Motivations for SQM

Curious results of Catterall & Gregory. Naive discretization does not give correct continuum results for effective (bosonic) mass.

Susy-Quantum Mechanics (SQM) is a very simple system. Should be able to understand completely.

Basic test for lattice susy approach.

Analytically tractable, easily simulated (small system, positive semi-definite fermion matrix), exact nonperturbative results available (solve 1d Schrödinger eq.).

Concisely illustrates some of the issues that arise in more interesting systems.

"Exact" susy (1/2) action available for comparison and study.

Naive lattice

The continuum imaginary-time action is:

$$S = \int_0^L dt \left[\frac{1}{2}((\partial_t x)^2 + h'^2(x)) + \bar{\psi}(\partial_t + h''(x))\psi \right]$$

$\psi, \bar{\psi}$ are 1-component Grassmann and x is real. $h(x)$ is the superpotential.

The action is invariant under the imaginary-time continuation of the susy transformations, generated by infinitesmal Grassmann parameters ϵ_1, ϵ_2:

$$\delta x = \epsilon_1 \psi + \epsilon_2 \bar{\psi}, \qquad \delta \bar{\psi} = -\epsilon_1(\partial_t x + h'), \qquad \delta \psi = -\epsilon_2(\partial_t x - h')$$

To prove this invariance one need only make use of (i) the Leibnitz rule and (ii) periodic boundary conditions for all fields.

But (i) is violated at order a^p, $p > 0$, by discretization. This is the chief obstruction to lattice susy.

With $\delta = \epsilon_1 Q_1 + \epsilon_2 Q_2$**,**

$$Q_1 x = \psi \ , \qquad Q_1 \bar{\psi} = -(\partial_t x + h') \ , \qquad Q_1 \psi = 0$$

$$Q_2 x = \bar{\psi} \ , \qquad Q_2 \bar{\psi} = 0 \ , \qquad Q_2 \psi = -(\partial_t x - h')$$

defines the supercharges Q_1, Q_2**. Algebra:** $\{Q_1, Q_2\} = 2\partial_t$ **if aux. fields or eqs. of motion introduced.**

In what follows it will be convenient to distinguish as $\tilde{h}$ **the part of the superpotential that leads to interaction terms in the action:**

$$h = \frac{1}{2} m x^2 + \tilde{h} \ , \qquad \tilde{h} = \sum_{n>2} \frac{g_n}{n} x^n$$

A particularly simple case that we will concentrate on is the one studied by CG:

$$h = \frac{1}{2} m x^2 + \frac{1}{4} g x^4$$

Naive action corresponds to replacement $\partial_t \to \Delta$**, where** Δ **is some choice of finite diff. operator.**

Here we take $\partial_t \to \Delta^-$ **for bosons, and**

$$\partial_t \to \Delta^W = \Delta^S - \frac{1}{2} r a \Delta^2$$

for fermions.

Doublers at $r = 0$ **aid our interpretation of important effects below.**

(Also, in future considerations of $d > 1$ **susy-FTs we need some sort of operator that lifts doublers;** Δ^W **is the simplest choice.)**

At $r = \pm 1$**,** $\Delta^W = \Delta^S - \frac{1}{2} r a \Delta^2 \to \Delta^{\mp}$**.**

Note also $\Delta^- x_i \Delta^- x_i = \Delta^+ x_i \Delta^+ x_i$**.**

Thus both naive choices $\Delta^{\pm}$ **are included in the discretization that we study.**

Then the action is:

$$a^{-1}S = \frac{1}{2}\Delta^- x_i \Delta^- x_i + \frac{1}{2}h_i' h_i' + \bar{\psi}_i(\Delta_{ij}^W + h_i''\delta_{ij})\psi_j$$

Here, $a = L/N$**, where** N **is the number of sites on the (periodic) lattice.**

