Schwinger pair creation in inhomogeneous electric fields

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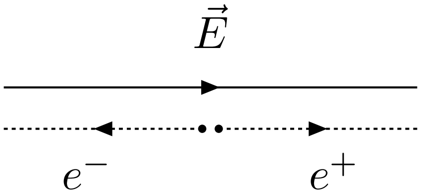
KITP, Frontiers of Intense Laser Physics, 30. 7. 2014

1. History

1931 F. Sauter: Dirac's theory of the electron predicts that an electric field of sufficient strength and extent can induce spontaneous creation of electron – positron pairs from the vacuum.
By a statistical fluctuation, a virtual pair separates out far enough to draw its rest mass energy from the field (vacuum tunneling).



The QED effective Lagrangian

1936 W. Heisenberg and H. Euler: One-loop QED effective Lagrangian in a constant field ("Euler-Heisenberg Lagrangian")

$$\mathcal{L}^{(1)}(a,b) \quad = \quad -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3}\,\mathrm{e}^{-m^2T}\left[\frac{(eaT)(ebT)}{\tanh(eaT)\tan(ebT)} - \frac{e^2}{3}(a^2-b^2)T^2 - 1\right]$$

Here a, b are the two invariants of the Maxwell field, related to **E**, **B** by $a^2 - b^2 = B^2 - E^2, \quad ab = \mathbf{E}\cdot\mathbf{B}$.

1936 V. Weisskopf: Analogously for Scalar QED.

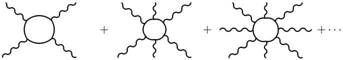
$$\mathcal{L}^{(1)}_{\mathrm{scal}}(a,b) \quad = \quad \frac{1}{16\pi^2}\int_0^\infty \frac{dT}{T^3}\,\mathrm{e}^{-m^2T}\left[\frac{(eaT)(ebT)}{\sinh(eaT)\sin(ebT)} + \frac{e^2}{6}(a^2-b^2)T^2 - 1\right]$$

N-photon amplitudes

The Euler-Heisenberg Lagrangian has the information on the N photon amplitudes in the low energy limit (where all photon energies are small compared to the electron mass, $\omega_i \ll m$). The amplitudes can be constructed explicitly from the weak field expansion coefficients c_{kl}, defined by

$$\mathcal{L}(a,b) = \sum_{k,l} c_{kl}\, a^{2k} b^{2l}$$

Diagrammatically, this corresponds to



Imaginary part of the effective action

If the field has an electric component ($b \neq 0$) there are poles on the integration contour at $ebT = k\pi$ which create an imaginary part. For the purely electric case one gets (J. Schwinger 1951)

$$
\begin{aligned}
\mathrm{Im}\mathcal{L}^{(1)}(E) &= \frac{m^4}{8\pi^3}\beta^2 \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\left[-\frac{\pi k}{\beta}\right] \\
\mathrm{Im}\mathcal{L}^{(1)}_{\mathrm{scal}}(E) &= -\frac{m^4}{16\pi^3}\beta^2 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \exp\left[-\frac{\pi k}{\beta}\right]
\end{aligned}
$$

($\beta = eE/m^2$).

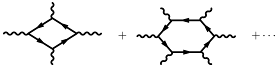
- The kth term relates to coherent creation of k pairs in one Compton volume.
- Weak field limit $\beta \ll 1 \Rightarrow$ only $k = 1$ relevant.
- $\mathrm{Im}\mathcal{L}(E)$ depends on E nonperturbatively, which is a confirmation of the tunneling picture.

Relation to pair creation

For not too strong fields, the imaginary part of the effective action relates to the total pair production probability P as

$$P \approx 2\mathrm{Im}\Gamma(E)$$

This is based on the Optical Theorem, which relates



to the "cut diagrams"

However, the latter individually all vanish for a constant field, which can emit only zero-energy photons. Thus what counts is the asymptotic behaviour for a large number of photons.

