Twisting, KD Fermions & Lattice SUSY

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Motivation

Try to preserve part of cont^{m} SUSY on lattice → reduce/evade fine tuning problems

Idea

- Models with extended SUSY twisted

  $S = Q\Lambda(\Phi)$ $\quad Q^2 = 0$ (mod G.T)

  – change of variables $\quad Q \sim$ built from supercharges

  – easier to keep $Q^2 = 0$ than any SUSY under discretization

- Fermions of twisted models → equiv in KD fields

  Dirac eq^{n} → KD action. – later.

- Geometrical ⇐ exploit to lattice without fermion doubling & keeping G.I & Q.

Twist (2D)

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$q_\alpha^{I, I'}$ global sy $SO(2) \times SO_A(2) \times SO_V(2)$

new rot group = diag subgroup $(SO(2) \times SO(2)_A)$

→ $I \equiv \alpha$

Supercharges $q_\alpha^{I, I'} = Q_{\alpha\beta}$ (drop I')

orig^{n} SUSY alg.

$\{Q_\alpha^i, Q_\beta^j\} = 2\delta^{ij} \gamma^\mu_{\alpha\beta} p^\mu$

↳ twisted alg.

$\{Q, Q\} = 0 \qquad Q = Q_{\alpha\alpha}$

$\{Q, G_\mu\} = P_\mu + \ldots$

P is Q-exact ⇒ $T_{\mu\nu}$ (maybe) Q-exact

→ S Q-exact ....

Fields $(D=2)$ 3

q matrix $\longrightarrow$ $\Psi_{\alpha\beta}$ matrix also

$$\Psi(x) = I\eta(x) + \gamma_\mu \psi_\mu(x) + \gamma_1\gamma_2 \chi_{12}(x)$$

$\eta, \psi_\mu\ \chi_{12}$ – twisted fermions

(comp of KD fields $\}$ – later)

if there is Q-exact expt

$$\Phi = (\overline{\phi}, A_\mu, B_{12})\quad (\phi)$$

$$Q^2\Psi = Q^2\overline{\Phi} = 0 \quad \text{up to G.T's}$$

expect

$$S = Q\,\mathrm{Tr}\, F(\Psi, \Phi)\ \ldots$$

Example $N=2$ SYM in $D=2$ 4

Cont: twisted form of action:

$$S = \beta\, Q\, \mathrm{Tr}\int d^2x\left(\tfrac{1}{4}\eta[\phi, \overline{\phi}] + 2\chi_{12}F_{12} + \chi_{12}B_{12} + \psi_\mu D_\mu \overline{\phi}\right)$$

fields in adjoint $f = \sum_a f^a T^a$ $\leftarrow$ A.H.

$U(N)$.

γ

$$Q A_\mu = \psi_\mu$$
$$Q \psi_\mu = D_\mu \phi$$
$$Q \phi = 0$$
$$Q \chi_{12} = B_{12}$$
$$Q B_{12} = [\phi, \chi_{12}]$$
$$Q \overline{\phi} = \eta$$
$$Q \eta = [\phi, \overline{\phi}].$$

note

$$Q^2 = \delta^\phi$$

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SYM action continued

Vary, integrate out B_{12}:

$$\begin{aligned} S &= \beta \mathrm{Tr} \int d^2x \left(\frac{1}{4}[\phi,\overline{\phi}]^2 - \frac{1}{4}\eta[\phi,\eta] - F_{12}^2 \right. \\ &+ D_\mu\phi D_\mu\overline{\phi} - \chi_{12}[\phi,\chi_{12}] \\ &- 2\chi_{12}\left(D_1\psi_2 - D_2\psi_1\right) - \psi_\mu D_\mu \eta \\ &+ \left. \psi_\mu[\overline{\phi},\psi_\mu] \right) \end{aligned}$$

But this is indeed just $N=2$ SYM

$$\Psi = \begin{pmatrix} \eta/2 - i\chi_{12} \\ \psi_1 - i\psi_2 \end{pmatrix}$$

$$S = \Psi^\dagger \gamma.D\Psi + \Psi^\dagger \frac{(1+\gamma_5)}{2}[\overline{\phi},\Psi] - \Psi^\dagger \frac{(1-\gamma_5)}{2}[\phi,\Psi]$$

use Euclidean chiral rep. for gamma matrices
$\overline{\phi}$,ϕ taken as complex conjugates for action real, positive definite.

