal{O}_1 \ldots \mathcal{O}_N \rangle$ - depends only on Q-coh. class

$H^{\cdot}_{Q}(\text{model}) = H^{\cdot}_{dR}(X)$

$H^{\cdot}_{dR}(X)^{\otimes N} \rightarrow \cancel{\mathbb{C}}\ H^{\cdot}(\overline{\mathcal{M}}_{g,N}, \mathbb{C})$

Gromov-Witten invariants of X

4D SYM $\Rightarrow$ Donaldson invariants

$H \equiv 0$ on the type observables

Integration over f.d. moduli space

(2D holo. maps $\Sigma \to X$)

$\prod T(\text{Hol}(\Sigma, X))$

<u>Question</u>: construct a full fledged QFT (without T!) for which <u>all</u> corr. fns are given by integrals over f-d. moduli spaces

<u>Answer</u>: yes! This theory is a special "weak coupling" limit (2D sigma model - inf. radius) of the "standard" theories

<u>Motivation</u>: Understanding non-supersymmetric models (purely bosonic $N=(0,0)$ or $N=(0,2)$)

<u>Langlands</u>: bosonic sigma models with target $X=\mathbb{P}^1$ (more generally, flag manifolds $U/T = G/B$)

Not CFT, but possess $\hat{\mathfrak{g}}$-symmetry of critical level (in this limit)

Two types of coupling constants:

- $-g$ - 4D YM
  $-g_{\mu\nu}$ - metric on X in 2D sigma models
- $-\theta$ θ-angle in YM $\quad \int e^{-S} \ldots \sim \sum_n e^{in\theta} \int_{c_2=n} e^{-S}$
  $-B_{\mu\nu}$ - B-field in 2D

$$\sum_{\beta \in H_2(X)} \sim e^{\int_{\Phi_*[\Sigma]=\beta} B} -$$

Complex coupling constant:

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \quad \text{in 4D}$$

$$\bar{\tau} = \frac{\theta}{2\pi} - \frac{4\pi i}{g^2}$$

$$\tau_{a\bar{b}} = B_{a\bar{b}} + i\frac{g_{a\bar{b}}}{2}$$

$$\bar{\tau}_{\bar{a}b} = B_{a\bar{b}} - i\frac{g_{a\bar{b}}}{2}$$

X – Kähler mfld

$g_{a\bar{b}}$ – metric components

$B = B_{a\bar{b}}\, dx^a \wedge dx^{\bar{b}}$

$\{x^a\}$ – cmplx coord on X

Look at $\tau, \bar{\tau}$ as two independent complex variables

drop the reality condition on θ and $B_{a\bar{b}}$

Take the limit $g \to 0$, $\theta \to \infty$ so that τ is fixed but

$\bar{\tau} \to -i\infty$

$$\theta = -\frac{8\pi^2}{g^2} i + 2\pi\tau$$

$$\bar{\tau} = \tau - \frac{8\pi i}{8g} \to -i\infty \text{ as } g \to 0$$

Likewise, in 2D

$$B_{a\bar{b}} = \frac{-i\lambda g_{a\bar{b}}}{2} + \tau_{a\bar{b}} \qquad \lambda \to \infty$$

$$\omega_K = \frac{i}{2} g_{a\bar{b}}\, dx^a \wedge dx^{\bar{b}} \qquad B = -\lambda\omega_K + \tau$$

$$\tau = \tau_{a\bar{b}}\, dx^a \wedge dx^{\bar{b}} \qquad \bar{\tau}_{\bar{a}b} \sim -i\lambda g_{a\bar{b}} \to i\infty$$

$$\partial_{\bar z} X^a \, \partial_z X^{\bar b} = |\partial_{\bar z} \Phi|^2$$
$$\partial_{\bar z} X^b \, \partial_{\bar z} X^{\bar b} = |\partial_z \Phi|^2$$

Action in this limit (2D) :

$$\Phi : \Sigma \to X$$

$$S(\Phi, \ldots) = \int_\Sigma \frac{1}{2} \langle d\Phi, * d\Phi \rangle_{\lambda g} + \int_\Sigma \Phi^*(\underbrace{-\lambda\omega_c + C}_{B})$$