Borel dispersion relation

Thus for a constant field we cannot use dispersion relations for individual diagrams; the appropriate generalization is a Borel dispersion relation: define the weak field expansion by

$$\mathcal{L}(E) = \sum_{n=2}^{\infty} c(n) \left(\frac{eE}{m^2}\right)^{2n}$$

$$c(n) \overset{n\to\infty}{\sim} c_\infty \Gamma[2n-2]$$

(G.V. Dunne & CS 1999):

$$\mathrm{Im}\mathcal{L}(E) \sim c_\infty \, \mathrm{e}^{-\frac{\pi m^2}{eE}}$$

for $\beta \to 0$.

Critical field strength

For a constant field the pair creation probability is exponentially small for

$$E \ll E_{\mathrm{crit}} = \frac{m^2}{eE} \approx 10^{18} \mathrm{V}/\mathrm{m}$$

Here the critical field strength is such that an electron will collect its rest energy moving along the field lines a distance of one Compton wave length.

Laser field configurations

To have any chance at seeing pair creation soon, complicated laser configurations must be used to lower the pair creation threshold. For example,

- Counterpropagating lasers beams with linear polarization (M. Ruf, G. R. Mocken, C. Müller, K. Z. Hatsagortsyan & C. H. Keitel 2009).
- Superimposing a plane-wave X-ray beam with a strongly focused optical laser pulse (G.V. Dunne, H. Gies & R. Schützhold 2009).
- . . . (many more).

Approximation methods for Schwinger pair creation

The calculation of pair creation rates for generic electric fields requires approximative methods:

- Until recent years, practically all such results were obtained using WKB (L. Keldysh 1965, E. Brézin and C. Itzykson 1970, N.B. Narozhnyi and A. I. Nikishov 1970, V.S. Popov 1972, . . .). A more sophisticated version of WKB is the worldline instanton formalism (I.K. Affleck, O. Alvarez and N.S. Manton 1982, G. V. Dunne and C. S. 2005, . . .).
- The quantum kinetic approach, based on some Vlasov-type equation (Y. Kluger et al. 1991, 1992, S.M. Schmidt et al. 1998, R. Alkofer, F. Hebenstreit and H. Gies 2008 . . .).
- The Dirac-Heisenberg-Wigner formalism (F. Hebenstreit, A.Ilderton, M. Marklund and J. Zamanian 2011, . . .).

2. The Quantum Kinetic Approach

Properties:

- Evolution equations for arbitrary fields.
- Straightforward to include the backreaction on the field.
- Usually can be solved only numerically ($\rightarrow$ Florian Hebenstreit's talk).
- Particularly simple for a purely time-dependent field $E(t)$.

Purely time-dependent fields

For a purely time-dependent electric field, the spatial momentum $\mathbf{k}$ is a good quantum number, so that one has a mode decomposition (for a scalar particle at one loop)

$$2\mathrm{Im}\mathcal{L}(t) = \sum_{\mathbf{k}} \ln\big(1 + \mathcal{N}_{\mathbf{k}}(t)\big)$$

The $\mathcal{N}_{\mathbf{k}}(t)$ are densities of created pairs of momentum $\mathbf{k}$.
Using the in-out formalism and a Bogoliubov transformation, one can derive the Quantum Vlasov Equation (Y. Kluger et al. 1991, 1992, S.M. Schmidt et al. 1998, R. Alkofer, F. Hebenstreit and H. Gies 2008 . . .).

The Quantum Vlasov equation is an evolution equation at fixed $\mathbf{k}$ for the density of pairs $\mathcal{N}_{\mathbf{k}}(t)$ (scalar case):

$$\dot{\mathcal{N}}_{\mathbf{k}}(t) = \frac{\dot{\omega}_{\mathbf{k}}(t)}{2\omega_{\mathbf{k}}(t)} \int_{t_0}^{t} dt' \frac{\dot{\omega}_{\mathbf{k}}(t')}{\omega_{\mathbf{k}}(t')} (1 + 2\mathcal{N}_{\mathbf{k}}(t')) \cos\left[2 \int_{t'}^{t} dt'' \omega_{\mathbf{k}}(t'')\right]$$

where t_0 is the initital time, usually $-\infty$, and

$$\omega_{\mathbf{k}}^2(t) = (k_{\parallel} - qA_{\parallel}(t))^2 + \mathbf{k}_{\perp}^2 + m^2$$

$\mathcal{N}_{\mathbf{k}}(t)$ is zero at $t = -\infty$, and for $t \to \infty$ turns into the density of created pairs with fixed momentum k.