Naive discretization of (imaginary-time) susy algebra $\{Q_1, Q_2\} = 2\partial_t \to 2\Delta^+$**:**

$$Q_1 x_i = \psi_i, \qquad Q_1\bar{\psi}_i = -(\Delta^+ x_i + h_i'), \qquad Q_1\psi_i = 0$$

$$Q_2 x_i = \bar{\psi}_i, \qquad Q_2\bar{\psi}_i = 0, \qquad Q_2\psi_i = -(\Delta^+ x_i - h_i')$$

The variation of the action, say under Q_1**:**

$$\begin{aligned} a^{-1}Q_1 S &= -\frac{a}{2}(1+r)x_i\Delta^-\Delta^2\psi_i - \frac{a}{2}(1-r)mx_i\Delta^2\psi_i \\ &\quad + \frac{ra}{2}\tilde{h}_i'\Delta^2\psi_i + (\Delta^S\tilde{h}_i' - \tilde{h}_i''\Delta^+ x_i)\psi_i \end{aligned}$$

All terms have an $\mathcal{O}(a)$ **suppression. To see this in the last term, we note:**

$$\Delta^S\tilde{h}_i' - \tilde{h}_i''\Delta^+ x_i = -\frac{a}{2}\tilde{h}_i''\Delta^2 x_i + \mathcal{O}(a^2)$$

The overall $\mathcal{O}(a)$ **suppression is consistent with the fact that the violation of susy is due to the violation of the Leibnitz rule.**

Power counting

In considerations below, we take

$$h = \frac{1}{2}mx^2 + \frac{1}{4}gx^4$$

Then Feynman vertices read of from

$$a^{-1}S_{\text{int}} = \sum_i \left(mgx_i^4 + \frac{1}{2}g^2x_i^6 + 3gx_i^2\bar{\psi}_i\psi_i \right)$$

Vertex rules indep. of a**; so no contrib. to** D**:**

$$\begin{aligned} D &= L - 2I_B - I_F \\ L &= 1 + I_B + I_F - V_4 - \tilde{V}_4 - V_6 \\ E_B + 2I_B &= 4V_4 + 6V_6 + 2\tilde{V}_4 \\ E_F + 2I_F &= 2\tilde{V}_4 \end{aligned}$$

The only $D \geq 0$ solution with $L > 0$ is:

$$D = E_F = I_B = V_4 = V_6 = 0$$

$$L = I_F = \tilde{V}_4 = 1, \quad E_B = 2$$

Boson 2-pt. fcn. w/ 1 fermion loop [Fig. 1a].

In the continuum, the corresponding expression is:

$$6g \int_{-\pi/a}^{\pi/a} \frac{dp}{2\pi} \frac{-ip + m}{p^2 + m^2} = 6g \left(\frac{1}{2} + \mathcal{O}(ma) \right)$$

In the lattice theory, the fermion loop contribution to the boson 2-point function is twice as big

$$6g \frac{1}{Na} \sum_{k=0}^{N-1} \frac{-ia^{-1} \sin(2\pi k/N) + m + 2ra^{-1} \sin^2(\pi k/N)}{a^{-2} \sin^2(2\pi k/N) + (m + 2ra^{-1} \sin^2(\pi k/N))^2}$$
$$= 6g \left(1 + \mathcal{O}(rma)\right)$$

Subtract off w/ CT:

$$\begin{aligned} a^{-1} S_{B,c} &= a^{-1} S_B + \frac{1}{2} \sum_i 3 g x_i^2 + \text{const.} \\ &= a^{-1} S_B + \frac{1}{2} \sum_i h_i'' \end{aligned}$$

For the h we study, this does not introduce any new $D \geq 0$ diagrams since it is just a mass shift.

The single $D \geq 0$ diagram+CT now has the right continuum limit. All $D < 0$ digrams have the right continuum limit by Reisz's thm. Thus we recover the complete perturbation series (all orders) of the continuum theory.

Still must check that the CT suffices at the nonperturbative level.