+ fermions

$$= \int_\Sigma \frac{1}{2} \lambda g_{a\bar b} \left(\partial_{\bar z} X^a \partial_z X^{\bar b} + \cancel{\partial_z X^a \partial_{\bar z} X^{\bar b}} \right) + \int_\Sigma \Phi^*(C)$$

$$+ \frac{1}{2} \lambda g_{a\bar b} \left(- - \; - \; \cancel{- -} \right)$$

$$= \int_\Sigma \lambda |\partial_z \Phi|^2 + \int_\Sigma \Phi^*(C)$$

$(*)$ $$S(\Phi) = \int \lambda g_{a\bar b} \, \partial_{\bar z} X^a \partial_z X^{\bar b} + \int \Phi^*(C) + \text{fermions}$$

$$\int e^{-S} \mathcal{O}_1 \cdots \mathcal{O}_N$$

$$\downarrow$$

$$e^{-\lambda \int |\partial_z \Phi|^2_g}$$

First order formalism: introduce new fields $p_a, p_{\bar a}$

$$p_a \in \Gamma(\Phi^*(T^*X) \otimes T^*\Sigma)$$

$$p_{\bar a} \in \ldots$$

$$S'(\Phi, p_a, p_{\bar a}, \ldots) = \int \Big(-i p_a \partial_{\bar z} X^a - i p_{\bar a} \partial_z X^{\bar a} + \underbrace{\tfrac{2}{\lambda} g^{a\bar b} p_a p_{\bar b}}_{\to 0 \text{ as } \lambda \to \infty}\Big)$$

$$+ \int \Phi^*(\omega) + \text{fermions}$$

E-L eqns: $p_a = \frac{\lambda}{2} i \partial_{\bar z} X^a$

$p_{\bar a} = \frac{\lambda}{2} i \partial_z X^{\bar a}$ → plug-in, get (*).

Our limit

$$\boxed{\partial_{\bar z} X^a = 0} \quad \text{E-L eq.}$$

path integral

$$\int e^{\int i p_a \partial_{\bar z} X^a + i p_{\bar a} \partial_z X^{\bar a}) + \ldots}$$

Toy model: $f : \mathbb{C}^N \to \mathbb{C}$, $\omega_1, \ldots, \omega_k$ on $\mathbb{C}^N$.

$$\int dx_1 \ldots dx_N \, dp \, \underbrace{e^{ipf + i\bar p \bar f}} \omega_1 \ldots \omega_k$$

$$= \int_{\{f=0\}} \omega_1 \ldots \omega_k$$

By def,

$$\int \mathcal{D}x^{\mu}\mathcal{D}p_a \, e^{\int i p_a \partial_{\bar z} x^a + i p_{\bar a} \partial_z x^{\bar a}) + \cdots} \mathcal{O}_1 \cdots \mathcal{O}_k$$

$$:= \int_{\{\partial_{\bar z} x^a = 0\}} \mathcal{O}_1 \cdots \mathcal{O}_k \, e^{-\int \phi^*(\tau)}$$

$\mathcal{M}(\Sigma, X)$

$\mathcal{M}_\Sigma(X, \beta)$ - holo maps $\phi : \Sigma \to X$ s.t.

$\phi_*([\Sigma]) = \beta, \ \beta \in H_2(X, \mathbb{Z})$.

$N=(2,2)$ type A twisting

$$= \sum_{\beta \in H_2(X,\mathbb{Z})} e^{-\int_\beta \tau} \int_{\mathcal{M}_\Sigma(X,\beta)} \mathcal{O}_1 \cdots \mathcal{O}_k$$

finite dim!

no assumption on $\mathcal{O}_i$'s $(Q \cdot \mathcal{O}_i = 0)$

Usual obs: closed diff form on X

Suppose all $\mathcal{O}_i$'s are type $\langle \mathcal{O}_1 \cdots \mathcal{O}_k \rangle$ - old answer, usual G-W

Many more observables: differential form on jet space of X.

- QFT at weak coupling limit (2D and 4D)

  $\cup$

  TQFT

- Logarithmic CFT $\left(X - \begin{smallmatrix}\text{compact}\\ \text{Kähler}\end{smallmatrix}\right)$

- Space of states: connection to chiral de Rham(X)