Solitonic example

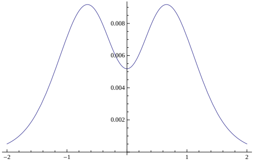
S.P. Kim and C.S. 2011: infinite family of analytic solutions to the Vlasov equation related to the Korteweg-de-Vries equation.
The simplest one has the gauge potential

$$qA(t) = k_{\parallel} - \sqrt{k_{\parallel}^2 + \frac{2\omega_0^2}{\cosh^2(\omega_0 t)}}.$$

Pair non-creation

The exact solution of the Vlasov equation for this field is

$$\mathcal{N}_{\mathbf{k}}(t) \quad = \quad \frac{4 + \mathrm{sech}^4(\omega_0 \mathrm{t})(1 + 2\cosh(2\omega_0 \mathrm{t}))}{8\sqrt{1 + 2\mathrm{sech}^2(\omega_0 \mathrm{t})}} - \frac{1}{2}$$



$\mathcal{N}_{\mathbf{k}}(t)$ vanishes for $t \to \infty$, thus there is no pair creation (at that particular momentum $\mathbf{k}$).
The external field excites the vacuum, but no particles materialize.

3. Worldline instantons

Feynman's worldline representation of the scalar QED effective action
R.P. Feynman, Phys. Rev. **80** (1950) 440.

$$
\begin{aligned}
\Gamma_{\rm scal}[A] &= \int_0^\infty \frac{dT}{T}\, \mathrm{e}^{-m^2 T} \int \mathcal{D}x(\tau)\, \mathrm{e}^{-S[x(\tau)]} \\
S[x(\tau)] &= \int_0^T d\tau \Big(\frac{\dot{x}^2}{4} + ieA\cdot\dot{x} \Big)
\end{aligned}
$$

Here m and T are the mass and proper time of the loop scalar, and the path integral $\int \mathcal{D}x(\tau)$ is over closed trajectories in Euclidean spacetime.

I. A. Affleck et al. 1982: Semiclassical approximation: replace the path integral by a single stationary trajectory = worldline instanton.

G.V. Dunne & CS 2005
G.V. Dunne, Q.-h. Wang, H. Gies & CS 2006

$$\int_{x(T)=x(0)=x^{(0)}} \mathcal{D}x(\tau)\, e^{-S[x(\tau)]} \approx \frac{\mathrm{e}^{-S[x^{\mathrm{cl}}](T)}}{(4\pi T)^2} \sqrt{\frac{\left|\det\left[\eta^{(\lambda)}_{\mu,\mathrm{free}}(T)\right]\right|}{\left|\det\left[\eta^{(\lambda)}_{\mu}(T)\right]\right|}}$$

- The extremal action trajectory $x^{\mathrm{cl}}(u)$ is a periodic solution of the (euclidean) Lorentz force equation.
- The $\eta^{(\lambda)}_{\mu}$ are zero modes of the fluctuation operator.

Constant field case

$\vec{E} = (0, 0, E) = const.$

$$
\begin{aligned}
x^{\mathrm{cl}}(u) &= \frac{m}{eE}\Big(x_1, x_2, \cos(2k\pi u), \sin(2k\pi u)\Big) \\
S[x^{\mathrm{cl}}] &= k\pi\,\frac{m^2}{eE}
\end{aligned}
$$

Winding number $k \in \mathbf{Z}^+$

kth worldline instanton $\Rightarrow$ kth Schwinger exponential

Prefactor deteminant $\Rightarrow$ correct normalization

Timelike Sauter case

Single pulse time dependent electric field

$$E(t) = E\,\mathrm{sech}^2(\omega\, t)$$

$$\begin{aligned} x_3(u) &= -\frac{1}{\omega}\,\frac{1}{\sqrt{1+\gamma^2}}\,\mathrm{arcsinh}\left[\gamma\, \cos\left(2n\pi u\right)\right] \\ x_4(u) &= \frac{1}{\omega}\,\arcsin\left[\frac{\gamma}{\sqrt{1+\gamma^2}}\,\sin\left(2n\pi\, u\right)\right] \end{aligned}$$

$$\gamma \equiv \frac{m\omega}{eE} \qquad\qquad (\text{adiabaticity parameter})$$