That is, for any diagram Γ we still could have nonperturbative terms that do not match those of the continuum:

$$\Gamma = \Gamma_{cont.} + \mathcal{O}(a) + \mathcal{O}(e^{-m^2/g})$$

We next show that the $\mathcal{O}(e^{-m^2/g})$ terms are absent, and make an $\mathcal{O}(a)$ improvement to the lattice action, using transfer matrix techniques.

T matrix analysis

Assume local CTs

$$h'^2(x) \to h'^2(x) + k(x), \quad h''(x)\bar{\psi}\psi \to (h''(x) + \ell(x))\bar{\psi}\psi$$

Then $Z = \mathbf{Tr}\,(-1)^F T^N$ **obtained from**

$$\begin{aligned} T[k,l] &= \mathcal{N}(a) \int_{-\infty}^{\infty} dz \exp\left[-\frac{z^2}{2a} - \frac{a}{2} h'^2(q) - \frac{a}{2} k(q) \right] \\ &\quad \times e^{izp} \left[1 + a(h''(q) + \ell(q)) b^\dagger b \right] \end{aligned}$$

where the operators satisfy

$$[q,p] = i, \qquad \{b, b^\dagger\} = 1, \qquad b^2 = (b^\dagger)^2 = 0$$

Straightforward manipulations in the $a \ll 1$ **limit yield the effective hamiltonian, defined by**

$$T[k,\ell] = e^{af(q)}\, e^{-aH[k,\ell]}\, e^{-af(q)}$$

equal to:

$$\begin{aligned} H[k,l] &= \frac{1}{2}p^2 + \frac{1}{2}h(q)'^2 + \frac{1}{2}k(q) \\ &\quad - (h''(q) + \ell(q)) \left(1 - \frac{1}{2} a (h''(q) + \ell(q)) \right) \frac{1}{2} [b^\dagger, b] \\ &\quad - \frac{1}{2}(h''(q) + \ell(q)) \left(1 - \frac{1}{2} a (h''(q) + \ell(q)) \right) + \mathcal{O}(a^2) \end{aligned}$$

with conjugation factors

$$f(q) = -\frac{1}{4}(h'^2(q) + k(q)) - \frac{1}{2}(h''(q) + \ell(q)) b^\dagger b$$

Then to match the continuum hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}h'^2(q) - \frac{1}{2}h''(q)[b^\dagger, b]$$

through $\mathcal{O}(a)$**, we must choose CTs**

$$k(q) = h''(q) + \mathcal{O}(a^2), \qquad l(q) = \frac{1}{2} a h''^2(q) + \mathcal{O}(a^2)$$

We see that the $\mathcal{O}(1)$ **CT** $k(q) = h''(q)$ **suffices at a non-perturbative level too.**

We also obtain the $\mathcal{O}(a)$ improved naive action:

$$\begin{aligned} a^{-1} S_{ca} &= \frac{1}{2}\sum_i \left(\Delta^- x_i \Delta^- x_i + h_i' h_i' + h_i''\right) \\ &\quad + \sum_{ij} \bar{\psi}_i (\Delta_{ij}^- + h_i'' \delta_{ij} + \frac{1}{2} a h_i''^2 \delta_{ij}) \psi_j \end{aligned}$$

Ward identity

We know that the $\mathcal{O}(1)$ CT will fix, but it is interesting to see how it occurs.

