Quantum geometric Langlands, $SL(2,\mathbb{R})$-WZNW and Liouville Theory

1 Motivation

Ph.: Solve $SL(2,\mathbb{R})$-WZNW (with S. Ribault)

M: Deformation of geom. Langlands:

$$\begin{array}{ccc} \boxed{{}^LG\text{- local syst.}} & \longrightarrow & \boxed{\mathcal{D}\text{-mod on } Bun_G} \\ & \searrow^{\text{or}} & \\ & & \boxed{\hat{\mathfrak{g}}_k\text{- conf. bl. at } k=-h^\vee} \end{array}$$

B-branes

A-branes

Pragmatic POV: View l.h.s as parametrization for r.h.s.

Observe: r.h.s. has obvious deformation: $k \neq -h^\vee$.

Proposal:

$$\boxed{\text{Conf. bl. for } W({}^L\mathfrak{g})} \longrightarrow \boxed{\hat{\mathfrak{g}}_k\text{-conf. bl.}}$$

parametrized (?) by local systems

"hard"

defd. by q-Hamiltonian

for $\mathfrak{g}=sl_2$: Virasoro-alg.

$\mathfrak{g}=sl_2$: $\boxed{\text{Liouville Th.}} \longrightarrow \boxed{sl(2,\mathbb{R})\text{-WZNW}}$

2 Virasoro → WZNW

a) Set-up : $\hat{\mathfrak{g}} = \widehat{sl}_{2,k}$, Generators J^a_n , $a=+,-,0$.

Reprs. $\mathcal{V}_{j,w}$ generated from $v_{j,w} \in \mathcal{V}_{j,w}$ s.t.

$$J^+_{n+w} v_{j,w} = 0 \qquad J^0_n v_{j,w} = 0 \qquad n>0$$
$$J^-_{n-w-1} v_{j,w} = 0$$
$$J^0_0 v_{j,w} = (j + \tfrac{k}{2} w) v_{j,w}$$

Let $\mathfrak{b}_w$ be subalg. gen. by $J^+_{n+w}, J^-_{n-w-1}, J^0_n, n\leq 1$

Then $\mathcal{V}_{j,w} = \mathcal{U}(\mathfrak{b}_w) \cdot v_{j,w}$.

b) $\hat{\mathfrak{g}}$-conf. bl. : Inv. lin. fct. on $\mathcal{V}^{(n)} = \bigotimes_{r=1}^n \mathcal{V}_r$, $\mathcal{V}_r = \mathcal{V}_{j_r w_r}$

$$f(\eta \cdot v) = 0 \qquad \forall \; v \in \mathcal{V}^{(n)}$$
$$\forall \; \eta \in \mathfrak{g}^{\mathcal{B}}_{out} , \quad \mathcal{B} : GL(2)\text{-bd.}$$

$$\mathfrak{g}^{\mathcal{B}}_{out} = \Gamma_{mer}(X \setminus \{z_1 \ldots z_n\}, \mathfrak{g}_{\mathcal{B}}), \quad \mathfrak{g}_{\mathcal{B}} = \mathcal{B} \times_{GL(2)} \mathfrak{g}$$

Varying $\mathcal{B}$ ⇝ sheaf of conf. bl. on Bun_G !

"Poor physicists sheaf (section) - function corresp." :

$$f \longmapsto f\Big(\bigotimes_{r=1}^n v_{j_r, w_r}\Big) \equiv f(v_0)$$
$$\equiv \langle \Psi^{j_1}_{w_1}(z_1) \ldots \Psi^{j_n}_{w_n}(z_n) \rangle_{X,\mathcal{B}}$$

function on subset $\mathcal{U} \subset \mathcal{M}_{g,n} \times \mathrm{Bun}_G$.

D-module : $f(X \cdot v_0) \equiv \delta_X f(v_0)$, $X \in \bigotimes_{r=1}^n \Pi^{(r)}_{j,w}(\hat{\mathfrak{g}})$

or : $\langle J^a(z) \Psi^{j_1}_{w_1}(z_1) \ldots \rangle = \mathcal{D}^a_{\mathcal{B}}(z) \langle \Psi^{j_1}_{w_1}(z_1) \ldots \rangle_{X,\mathcal{B}}$

$\mathcal{D}$-module structure $\Rightarrow$ PDE for fctn. $f(v_0)$:

Still highly nontrivial for X: genus 0, $\mathcal{B}$: bundles with parabolic struct. at z_r: Gauge group reduced to $B(x_r) = \begin{pmatrix} 1 & 0 \\ x_r & 1 \end{pmatrix}\begin{pmatrix} a & u \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -x_r & 1 \end{pmatrix} \subset GL(2)$ at z_r.

