Reduced Matrix Models and Noncommutative Lattice Theories

by

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Based on:

- pedagogical introduction (TEK vs. NCYM)
- original part

Ambjørn, Nishimura, Szabo, Y. M. (1999 – 2001)

Ambjørn, Dubin, Y. M. (2004)

Reduced Matrix Models

Eguchi, Kawai (1982)

Large-N reduction:

d-dimensional LGT is equivalent at $N = \infty$ to the one on a hypercube

$$L^d \cdot \infty = 1^d \cdot \infty$$

Subtleties for gauge theories for $d > 2$:

- quenching prescription

Bhanot, Heller, Neuberger (1982)

- twisting prescription

González-Arroyo, Okawa (1983)

work both on the lattice and in the continuum

Reduction of Scalar Field

Parisi prescription:

Matrix-valued $N \times N$ scalar field $\varphi_{kj}(x)$

$$\varphi(x) \overset{\text{red.}}{\to} D^\dagger(x)\varphi D(x)$$

$$D(x) = \mathrm{e}^{-\mathrm{i}P_\mu x_\mu}$$

$$P^\mu = \mathrm{diag}\,(p_1^\mu, \ldots, p_N^\mu)$$

— diagonal Hermitian. Explicitly

$$\varphi_{kj}(x) \overset{\text{red.}}{\to} \mathrm{e}^{\mathrm{i}(p_k - p_j)^\mu x_\mu}\varphi_{kj}$$

where φ_{kj} is x-independent

like a gauge transformation of a *constant* field

Reduction of Scalar Field (cont.)

Reduced action

$$S_{\mathrm{R}} = -N\sum_{ij}\left|\varphi_{ij}\right|^2\sum_\mu \cos\left[(p_i^\mu - p_j^\mu)a\right] + N\,\mathrm{tr}\,V(\varphi)$$

is *equivalent* at $N=\infty$ to the original one

$$S = \sum_x N\,\mathrm{tr}\,[-\sum_\mu \varphi(x)\varphi(x+a\hat{\mu}) + V(\varphi(x))]$$

Averages coincide after integration over p_i^μ

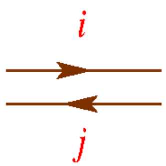
$$\langle F[\varphi(x)]\rangle \overset{\text{red.}}{=} a^{Nd}\int_{-\pi/a}^{\pi/a}\prod_{\mu=1}^{d}\prod_{i=1}^{N}\frac{\mathrm{d}p_i^\mu}{2\pi}\langle F[D^\dagger(x)\varphi D(x)]\rangle_{\mathrm{R}}$$

p_i^μ are *quenched*

can instead be distributed over a hypercube

Recovering Planar Diagrams

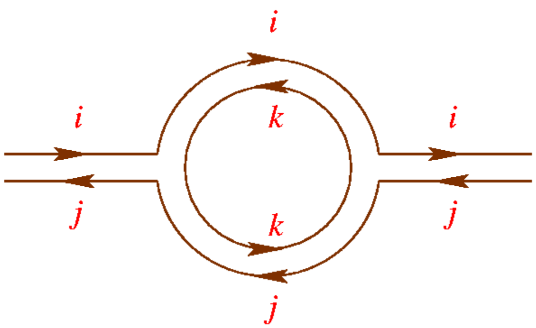
Propagator in Reduced Model



$$= \frac{1}{N} G\big(p_i - p_j\big)$$

$$G\big(p_i - p_j\big) = \frac{1}{M - 2\sum_\mu \cos\left[(p_i^\mu - p_j^\mu)a\right]} .$$

Simplest planar diagram



$$= \frac{\lambda^2}{N^2} G^2\big(p_i - p_j\big) \sum_k G(p_i - p_k)\, G\big(p_k - p_j\big)$$

Recovering Planar Diagrams (cont.)