The measure is invariant, so for any operator $\mathcal{O}$

$$\langle Q_A \mathcal{O} \rangle = \langle (Q_A S) \mathcal{O} \rangle = a \langle \mathcal{Y}_A \mathcal{O} \rangle$$

Recall that (below we set $r = 1$)

$$\begin{aligned} a^{-1} Q_1 S &= -\frac{a}{2}(1+r) x_i \Delta^- \Delta^2 \psi_i - \frac{a}{2}(1-r) m x_i \Delta^2 \psi_i \\ &\quad + \frac{ra}{2} \tilde{h}_i' \Delta^2 \psi_i + (\Delta^S \tilde{h}_i' - \tilde{h}_i'' \Delta^+ x_i) \psi_i \end{aligned}$$

It is convenient to separate into 2-pt and 4-pt vertices:

$$Q_1 S = (Q_1 S)_{(2)} + (Q_1 S)_{(4)} \equiv a \mathcal{Y}_{1,(2)} + a \mathcal{Y}_{1,(4)}$$

W/ the $\mathcal{O}(1)$ CT, the susy variation is modified to

$$Q_1 S \to Q_1 S + a \sum_i 3 g x_i \psi_i$$

This extra term suffices to cancel the finite (a-indep.) violation of Ward id. [Fig. 2].

Susy lattice. Power-counting

The susy lattice:

$$a^{-1}S = \frac{1}{2}\sum_i (\Delta^W x_i + h_i')^2 + \sum_{ij} \bar{\psi}_i(\Delta^W_{ij} + h_i''\delta_{ij})\psi_j$$

1st by Nicolai map [Catterall & Gregory].

$$a^{-1}S = \frac{1}{2}\mathcal{N}_i\mathcal{N}_i + \bar{\psi}_i \frac{\partial \mathcal{N}_i}{\partial x_k}\psi_k, \qquad \mathcal{N}_i = \Delta^W x_i + h_i'$$

This form makes the exact susy rather obvious:

$$\delta_1 x_i = \epsilon_1 \psi_i, \qquad \delta_1 \psi_i = 0, \qquad \delta_1 \bar{\psi}_i = -\epsilon_1 \mathcal{N}_i$$

Later by superfields [Giedt & Poppitz].

New vertices occur in the susy theory.

$$\begin{aligned} a^{-1}S^{\text{susy}}_{\text{int}} &= \sum_i \Big(mgx_i^4 + \frac{1}{2}g^2 x_i^6 + 3gx_i^2 \bar{\psi}_i \psi_i \\ &\quad + gx_i^3 \Delta^S x_i - \frac{rag}{2} x_i^3 \Delta^2 x_i \Big) \end{aligned}$$

In power-counting, we combine boson 4-point into one $D = 1$ vertex:

$$V_4 \leftrightarrow mgx_i^4 + gx_i^3 \Delta^S x_i - \frac{rag}{2} x_i^3 \Delta^2 x_i$$

It is $D = 1$, since $\Delta^S \sim a\Delta^2 \sim a^{-1}$.

In addition, the boson propagator is now the modulus-squared of the fermion propagator. (Will lead to precise cancellations.)

Altogether,

$$D = L + V_4 - 2I_B - I_F$$

$$L = 1 + I_B + I_F - V_4 - \tilde{V}_4 - V_6$$

$$E_B + 2I_B = 4V_4 + 6V_6 + 2\tilde{V}_4$$

$$E_F + 2I_F = 2\tilde{V}_4$$

New $D=0$ diagram, the scalar loop [Fig. 1(b)]:

$$D = E_F = I_F = \tilde{V}_4 = V_6 = 0$$

$$L = I_B = V_4 = 1, \quad E_B = 2$$

Gives:

$$-6g \frac{1}{Na} \sum_{k=0}^{N-1} \frac{2m + 2ra^{-1}\sin^2(\pi k/N)}{a^{-2}\sin^2(2\pi k/N) + (m + 2ra^{-1}\sin^2(\pi k/N))^2}$$

Added to the fermion loop diagram,

$$6g \frac{1}{Na} \sum_{k=0}^{N-1} \frac{-ia^{-1}\sin(2\pi k/N) + m + 2ra^{-1}\sin^2(\pi k/N)}{a^{-2}\sin^2(2\pi k/N) + (m + 2ra^{-1}\sin^2(\pi k/N))^2}$$

the $D=0$ contributions cancel: only the $-m$ in the numerator survives.

