Sneaking up on finite density

(An incitement toward irresponsible thinking.)

Understanding QCD @ finite density is ~~one of~~ the most important unsolved problem~~s~~ in the Standard Model...

... but that is very hard!

Is it possible to attack different problems that could help us toward this goal?

Various ideas I will touch on here:

- Different density matrices
- Non-Lorentz-invariant theories
- Non-QCD theories
  Scalars, nonrelativistic fermions

Standard approach:

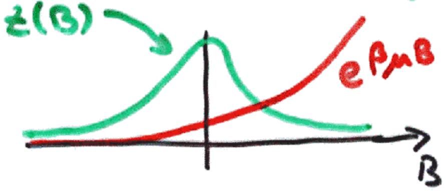
$$Z(\mu) = \mathrm{Tr}\, e^{-\beta\hat{H}} e^{\beta\mu\hat{B}} \leftarrow \text{baryon \#}$$

Define $Z(B) = \mathrm{Tr}\, e^{-\beta H} \delta(\hat{B} - B)$

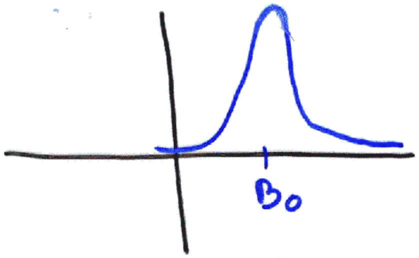
$$Z(\mu) = \sum_B Z(B) e^{\beta\mu B}$$

$$Z(B) = e^{-V\beta\mathcal{F}(B)} \leftarrow \text{grows faster than } B$$

eg $B^{4/3}$ for free relativistic fermions



multiply together

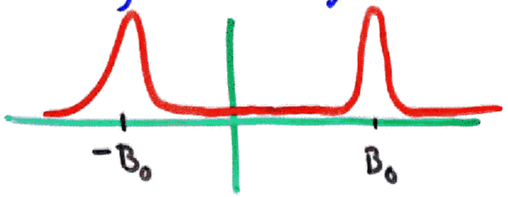


Path integral:

$$Z(\mu) = \int dA\, e^{-S_{YM}} \Delta_A(\mu) \longrightarrow \det[\not{D} + m + \mu\gamma^0]$$

$$\Delta_A^*(\mu) = \Delta_A(-\mu) \quad \ldots \text{complex}$$

Since $\Delta(\mu)^* = \Delta(-\mu)$, this suggests that the explicit breaking of charge symmetry is the source of the problem. Would it help to have a density matrix symmetric under C ?



This would be just as good for most applications...

if you considered the ~~wavefunction~~ density matrix

$$\frac{|N\rangle\langle N| + |\bar{N}\rangle\langle\bar{N}|}{\sqrt{2}}$$

neutron star!

anti-neutron star!

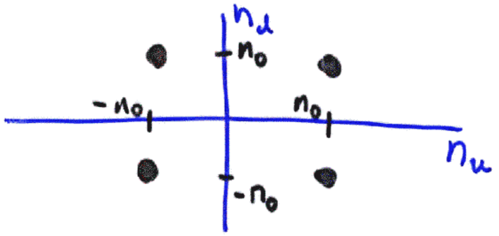
you could compute all matrix elements of interest (eg, local crcdlat correlation functions)

Bimodal density matrix

- $Z \equiv Z(\mu) + Z(-\mu)$?

No good. Real, but not positive integrand in path integral

- $Z_4 \equiv (Z(\mu) + Z(-\mu))^2$? = 2 flavor theory



Partition function peaks at 4 places

$n_u = \pm n_0, \quad n_d = \pm n_0$

$$Z_4 = Z_B + \underset{\text{baryon \#}}{Z_{-B}} + \underset{\text{isospin}}{Z_I} + Z_{-I}$$

Problem: $Z_{\pm I} \gg Z_{\pm B}$

matrix elements are insensitive to $Z_{\pm B}$.

Could we use Z_4 to compute

$$\langle B\,\mathcal{O}\rangle \text{ instead of } \langle\mathcal{O}\rangle\ ?$$

kills contributions from States with nonzero I but zero B ?

No. Suppresses matrix elements from I states by $\sim \frac{1}{\sqrt{n_0}}$, not enough* to counter relative enhancement by $\sim e^{+n_0^p}$

Would need to compute

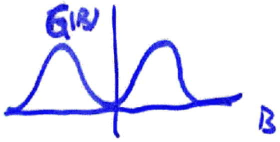
$$\langle e^{B}\,\mathcal{O}\rangle \quad \text{or} \quad \langle e^{-I}\,\mathcal{O}\rangle$$

To access the B part of the density matrix... not feasible.