$\Rightarrow$ $f(v_0) \equiv g(v_0; x_1, \dots, x_n; z_1 \dots z_n)$ satisfies

$$(KZ) \qquad (k+2)\frac{\partial}{\partial z_i} g = \sum_{j \neq i} \frac{\mathcal{D}^{(2)}_{ji,x}}{z_j - z_i} g \qquad \begin{pmatrix} \text{for } w_r = 0 \\ r = 1, \dots, n \end{pmatrix}$$

$\mathcal{D}^{(2)}_{ji,x}$: 2nd order DO on x

Rem. $\mathcal{D}$-module struct. allows to reconstruct $f(v)$ from $f(v_0)$ (away from sing.) "function - sheaf - correspondence".

c) Left hand side : $\mathcal{W}({}^L\mathfrak{g}) = Vir$, gen. L_n

reprs. $\mathcal{W}_\alpha$: hwr., generated from $w_\alpha \in \mathcal{W}_\alpha$

$L_n w_\alpha = 0 \qquad n > 0$

$L_0 w_\alpha = \alpha(Q-\alpha) w_\alpha$

Conf. bl.: lin fct. $\mathcal{S} : \bigotimes_{r=1}^{m} \mathcal{W}_{\alpha_r} \to \mathbb{C}$

$\mathcal{S}(\eta \cdot v) = 0 \qquad \forall\ \eta \in \Gamma(X \setminus \{z_1, \dots, z_m\}, \Theta_X)$

or: $\mathcal{S}(v_0) \equiv \langle \Psi_{\alpha_1}(z_1) \dots \Psi_{\alpha_m}(z_m) \rangle_X$, $v_0 = \bigotimes_{r=1}^{m} w_{\alpha_r}$

If $\alpha_r = -\frac{1}{2b} \Rightarrow \mathcal{S}(v_0)$ satisfies PDE $\mathcal{D}^{BPZ}_r \mathcal{S} = 0$,

$$\mathcal{D}^{BPZ}_r = b^2 \frac{\partial^2}{\partial z_r^2} + \sum_{s \neq r} \left(\frac{1}{z_r - z_s} \frac{\partial}{\partial z_s} + \frac{\Delta_{\alpha_s}}{(z_r - z_s)^2} \right)$$

d) Correspondence : (for $g=0$, $\omega_r = 0$)

Key observation (Feigin, Frenkel, Stoyanovsky) :

There ex. map $\mathcal{G} : \mathrm{Sol}_{BPZ} \to \mathrm{Sol}_{KZ'}$,

where (KZ') obtained from (KZ) by Radon-Trf.

$$x_i \to \frac{\partial}{\partial \mu_i}, \quad \frac{\partial}{\partial x_i} \to -\mu_i$$

Key ingredient : Ch. of var. $\underline{\mu} = (\mu_1, \ldots, \mu_n)$ to $(\underline{y}, u)$

$$(\mathrm{SOV}) \qquad \sum_{r=1}^{n} \frac{\mu_r}{t - z_r} = u \, \frac{\prod_{a}^{n-2} (t - y_a)}{\prod_{r=1} (t - z_r)}$$

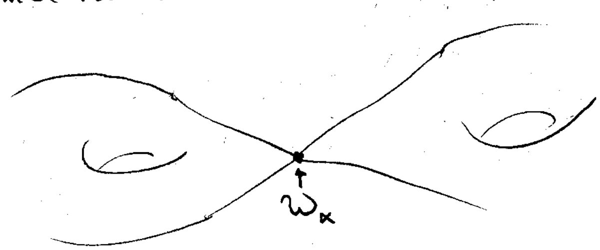
Rem. : There ex. gen. of (SOV) to $g>0$ (Hikida, Schomerus)

and to $\omega_r \neq 0$, $g=0$ (Ribault)

e) What does it help us ?