Net result: $(a) + (b) = -3g(1 + \mathcal{O}(ma)) \to -3g$.

(Recall that in the continuum, the two diagrams yield: $3g - 6g = -3g$.)

Actually, Reisz's theorem guarantees correct continuum limit for $D<0$ part (easy to verify here), so no need to do calculation once we see $D=0$ parts cancel.

Entire perturbation series guaranteed to approach continuum value.

In summary, the chief advantage of the 1 exact susy is that the lattice perturbation series is $D<0$ and no CTs are required.

Implications and outlook

In $d>2$ theories, expect that some exact lattice susy will similarly improve the UV behavior of the lattice perturbation series. Then no/less CTs required, since less $D \geq 0$ diagrams to cure.

What happens at nonperturbative level far less clear.

For a simple 2d (2,2) Wess-Zumino model, we showed that 1 exact lattice susy leads to continuum pert. series w/o CTs.

This is highly nontrivial, since 3 of 4 susys, $SO(2)$ rotation invariance, and chiral $U(1)_R$ all recovered as long distance "accidental" symmetries, w/o fine-tuning.

Presently studying (via finite-size scaling in Monte Carlo simulations) the critical exponents (known from SCFT) as nonperturbative check.

MC Simulation

We have extracted excitation energies, or, effective masses, from connected Green functions:

$$G^{1B}(t) = \langle x_1 x_{1+t/a} \rangle \ , \qquad G^{1F}(t) = \langle \psi_1 \bar{\psi}_{1+t/a} \rangle$$

$$G^{2B}(t) = \langle x_1^2 x_{1+t/a}^2 \rangle_{conn.} \ , \qquad G^{2F}(t) = \langle x_1 \psi_1 x_{1+t/a} \bar{\psi}_{1+t/a} \rangle$$

Here $t = a, 2a, \ldots, Na$ is the imaginary-time of points on the lattice.

Due to the symmetry $x \to -x$ of the action, as well as fermion number, the states that contribute to each of these Green functions come from different sectors of the state space.

For $t \ll Na$ and $N \gg 1$, we have for example

$$G^{1B}(t) = c_{1B}e^{-m_{1B}t} + c_{3B}e^{-m_{3B}t} + \dots$$
$$G^{2B}(t) = c_{2B}e^{-m_{2B}t} + c_{4B}e^{-m_{4B}t} + \dots$$

and similar equations for the fermions. Here $m_{1B} < m_{2B} < m_{3B} < m_{4B}$.

All of our simulations are performed using hybrid Monte Carlo techniques.

Our chief impediment was the rapid increase of auto-correlation time (1000's of updates as we approach continuum).

Regarding parameters, note that

$$L = Na \equiv 1 \gg m^{-1} = 0.1 \approx m_{eff}^{-1} \gg a = LN^{-1} \equiv N^{-1}$$

I.e., $m = 10$, and $N \gg 10$ is the continuum limit.

Results (figures) confirm our analytical analysis.

Conclusions

Lattice power-counting analysis is a crucial pre-simulation step, since continuum power-counting generally modified.

In a super-renormalizable lattice FT, CTs are manageable; perturbative quantum continuum limit reliably obtained.

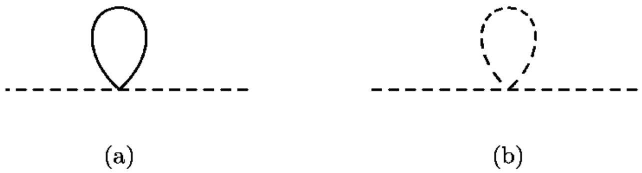
2d super-Yang Mills, nonlinear sigma models, etc. perhaps do-able if the lattice pert. series is at least super-renormalizable.

Of course, there are nonperturbative questions. Here we were able to address with transfer matrix analysis. Harder in $d > 1$ FT.

MC simulations support the analytical arguments. Shows that some susy systems can—in practice—be reliably studied on the lattice. But more interesting systems...we will see.