Worldline action

Stationary worldline action:

$$S_0 \quad = \quad n \frac{m^2\pi}{eE} \left(\frac{2}{1 + \sqrt{1+\gamma^2}} \right)$$

S_0 decreases with increasing γ, thus the pair creation rate increases.

Timelike Sauter instantons

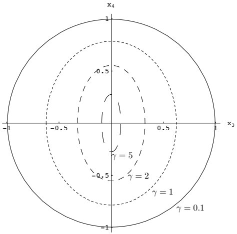


Figure: Plot of the worldline instanton paths in the (x_3, x_4) plane for the case $E(t) = E\mathrm{sech}^2(\omega t)$. The paths are shown for various values of the adiabaticity parameter γ. $x_{3,4}$ have been expressed in units of $\frac{m}{eE}$.

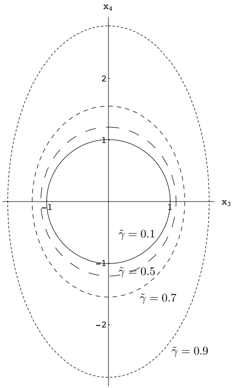
Spacelike Sauter case

$$E(x_3) = \frac{E}{\cosh^2(x_3/d)}$$

$$\begin{aligned} x_3(u) &= \frac{m}{eE}\frac{1}{\tilde{\gamma}}\,\mathrm{arcsinh}\left(\frac{\tilde{\gamma}}{\sqrt{1-\tilde{\gamma}^2}}\,\sin(2k\pi u)\right) \\ x_4(u) &= \frac{m}{eE}\frac{1}{\tilde{\gamma}\sqrt{1-\tilde{\gamma}^2}}\,\arcsin(\tilde{\gamma}\cos(2n\pi u)) \end{aligned}$$

$$\tilde{\gamma} \equiv \frac{m}{eEd} \qquad \text{(inhomogeneity parameter)}$$

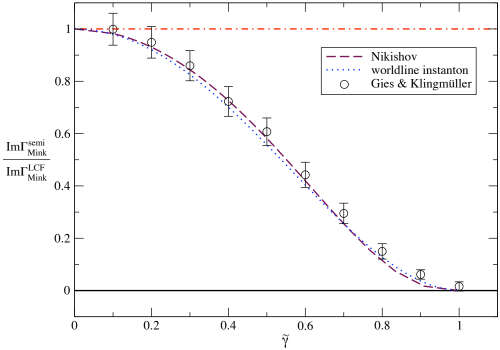
Worldline instantons for the spacelike Sauter field



Stationary action:

$$S_0 = n \frac{m^2\pi}{eE}\left(\frac{2}{1+\sqrt{1-\tilde{\gamma}^2}}\right)$$

S_0 increases with $\tilde{\gamma}$ $\rightarrow$ decrease of pair creation rate.
Comparison of the pair creation rate with exact results:



Note:

- No pair creation for $\tilde{\gamma} > 1$! This fits into the vacuum tunneling picture - for such $\tilde{\gamma}$ the field has insufficient extent to provide a virtual particle with its rest mass energy.
- The limiting case $\tilde{\gamma} = 1$ corresponds to one particle running from $x = -\infty$ to ∞ and the other one from $x = \infty$ to $-\infty$.

General rule

From the worldline instanton formalism we can learn:

- Inhomogeneity in space tends to reduce the pair creation rate. An insufficiently extended field, no matter how intense, will not pair-produce.
- Inhomogeneity in time tends to enhance the pair creation rate. A purely time-dependent field will *always* give a non-zero pair creation rate.