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Twisted Fields=Kähler-Dirac Fields I

In D dimensions a Kähler-Dirac field ω takes form

$$\omega = (f, f_\mu, f_{\mu\nu}, \ldots, f_{\mu_1\ldots\mu_d})$$

component fields $\psi_{\mu\nu\ldots}$ are antisymmetric tensors or p-forms
Exterior derivative d acts on the components as

$$d\omega = (0, \partial_\mu f, \partial_\mu f_\nu - \partial_\nu f_\mu, \ldots)$$

Natural dot product

$$< A|B >= \int d^D x \sqrt{g} \sum_p A^{\mu_1\ldots\mu_p} B_{\mu_1\ldots\mu_p}$$

Adjoint $d^\dagger$

$$-d^\dagger \omega = \left(f^\nu, f^\nu_\mu, \ldots, 0\right)_{;\nu}$$

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Twisted Fields=Kähler-Dirac Fields II

Using these fields form matrix

$$\Psi^{(\omega)}_{\alpha\beta}(x)=\sum_{p=0}^{D}(\gamma^{\mu_1}\ldots\gamma^{\mu_p})_{\alpha\beta}\,\psi_{\mu_1\ldots\mu_p}(x)$$

Straightforward to show that

$$\gamma^{\mu}_{\alpha\alpha'}\partial_\mu\Psi^{(\omega)}_{\alpha'\beta}=0$$

equivalent to

$$(d-d^\dagger)\omega=0$$

Interpretation: each column of $\Psi^{(\omega)}$ corresponds to a Dirac fermion

Equally true when gauged $d\to D$

Kähler-Dirac action:

$$<\omega^\dagger|(D-D^\dagger)\omega>$$

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KD action $(D=2)$

= twisted action

$\leftarrow \omega$

$$d\,(\eta/2,\ \psi_\mu,\ \chi_{\mu\nu}) = (0,\ \partial_\mu\eta/2,\ \partial_\mu\psi_\nu-\partial_\nu\psi_\mu)$$

$$-d^\dagger(\eta/2,\ \psi_\mu\ \chi_{12}) = (\partial_\mu\psi_\mu,\ \partial_\mu\chi_{\mu\nu},\ 0)$$

$$\langle\omega^\dagger|(d-d^\dagger)\omega\rangle =$$

$\leftarrow$ covariant

$$\bar{\eta}/2\ \overset{D_\mu}{\partial_\mu}\psi_\mu+\bar{\psi}_\mu\left(\overset{D_\mu}{\partial_\mu}\eta/2+\overset{D_\nu}{\partial_\nu}\chi_{\nu\mu}\right)$$

$$+\bar{\chi}_{\mu\nu}\left(\overset{D_\mu}{\partial_\mu}\psi_\nu-\overset{D}{\partial_\nu}\psi_\mu\right)$$

apply reality condition to fields

$$\psi_\mu D_\mu\eta+2\chi_{\mu\nu}(D_\mu\psi_\nu-D_\nu\psi_\mu)$$

= twisted fermion action!

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D.O.F

D=2 DK action naturally describes

2 Dirac fermions

~~#~~ but by applying a reality condition on component twisted fields →

~~2 Mo~~ 1 Dirac spinor.
(2 Majorana)

but this just what ~~on~~ one needs for N=2 SYM!

(no fractional part of fermion det encountered ...)

also well-known result that DK actions may be discretized without spectrum doubling

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Very concise way of rewriting SYM action:

$\Psi = (\eta/2, \psi_\mu, \chi_{12})$ $\quad \Phi = (\bar{\phi} - \phi, A_\mu, B_{12})$

$$S_{twist} = Q \langle \Psi | d_A \Phi + Q\Psi \rangle$$

$$\eta/2\,[\phi, \bar{\phi}] + \psi_\mu D_\mu(\bar{\phi} - \phi)$$
$$+ \psi_\mu D_\mu \phi$$
$$+ 2\chi_{12}\,(\cancel{\ldots})\,F_{12}$$
$$+ \chi_{12} B_{12}$$

QED.