Total momenta along the double lines

$$\left.\begin{aligned} p_i - p_j &= p\,, \\ p_j - p_k &= q\,, \\ p_i - p_k &= p + q\,. \end{aligned}\right\}$$

recover

$$G^{(2)}(p) = a^d \frac{\lambda^2}{N} G^2(p) \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^d q}{(2\pi)^d} G(q)\, G(p+q)$$

after

$$\frac{1}{N} \sum_k f(p_k) \Rightarrow a^d \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^d q}{(2\pi)^d} f(q)$$

Similarly it works in the continuum

(for spherical Gaussian regularization)

Gross, Kitazawa (1982)

Reduction in the Continuum

Gross, Kitazawa (1982)

Continuum reduced action

$$S_{\mathrm{R}} = vN\,\mathrm{tr}\left\{-\frac{1}{2}[P_\mu,\varphi]^2 + V(\varphi)\right\}$$

Unit volume v depends on regularization:

$$v = a^d \qquad \text{lattice regularization}$$

$$v = \left(\frac{2\pi}{\Lambda^2}\right)^{d/2} \qquad \text{spherical regularization}$$

$$\int^\Lambda \mathrm{d}^d p \cdots = \int \frac{\mathrm{d}^d p}{(\Lambda\sqrt{\pi})^d}\,\mathrm{e}^{-p^2/\Lambda^2}\cdots$$

Averages coincide

$$\langle F[\varphi(x)]\rangle \stackrel{\text{red.}}{=} \int^\Lambda \prod_{i=1}^N \mathrm{d}^d p_i \,\langle F[D^\dagger(x)\varphi D(x)]\rangle_{\mathrm{R}}$$

or distribute $\rho(p) = \left(\sqrt{\pi}\Lambda\right)^{-d}\mathrm{e}^{-p^2/\Lambda^2}$

Eguchi–Kawai model

Eguchi, Kawai (1982)

Reduction of Yang–Mills fields (lattice)

$$U_\mu(x) \stackrel{\text{red.}}{\to} D^\dagger(x+a\hat{\mu})U_\mu D(x)$$

compatible with the reduced gauge transform.

$$\Omega(x) \stackrel{\text{red.}}{\to} D^\dagger(x)\Omega D(x)$$

If first to gauge transform and then to reduce

$$\Omega(x+a\hat{\mu})\,U_\mu(x)\,\Omega^\dagger(x) \stackrel{\text{red.}}{\to} D^\dagger(x+a\hat{\mu})\Omega U_\mu \Omega^\dagger D(x)$$

then $U_\mu \stackrel{\text{g.t.}}{\longrightarrow} \Omega U_\mu \Omega^\dagger$

Subtleties for the large-N reduction on a torus

(when U_μ is to be constrained)

Eguchi–Kawai model (cont.)

Reduced gauge action (lattice)

$$S_{\rm R} = \frac{1}{2}\sum_{\mu\neq\nu}\left\{1-\frac{1}{N}{\rm tr}\,[U_\nu^\dagger U_\mu^\dagger D_\mu D_\nu D_\mu^\dagger D_\nu^\dagger U_\nu U_\mu]\right\}$$
$$= \frac{1}{2}\sum_{\mu\neq\nu}\left(1-\frac{1}{N}{\rm tr}\,U_\nu^\dagger U_\mu^\dagger U_\nu U_\mu\right)$$

because D_ν and $D_\mu^\dagger$ *commute*

$$D_\mu \stackrel{\rm def}{=} D(x+a\hat{\mu})\,D^\dagger(x) = {\rm e}^{-{\rm i}P_\mu a}$$

Gauge-invariant averages

$$\langle F[U_\mu(x)]\rangle \stackrel{\rm red.}{=} \langle F[U_\mu]\rangle_{\rm EK} \qquad \boxed{\text{gauge invariant } F}$$

e.g. Wilson loops

$$\left\langle\frac{1}{N}{\rm tr}\,U(C)\right\rangle \stackrel{\rm red.}{=} \langle\frac{1}{N}{\rm tr}\,\boldsymbol{P}\prod_{i\in C}U_{\mu_i}\rangle_{\rm EK}$$

$$= \langle\frac{1}{N}{\rm tr}\,U_2^\dagger{}^R U_1^\dagger{}^T U_2{}^R U_1{}^T\rangle_{\rm EK} \qquad \text{(for rectangle)}$$