Nonplanar worldline instantons

G.V. Dunne and Q.-h. Wang 2006:

Spatially inhomogeneous fields $\vec{E}(\vec{x})$, using the gauge

$$A_4(\vec{x}) = -i\frac{E}{k}\, f(\vec{x})$$

Inhomogeneity parameter $\gamma = \frac{mk}{eE}$

Examples:

$$\begin{aligned} f(\vec{x}) &= \frac{k(x_1 + x_2)}{1 + k^2(x_1^2 + x_2^2)} \\ f(\vec{x}) &= k(x_1 + x_2)\, \mathrm{e}^{-k^2(x_1^2+x_2^2)} \end{aligned}$$

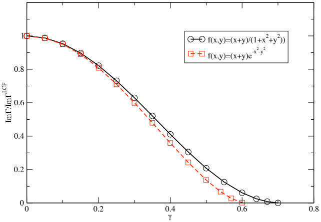


Figure: $\mathrm{Im}\Gamma/\mathrm{Im}\Gamma^{\mathrm{LCF}}$ for $f(\vec{x}) = \frac{k(x_1+x_2)}{1+k^2(x_1^2+x_2^2)}$ and $f(\vec{x}) = k(x_1 + x_2)\,\mathrm{e}^{-k^2(x_1^2+x_2^2)}$

The double Sauter case

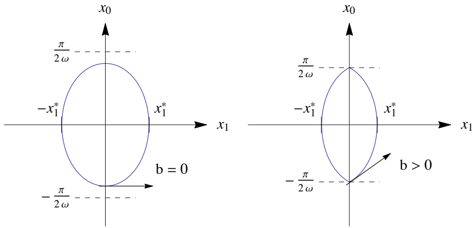
C. Schneider & R. Schützhold 2014

Combine a strong spacelike with a weak timelike Sauter field:

$$\mathbf{E}(t,x) = \left(\frac{E}{\cosh^2(kx)} + \frac{E'}{\cosh^2(\omega t)}\right)\mathbf{e}_x$$

where $E' \ll E \ll E_{\mathrm{crit}}$.

For sufficiently large ω the weak temporal pulse squeezes the worldline instanton in the x_0 direction, leading to a significant enhancement of the pair creation rate.

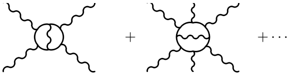


4. Higher-loop corrections to Schwinger's formula

So far all our discussion was at the one-loop level. Higher-loop corrections are not likely to be measured any time soon, but of great theoretical interest.

Two loop (one-photon exchange) corrections:

Euler-Heisenberg Lagrangian:



Schwinger pair creation:

2-Loop Euler-Heisenberg Lagrangian

V. I. Ritus 1975, S.L. Lebedev & V.I. Ritus 1984, W. Dittrich & M. Reuter 1985, M. Reuter, M.G. Schmidt & C.S. 1997: The two-loop correction $\mathcal{L}^{(2)}(E)$ to the Euler-Heisenberg Lagrangian leads to rather intractable integrals. However, the imaginary part $\mathrm{Im}\mathcal{L}^{(2)}(E)$ becomes extremely simple in the weak-field limit:

Weak field limit:

$$\mathrm{Im}\mathcal{L}^{(1)}(E) + \mathrm{Im}\mathcal{L}^{(2)}(E) \overset{\beta\to 0}{\sim} \frac{m^4\beta^2}{8\pi^3}\left(1+\alpha\pi\right) \mathrm{e}^{-\frac{\pi}{\beta}}$$

S.L. Lebedev & V.I. Ritus 1984: Assuming that higher orders will lead to exponentiation

$$\mathrm{Im}\mathcal{L}^{(1)}(E) + \mathrm{Im}\mathcal{L}^{(2)}(E) + \mathrm{Im}\mathcal{L}^{(3)}(E) + \ldots \overset{\beta\to 0}{\sim} \frac{m^4\beta^2}{8\pi^3} \mathrm{e}^{\alpha\pi}\, \mathrm{e}^{-\frac{\pi}{\beta}}$$