$g=0$ ✓
$g>0$ (✓)

There ex. constr. of Vir-conf. bl. (J.T.) with parametrization in terms of behavior at $\partial \mathcal{M}_{g,n}$



Varying $x \in \mathbb{C} \rightsquigarrow$ moduli sp. of conf. bl. $\mathcal{ME}_x$

$$\dim \mathcal{ME}_x = 3g-3+n = \dim \mathrm{Op}_{{}^L\mathfrak{g}}(x)$$

[Inserting $3g-3+n$ extra "fake" singularities $j_s \in \frac{1}{2}\mathbb{Z}_+$

$$\Rightarrow \dim \mathcal{ME}_x = 6g-6+2n = \dim \mathrm{Loc}_{G}(x)$$

Transported to "automorphic side" (r.h.s.) via $\mathcal{G}$?]

Alternative POV : Parametrize conf. bl. by BPZ-loc. syst.!

(3) Hecke-op.:

Recall geom. origin: el. bundle modif.:

$$(\mathcal{B}, \mathcal{B}', z, \beta : \mathcal{B} \to \mathcal{B}')$$

Sects of $\mathcal{B}'$: allowed to have poles along certain lines in the fibres.

Q. Can we define a reasonable operation mapping conf. bl. $f_{\mathcal{B}} \to (\mathcal{H}f)_{\mathcal{B}'}$?

Note: Defining inv. cond $f(\eta \cdot v)$ $\forall \eta \in \mathfrak{g}^{\mathcal{B}'}_{out}$ now involves η with poles at z !

$\rightsquigarrow$ Represent $\mathcal{H}$ by extra insertion

$$\left(\mathcal{V}^{(n)} \to \mathcal{V}^{(n+1)} \equiv \bigotimes_{r=0}^{n} \mathcal{V}_r \quad , \quad \mathcal{V}_0 \equiv \mathcal{V}_{j_0 w_0} \right)$$

Claim: There ex. a distinguished choice for $\mathcal{V}_0$, $(j_0 w_0) = (k/2, 0)$ s.t.

$$\mathcal{E}_{\mathcal{B}'} \simeq \mathbb{C}^2_z \otimes \mathcal{E}_{\mathcal{B}} ,$$

and variation of z described by certain local system (monodromy repr.)

In terms of fcts. on Bun_G:

$$\mathcal{H}\left(\langle \Psi^{j_1}_{w_1}(x_1, z_1) \dots \Psi^{j_n}_{w_n}(x_n, z_n) \rangle_{x, \mathcal{B}} \right)$$

$$\equiv \lim_{\mu \to 0} \underbrace{\mu^{2+k}}_{\text{becomes "pushforward" for } k=-2 \,?} \int dx \, e^{i\mu x} \langle \Psi^{k/2}_0(x|z) \Psi^{j_1}_{w_1}(x_1|z_1) \dots \rangle_{x, \mathcal{B}}$$

How to see this?

Key observation:

$$\mathcal{V}_0 \text{ degenerate} \qquad (J^-_{-1} v_{k/2,0} = 0)$$

$$\Rightarrow \langle \ldots \rangle^{\hat{sl}_2}_{S3'} \text{ represented by}$$

$$\langle \Psi_{-1/2b}(z)\, \Psi_{\alpha_1}(z_1) \ldots \Psi_{\alpha_m}(z_m) \rangle^{Vir}_{X}$$

And indeed:
$$\mathcal{C}^{Vir}_X (\mathcal{W}_{-1/2b} \otimes \mathcal{W}_{\alpha_1} \otimes \ldots \otimes \mathcal{W}_{\alpha_m})$$
$$= \mathbb{C}^2 \otimes \mathcal{C}^{Vir}_{X'} (\mathcal{W}_{\alpha_1} \otimes \ldots \otimes \mathcal{W}_{\alpha_m})$$

variation of z $\rightsquigarrow$ BPZ - local system. $\square$

(4) Concluding remarks

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- Longstanding problem for $SL(2,\mathbb{R})$-WZNW:
  Treatment of mixed "$x = z$" singularities
- Hecke-op. shed new light on this
  - related to spectral flow (FZZ, MO)
  - view spaces of conf. bl. as modules over Hecke-alg.
- There seems to exist "highest weight spaces" for Hecke-action (no deg. fields in $\langle \ldots \rangle^{Vir}$)

  ‖ These spaces carry natural scalar products ‖

  $\rightsquigarrow$ Honest harmonic analysis on Bun_G?