Lattice I (11)

(1) $A_\mu \to U_\mu$ (link)

(2) ψ_μ - link field

$\chi_{\mu\nu}$ - plaquette

$\eta, \bar{\phi}, \phi$ site fields

transform like link

$$\psi_\mu \to G(x)\,\psi_\mu(x)\,G(x+\mu)^{-1}$$

$$\eta \to G(x)\,\eta(x)\,G(x)^{-1}$$

$$\chi_{\mu\nu}(x) \to G(x)\,\chi_{\mu\nu}(x)\,G(x+\mu+\nu)^{-1} \quad \text{etc}$$

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* reduce to usual rules, as $a \to 0$
* to construct G.I. action need

$\psi, \psi^+ \sim G(x+\mu)\,\psi^+\,G(x)^{-1}$

Φ only poss if $\psi = \sum \psi^a T^a$

ψ^a = complex

same for $U_\mu = e^{A_\mu}$

Lattice II (12)

~~need covariant lattice de~~

Derivatives?

Following set work.

(g'need rules in D but lo simple)

$$D_\mu^+ \eta = U_\mu(x)\,\eta(x+\mu) - \eta(x)\,U_\mu(x)$$

transforms like

$$G(x)\,U_\mu(x)\,\cancel{G(x+\mu)^{-1}\,G(x+\mu)}\,\eta(x+\mu)\,G(x+\mu)^{-1}$$

$$G_x \ldots G_{x+\mu}^{-1}$$

ie transforms like link field.

$$D_\mu^+ \psi_\nu = U_\mu(x)\,\psi_\nu(x+\mu) - \psi_\nu(x)\,U_\mu(x+\nu)$$

transforms like plaquette field!

(13)

Similarly for adjoint

$$D_\mu^- \chi_{\mu\nu} = \chi_{\mu\nu}(x) U^\dagger_\mu(x+\nu) -$$

$$U_\mu^\dagger(x-\mu)\, \chi_{\mu\nu}(x-\mu)$$

↑ like link

$$D_\mu^- \psi_\mu = \psi_\mu(x) U^\dagger_\mu(x) - U_\mu^\dagger(x-\mu)\, \psi_\mu(x-\mu)$$

↑ like site

(14)

Field Strength

$$F_{\mu\nu}(x) = D_\mu^+ U_\nu(x) = f_{\mu\nu}^{\text{cont}} + O(a)$$

$$= U_\mu(x) U_\nu(x+\mu) - U_\nu(x) U_\mu(x+\nu)$$

(1) $F F^\dagger \rightarrow \left(U_\mu(x) U_\nu(x+\mu) - U_\nu(x) U_\mu(x+\nu)\right)$

$$\left[U_\nu^\dagger(x+\mu) U_\mu^\dagger(x) - U_\mu^\dagger(x+\nu) U_\nu^\dagger(x)\right]$$

$$= -U_\mu(x) U_\nu(x+\mu) U_\mu^\dagger(x+\nu) U_\nu^\dagger(x)$$

$$+ \text{h.c}$$

$$+ U_\mu(x) U_\nu(x+\mu) U_\nu^\dagger(x+\mu) U_\mu^\dagger(x+\mu)$$

$$= \underbrace{U_\nu^\dagger(x) U_\mu(x) U_\nu(x+\mu) U_\nu^\dagger(x+\mu)}_{= I\ !}$$

(2) $Q F_{\mu\nu} = D_\mu^+ \psi_\nu - D_\nu^+ \psi_\mu$ like cont !