The conjecture of Affleck, Alvarez and Manton

For Scalar QED, the corresponding conjecture was established already two years earlier by (I.K. Affleck, O. Alvarez, N.S. Manton 1982), using a naive extension of the above one-loop worldline instanton calcuation:

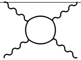
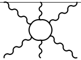
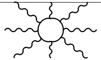
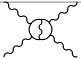
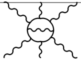
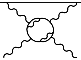
$$\begin{aligned}\sum_{l=1}^{\infty} \mathrm{Im}\mathcal{L}^{(l)}_{\mathrm{scal}}(E) \quad &\overset{\beta\to 0}{\sim} \quad -\frac{m^4\beta^2}{16\pi^3}\,\exp\Big[-\frac{\pi}{\beta}+\alpha\pi\Big] \\ &= \quad \mathrm{Im}\mathcal{L}^{(1)}_{\mathrm{scal}}(E)\;\mathrm{e}^{\alpha\pi}\end{aligned}$$

Remarkable:

- True all-loop result, receives contributions from an infinite set of graphs of arbitrary loop order (although non-quenched diagrams get suppressed in this limit).
- Includes mass renormalization! (!?)

Diagrams contributing to the AAM formula

In terms of Feynman diagrams:

	Number of external legs			
Number of loops	4	6	8	$\cdots$
1				$\cdots$
2			$\cdots$	$\cdots$
3		$\cdots$	$\ddots$	$\vdots$
$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\vdots$

(plus mass renormalization counterdiagrams!). Diagrams with more than one fermion loop get suppressed in the $E \to 0$ limit.

Discussion of the AAM conjecture

- Strange: an all-order loop summation has produced the finite analytical factor $\mathrm{e}^{\alpha\pi}$! This violates Dyson's theorem!
- But both conjectures have been verified only at two loops. A three-loop check is in order, but calculating the three-loop Euler-Heisenberg Lagrangian in $D = 4$ is too difficult.
- However, a verification of the (analogue of) the AAM formula in $D = 2$ QED seems in reach (I. Huet, D.G.C. McKeon and CS. 2010, I. Huet, M. Rausch de Traubenberg and CS, work in progress).

5. Field dependence of the electron mass

- In QED at tree-level, many arguments have been given for a field-dependence of the electron mass (N.D. Sengupta 1952, D.Volkov 1953, H. Reiss 1962, A.I. Nikishov & V.I. Ritus 1964, T.W.B. Kibble 1965 . . .).
- This mass-shift has been confirmed so far only indirectly (through the change in frequency of the radiation emitted by the electron).
- It is always positive but otherwise far from universal, depending on both intensity and pulse-shape (C. Harvey, T. Heinzl, A. Ilderton & M. Marklund 2012).

Ritus' "classical" one-loop mass shift

V.I. Ritus 1978: electron mass shift from the one-loop propagator in a constant electric field.

In the weak-field limit,

$$m(E) \approx m - \frac{\alpha}{2}\frac{eE}{m} + O(\hbar)$$

This mass shift is negative, and has a "classical" part that does not vanish for $\hbar \to 0$.

Lebedev-Ritus mass shift

S.L. Lebedev & V.I. Ritus 1984: assuming the exponentiation

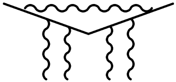
$$\sum_{l=1}^{\infty} \mathrm{Im}\mathcal{L}^{(l)} \quad \overset{\beta\to 0}{\sim} \quad -\frac{m^4\beta^2}{16\pi^3}\exp\Big[-\frac{\pi}{\beta}+\alpha\pi\Big]$$

then the result can be interpreted in the tunneling picture as the corrections to the Schwinger pair creation rate due to the pair being created with a negative Coulomb interaction energy

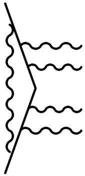
$$m(E) \approx m - \frac{\alpha}{2}\frac{eE}{m}$$

Ritus vs. Lebedev-Ritus

This mass shift is identical with the Ritus mass shift, as it should, since the processes are related by crossing:



$\Longleftrightarrow$



This lends further support to the exponentiation conjecture!
It would be interesting to calculate δm for laser fields ...

Thank you for your attention!