More on bimodal density matrices

What about $Z \equiv \sum_B Z(B)\,G(B)$

where



looks like spontaneous breaking of C, CPT ~~Lorentz sym.~~ ~~baryon #~~.

$G(B) \equiv e^{-g(B)}$

For path integral, need to express as bilinear fermion action... with positive det.

write $G(B) = \int \tilde{G}(\mu)\, e^{i\mu B}\, d\mu$

$$Z = \int d\mu\ \tilde{G}(\mu)\ \mathrm{Tr}\, e^{-\beta H}\, e^{i\mu\beta B}$$

$$= \int d\mu\ \underbrace{\tilde{G}(\mu)}_{\text{must be positive + real}}\ \underbrace{Z(i\mu)}_{\text{positive, real determinant}}$$

⟹ implies that max of G(B) is at B=0!

OK: · chemical potential not feasible
· Spontaneous B generation not feasible

How does the universe get its nonzero baryon number?

Baryogenesis: Non-~~thermal~~-equilibrium processes have left us in an unstable excited state (all the baryon # will eventually decay, but that will take a long time)

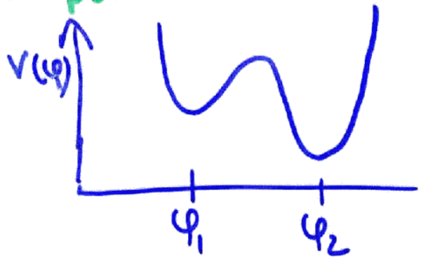
The problem is, we don't know how to incorporate departure from equilibrium into a lattice calculation... no room for real time in Euclidean path integral.

Could we take advantage of Monte Carlo time?

eg — could a bad simulation lead to good results?

An oversimplistic paradigm:

Consider a scalar field theory with potential:



Suppose I generate φ stochastically, starting with an initial configuration $\varphi = \varphi_1$ ... and measured my correlation functions after the system had acheive a local equilibration, but before it had found the true minimum at $\varphi = \varphi_2$.

Could ~~we~~ take the "Z_4" model, $Z_4 \equiv (Z(\mu) + Z(-\mu))^2$, start with a gauge field configuration for which $\langle \hat{B} \rangle_A \sim B_0$, and then do a "bad" simulation so that we never sample $\langle \hat{B} \rangle_A \simeq 0$ fields?

I guess analogy with scalar field is poor: no reason to expect $\overset{\text{action}}{\wedge}$ barrier around point in field space with large $\langle \hat{B} \rangle_A$ or $\langle \hat{B}^2 \rangle_A$.

... so no reason to "get stuck" there.

Still: I think it worth exploring further.

* * *

New topic: Non-lorentz invariant "QCD"?

Motivation: Theory of color superconductivity suggests that at high density $\langle qq \rangle$ condensate breaks color.

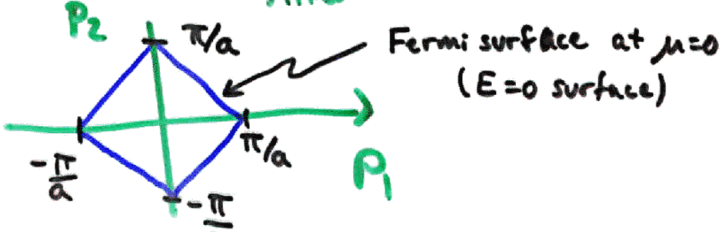
Pretty neat: a gauge symmetry causing itself to spontaneously break... similar to chiral gauge theories.

Given that we do not know how to study this on the lattice, can we study a related theory w/o a sign problem?

Some fermion theories in condensed matter ~~an~~ have no sign problem for "half filling" (Hubbard model?) Due to particle-hole symmetry?

Perhaps one can engineer an SU(3) Yang-Mills lattice theory with a non-conventional dispersion relation, to avoid the sign problem?

E.g. in 2D, if $E = -(\cos p_1 + \cos p_2)$



Looks like a possible candidate for positive det... don't know. Has a Fermi surface + may display pairing?