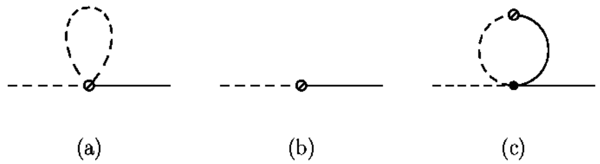
Upcoming work on (2,2) LG/WZ models vs. (2,2) minimal CFT models. Finite-size-scaling to extract critical exponents for comparison to CFT.

Figure 1



- $\mathcal{O}(g)$ corrections to boson propagator.
- Diagram (a) has only a $D = -1$ piece surviving in the continuum. In the continuum, $(a) + (b) = -3g$.
- Naive (uncorrected) lattice has $D = 0$ contribution coming from fermion doublers. Then $(a) + (b) = \mathcal{O}(ma) \to 0$.
- Corrected naive: $(a)+(b)+\mathrm{CT} = -3g(1+\mathcal{O}(ma)) \to -3g$.
- In the susy lattice action, additional interactions cause diagram (b) to also acquire a $D = 0$ contribution, which just cancels that of diagram (a). Susy lattice: $(a) + (b) = -3g(1 + \mathcal{O}(ma)) \to -3g$.

Figure 2



- Diagrams associated cancellation of the $\mathcal{O}(g)$ violation of the susy Ward identity.

- The 2-point and 4-point shaded vertices that violate fermion number are those arising from $Q_1 S_c$.

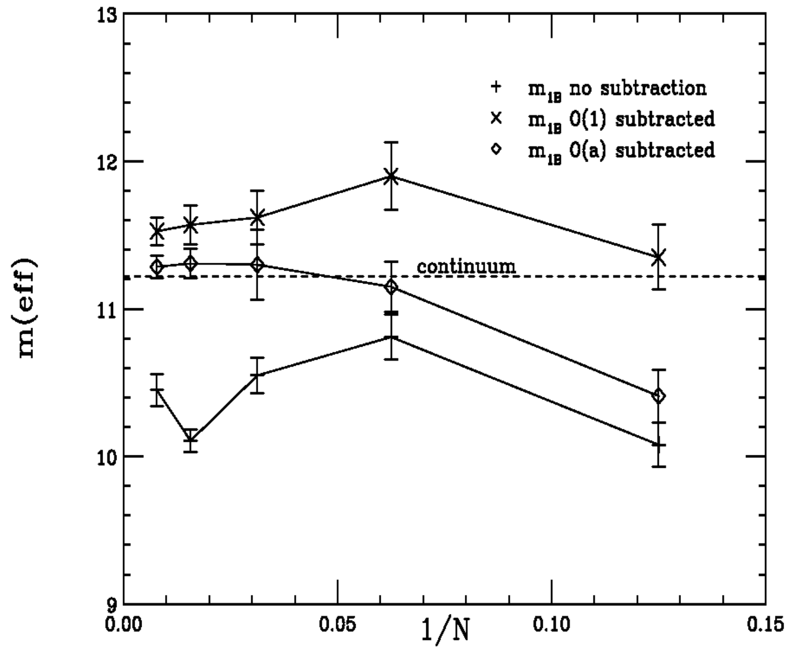
$$(a)\ :\ \mathcal{Y}_{1,(4)} = \frac{a}{2}\tilde{h}_i' \Delta^2 \psi_i + (\Delta^S \tilde{h}_i' - \tilde{h}_i'' \Delta^+ x_i)\psi_i$$