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Q-sym on lattice

$$QU_\mu = \psi_\mu$$
$$Q\psi_\mu = D_\mu^+\phi$$
$$Q\phi = 0$$
$$Q\chi_{12} = B_{12}$$
$$QB_{12} = [\phi, \chi_{12}]^{(12)}$$
$$Q\bar{\phi} = \eta$$
$$Q\eta = [\phi, \bar{\phi}]$$

$$[\phi, \chi_{\mu\nu}]^{(\mu\nu)} = \phi(x)\chi_{\mu\nu}(x) - \chi_{\mu\nu}(x)\phi(x+\mu+\nu)$$

arises from funny G.T of fields

again $Q^2 = \delta^\phi$ lattice fields

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Connection to SYM ?

$$\beta \mathrm{Tr} \sum_x F_{12}^\dagger(x) F_{12}(x)$$

$$\beta \mathrm{Tr} \sum_x \left(2I - U_P - U_P^\dagger\right) + \beta \mathrm{Tr} \sum_x \left(M_{12} + M_{21} - 2I\right)$$

where

$$U_P = \mathrm{Tr}\left(U_1(x)U_2(x+1)U_1^\dagger(x+2)U_2^\dagger(x)\right)$$

and

$$M_{12} = U_1(x)U_1^\dagger(x)U_2^\dagger(x+1)U_2(x+1)$$

2nd term is zero if $U_\mu^\dagger(x)U_\mu(x) = I$.
Think of path integral as contour integral
To target continuum SYM theory choose contour so that:

$$\begin{aligned} \mathrm{Im}A_\mu^a &= 0 \text{ all fields bar scalars} \\ \bar{\phi} &= \phi^\dagger \end{aligned}$$

Clearly preserves G.I
Conjecture also preserves twisted W.I
Q-exact S ensures W.I β-independent
$\beta \to \infty$ collapses to continuum model - OK

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Fermions ?

Form spinors:

$$\Psi = \begin{pmatrix} \chi_{12} \\ \eta \\ \psi_1 \\ \psi_2 \end{pmatrix}$$

Action:
$\Psi^\dagger M \Psi$

$$M = \begin{pmatrix} -[\phi,]^{(p)} & K \\ -K^\dagger & [\overline{\phi},]^{(p)} \end{pmatrix}$$

and

$$K = \begin{pmatrix} D_2^+ & -D_1^+ \\ -D_1^- & -D_2^- \end{pmatrix}$$

After integration – Pf(M).
In free limit Pf$(M) = \det(K) = \det(D_\mu^+ D_\mu^-)$
As advertized – no doubles!

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Twisting N=4 SYM in D=4

R-symmetry – $SU(4)$ contains $SU(2) \times SU(2)$.
Allows for twisting.
Equivalently

$$\Psi_\alpha^{I,I'} \to \Psi_{\beta\alpha}^{I'}$$

Anticommuting matrix – expand on basis of γ's again – components are DK (twisted) fields
$D = 4$ DK field includes $(\eta, \psi_\mu, \chi_{\mu\nu}, \lambda_\mu, \rho)$ plus superpartners $(\overline{\phi}, A_\mu, B_{\mu\nu}, V_\mu, C)$

φ 4D eps tensor used.

Integrate out $B_{\mu\nu}$ and C:
Fields of twisted $N = 4$ SYM!
Twisted supersymmetry:

$$\begin{array}{rlrl} QA_\mu = \psi_\mu & & Q\psi_\mu = D_\mu\phi \\ QV_\mu = \lambda_\mu & & Q\lambda_\mu = [\phi, V_\mu] \\ Q\eta = \overline{\phi} & & Q\overline{\phi} = [\phi, \overline{\phi}] \\ Q\chi_{\mu\nu} = B_{\mu\nu} & & QB_{\mu\nu} = [\phi, \chi_{\mu\nu}] \\ Q\rho = C & & QC = [\phi, \rho] \end{array}$$

Furthermore $S^{\text{twist}}_{N=4\ SYM} = Q\text{Tr}\Lambda$!
Translate to lattice ?

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Conclusions

- Theories with extended SUSY (in Eucl. space) can (often) be discretized while preserving G.I and a (twisted) SUSY
- Lattice theories are local and free of spectrum doubling.
- Discretization proceeds from reformulation in geometrical terms (Dirac-Kähler fields)
- Should be possible on curved spaces (simplicial manifolds)
- Truly non-perturbative formulation – numerical simulation possible