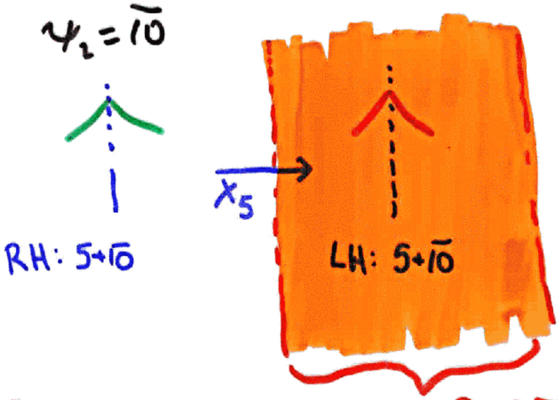
Yet another unrelated comment:

Could there be a connection between finite density + chiral gauge theories?

Eg, chiral gauge theory: SU(5) with LH fermions = $5+\overline{10}$

Domain wall + anti-wall: 5D theory with

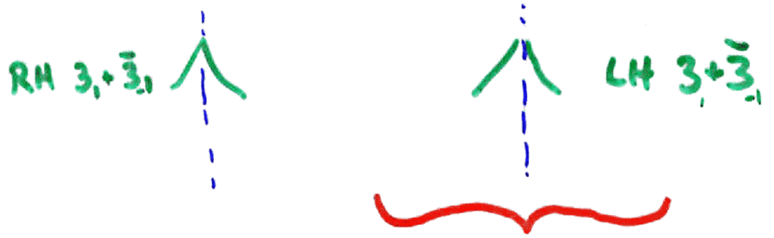
$\Psi_1 = 5$ $\Psi_2 = \overline{10}$



If we could turn on SU(5) only in this region, we would have a chiral gauge theory. (This breaks gauge sym. in 5d theory... perhaps sensible limits could still be defined)

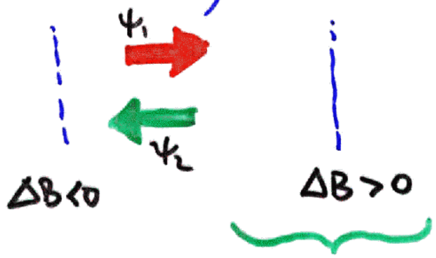
But now consider a 5d gauge symmetry

SU(3)×U(1): $\Psi_1 = 3_1$ $\Psi_2 = \bar{3}_{-1}$



localizing SU(3) makes this look like 1-flavor QCD

Now turn on 4D $F\tilde{F}$ for the U(1) (a classical background field). Get current flow:



localizing SU(3) ⇒ QCD with nonzero baryon #.

No reason to think this problem is easier... but it does look different!

Looking for other problems in finite density

- Nonrelativistic fermions
- Scalars

1. Nonrelativistic fermions

Why does a nonzero isospin density in QCD cause no problems, but a finite baryon # density does? Why does a finite density of fermions in a real representation of the gauge group cause no problems?

interaction between fermions most attractive when $R_1 \times R_2 = \text{Singlet} + \ldots$

eg: $3\times\bar{3}$, 8×8 of SU(3)
2×2 in SU(2)
⋮

Not 3×3 in SU(3)

Less attraction / repulsion

⇒ Z must be exponentially smaller

⇒ cancellations in path integral

Are there interesting systems where the fermion interactions are attractive (other than nonzero isospin)?

Yes, when one incorporates effective field theory.

Nonrelativistic fermions with large scattering length

- dilute neutron gas $|a| \sim 28\text{ fm} \gg m_\pi^{-1}$
- Trapped fermionic atoms $|a| \to \infty$ by tuning

↪ a universal, conformal theory

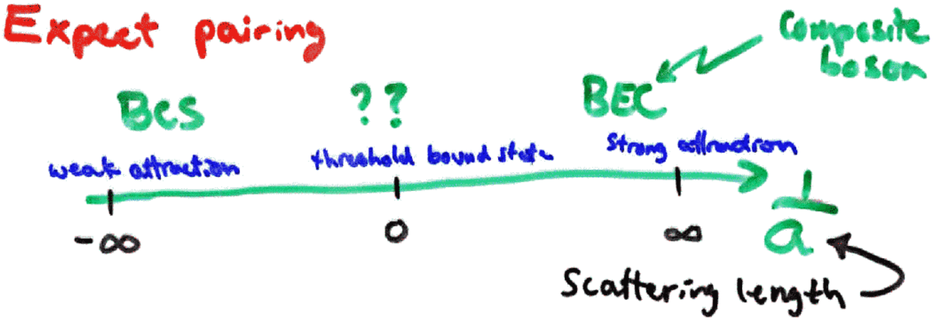
Dilute gas: inter atomic spacing >> atom size, but spacing << scattering length

An experimentally realizable, analytically intractable strongly coupled many body problem.