$$(b)\ :\ 3agx_i\psi_i$$

$$(c)\ :\ \mathcal{Y}_{1,(2)} = ax_i \Delta^- \Delta^2 \psi_i$$

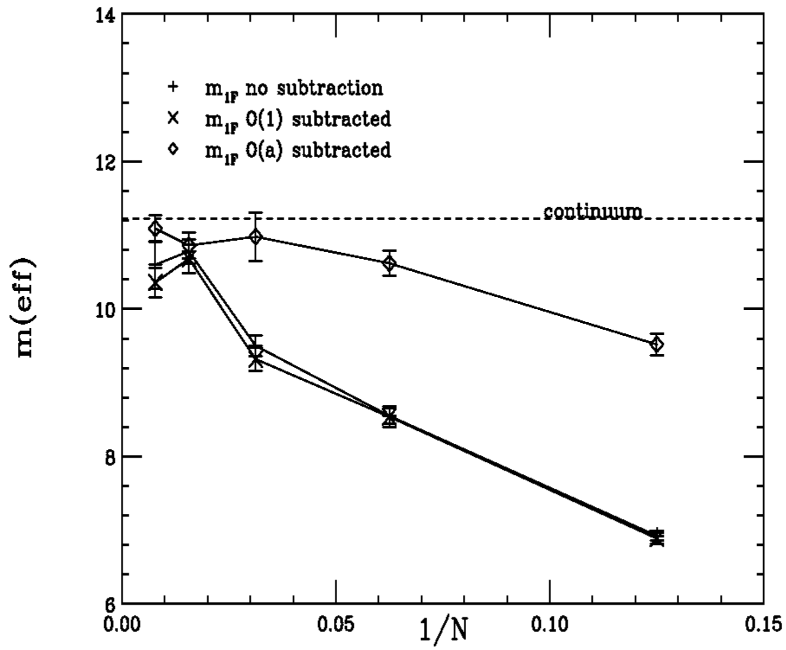
- The sum of these diagrams vanishes in the $a \to 0$ limit, provided the external momentum satisfies $|p_{ext}| \ll a$.

Figure 3



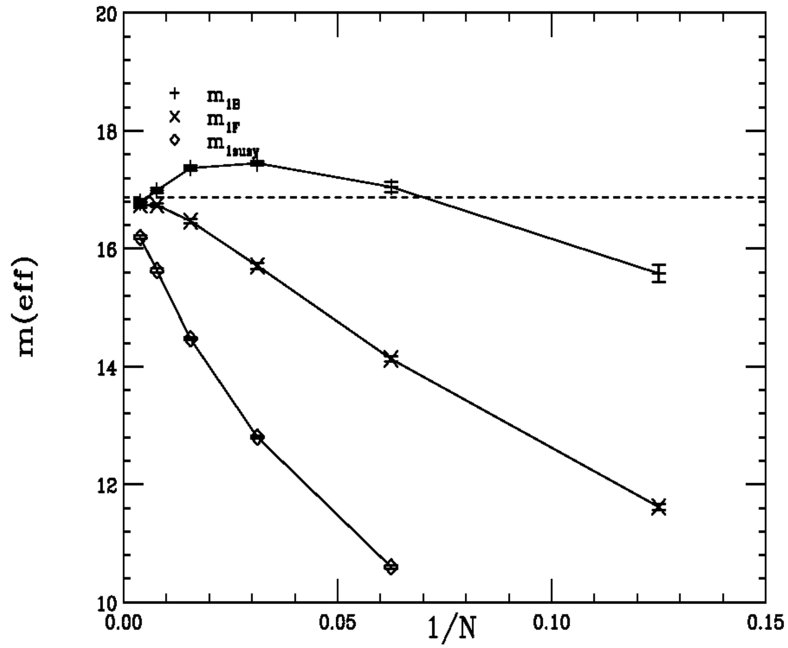
Leading boson mass for various forms of the action. Large N corresponds to the continuum limit. Here, $m = g = 10$.

Figure 4



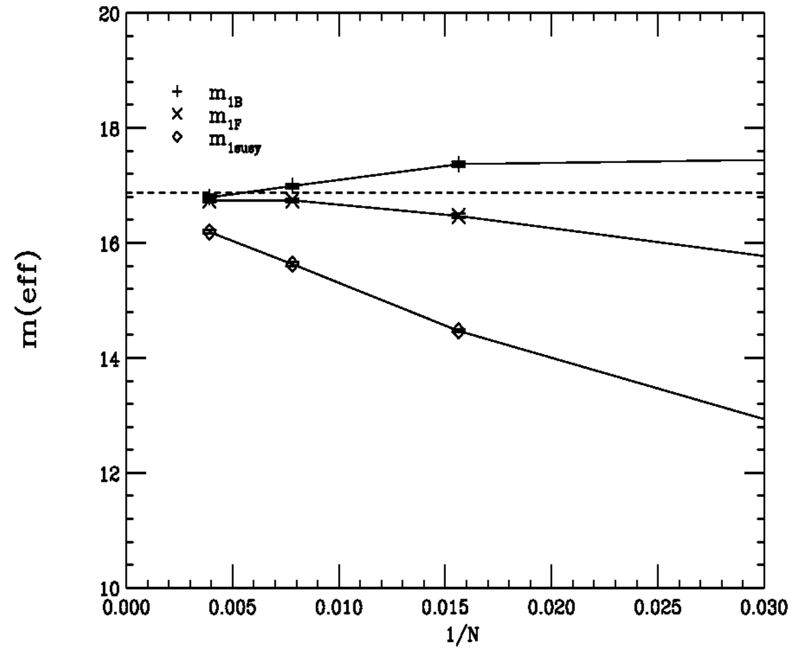
Similar to Fig. 4, except that fermion masses are displayed.

Figure 5



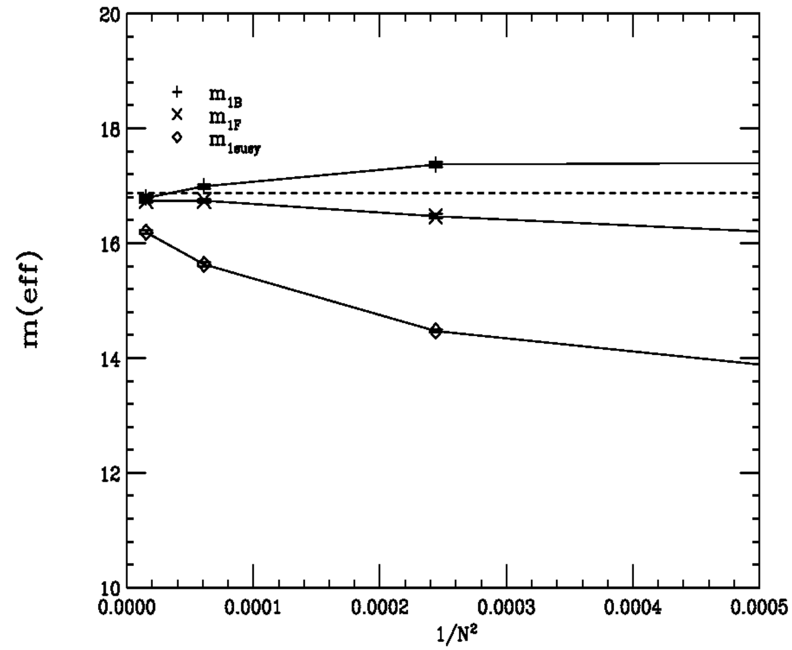
Strongly coupled, $\mathcal{O}(a)$ improved. $g = 100, m = 10$. Susy lattice results also shown. Larger discrepancy due to lack of $\mathcal{O}(a)$ improvement in that case.

Figure 6



Close-up of $g = 100, m = 10$ data. Susy lattice results shows $\mathcal{O}(a)$ scaling in continuum limit.

Figure 7



Close-up of $g = 100, m = 10$ data. The improved action data is not quite good enough to see $\mathcal{O}(a^2)$ scaling. Simple project to improve statistics, push to smaller a.