Chen + I recently showed:

- System can be formulated on lattice w/o a sign problem (even with source for $\psi\psi$)
- Lattice cutoff eliminates 1st order transition to high density one would find in the continuum with a purely attractive interaction
- This system is being numerically investigated by Matt Wingate.

Expect pairing



What about scalars at finite density?

These also suffer from an oscillating integrand in the path integral

$$S = \int |\partial_\mu \varphi|^2 + V(\varphi) + \underbrace{\mu \varphi^* \overleftrightarrow{\partial_\tau} \varphi}_{\uparrow \text{ imaginary}}$$

$\Rightarrow \mu^2 |\varphi|^2$

Interesting theory in its own right

- e.g. Q-balls ← Can have eq. of state indistinguishable from fermionic matter.
- Nonrelativistic trapped bosonic atoms.

Can try some of the same games I discussed with fermions — eg spontaneous generation of "baryon number" ... hasn't worked... phases appear.

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To gain insight, try dissecting the simplest model imaginable:

Quantum mechanics: a particle on a circle. Chemical potential for angular momentum $\hat{L}$

$$Z(\mu) = \mathrm{Tr}\, e^{-\beta \hat{H}} e^{\mu\beta\hat{L}} \quad , \quad \hat{H} = \tfrac{1}{2}\hat{L}^2$$

$$= \sum_m e^{-\beta\left(\frac{m^2}{2} - \mu m\right)}$$

a Jacobi Θ function.

construct the path integral

$$\int_0^{2\pi} d\theta \; |\theta\rangle\langle\theta| = \mathbb{1} \quad ,$$

$$\sum_m |m\rangle\langle m| = \mathbb{1}$$

$$\langle m|\theta\rangle = e^{im\theta}$$

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$$\langle\theta_i| e^{-\Delta\beta\left(\frac{1}{2}\hat{L}^2 - \mu\hat{L}\right)} |\theta_{i+1}\rangle$$

$$= \sum_m e^{-\Delta\beta\left(\frac{m^2}{2} - \mu m\right) + im(\theta_{i+1} - \theta_i)}$$

use Poisson resummation formula

$$\sum_n e^{ikn} = 2\pi \sum_n \delta(k - 2\pi n)$$

$$\Rightarrow \quad \sum_m e^{-am^2 - bm} = \sqrt{\frac{\pi}{a}} \sum_n e^{-\pi^2/a\,\left(n - ib/2\pi\right)^2}$$

So

$$\langle\theta_i| e^{-\Delta\beta\left(\hat{L}^2/2 - \mu\hat{L}\right)} |\theta_{i+1}\rangle$$

$$= \sum_n \sqrt{\frac{2\pi}{\Delta\beta}}\; e^{-2\pi^2/\Delta\beta\,\left(\underbrace{n - \frac{\theta_{i+1} - \theta_i}{2\pi}} + i\mu\frac{\Delta\beta}{2\pi}\right)^2}$$

get rid of $\sum_n$ by decompactifying θ

$$\int_0^{2\pi} d\theta_i \rightarrow \int_{-\infty}^{\infty} d\theta_i$$

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Result:

$$Z(\mu) = \mathcal{N} \underbrace{\int [d\theta]}_{\text{noncompact, periodic B.C. mod } 2\pi} e^{-\int_0^\beta d\tau\, \frac{1}{2}(\dot\theta - i\mu)^2}$$

noncompact, periodic B.C. mod 2π

Note: μ term is topological

$$e^{-\int d\tau\,(-i\mu\dot\theta)} = e^{i\mu\cdot 2\pi N}$$

winding number

Open question: In a scalar quantum field theory (eg. nonlinear σ-model) can the chemical potential be related to topology? Can its effects be summed analytically (or at least, factored out) from with numerical simulation only for non-topological modes? Any connection with fermions?

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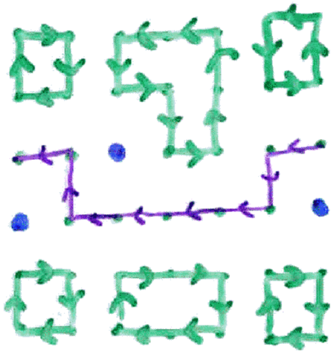
Fermion integral:

$$\int d\bar\Psi d\Psi\; e^{-\int \bar\Psi(\not{D} + m + \mu\gamma_0)\Psi}\Big|_{\text{lattice}}$$

$$= \sum$$



$$= \sum C_N e^{\mu}$$ (Net "winding number" of fermion paths)