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Continuum EK model

Continuum Eguchi–Kawai action

$$S_{\rm EK} = -\frac{v}{4g^2}{\rm tr}\,[A_\mu, A_\nu]^2$$

unit volume v is fixed by the regularization

is invariant under the gauge transformation

$$A_\mu \stackrel{\rm g.t.}{\longrightarrow} \Omega A_\mu \Omega^\dagger$$

Reduction of the *covariant derivative*

$${\rm i}\partial_\mu + \mathcal{A}_\mu(x) \stackrel{\rm red.}{\to} D^\dagger(x) A_\mu D(x)$$

rather than $\mathcal{A}_\mu(x)$ itself

Vacuum state given by *diagonal* matrices

$$A_\mu^{\rm cl} = -P_\mu \qquad \text{(modulo gauge transformation)}$$

is very degenerate: integration over P_μ

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Continuum EK model (cont.)

Gauge-invariant averages

$$\langle F[i\partial_\mu + \mathcal{A}_\mu(x)]\rangle \overset{\text{red.}}{=} \langle F[A_\mu]\rangle_{\text{EK}} \qquad \boxed{\text{gauge inv. } F}$$

and the Wilson loops

$$\left\langle \frac{1}{N} \operatorname{tr} \boldsymbol{P} \, e^{i \oint d\xi^\mu \mathcal{A}_\mu(\xi)} \right\rangle \overset{\text{red.}}{=} \left\langle \frac{1}{N} \operatorname{tr} \boldsymbol{P} \, e^{i \oint d\xi^\mu A_\mu} \right\rangle_{\text{EK}}$$

The reduction is valid *strictly* at $N = \infty$

Why the equivalence?

Open Wilson loops has to vanish:

- original theory: owing to gauge symmetry
- reduced model: owing to the R^d symmetry

$$A_\mu{}^{ij} \to A_\mu{}^{ij} + r_\mu \delta^{ij}$$

which was $U(1)^d$ on the lattice

This guarantees

$$W_{\text{EK}}(C_{yx}) \approx \frac{\delta_\Lambda^{(d)}(x-y)}{\delta_\Lambda^{(d)}(0)} W_{\text{EK}}(C_{xx})$$

$$= \delta_{xy} W_{\text{EK}}(C_{xx}) \qquad \text{(lattice)}$$

The reduced action is invariant under the R^d, but it should *not* be broken spontaneously

In fact it *is* broken for $d > 2$

Bhanot, Heller, Neuberger (1982)

Quenched EK model

Bhanot, Heller, Neuberger (1982)

Decompose $A_\mu = -V_\mu P_\mu V_\mu^\dagger$

with *diagonal* P_μ and *unitary* V_μ

Gauge symmetry: $V_\mu \to \Omega V_\mu$

Haar measure: $\mathrm{d}A_\mu = \mathrm{d}P_\mu \, \mathrm{d}V_\mu \, \Delta^2 (P_\mu)$

$\Delta (P_\mu)$ — Vandermonde determinant

The idea is to *quench* P_μ:

$$\langle F[\mathrm{i}\partial_\mu + \mathcal{A}_\mu(x)] \rangle \overset{\text{red.}}{=} \int^\Lambda \prod_{i=1}^N \mathrm{d}^d p_i \langle F\left[-V_\mu P_\mu V_\mu^\dagger\right] \rangle_V$$

Then the R^d symmetry is *not* broken and

planar diagrams are reproduced

Twisting Prescription

González-Arroyo, Okawa (1983)

Same reduction prescription as before but now

$$D(x) = \Gamma_1^{x_1/a} \Gamma_2^{x_2/a} \cdots \Gamma_d^{x_d/a}$$

where d (even) unitary $N \times N$ twisteaters Γ_μ

obey the Weyl–'t Hooft commutation relation

$$\Gamma_\mu \Gamma_\nu = Z_{\mu\nu} \Gamma_\nu \Gamma_\mu \qquad Z_{\mu\nu} = Z_{\nu\mu}^\dagger \in Z(N)$$

$d = 2$: clock and shift matrices: Weyl (1931)

$$\mathcal{Q} = \mathrm{diag}\left(1, \omega, \omega^2, \ldots, \omega^{L-1}\right) \qquad \omega \in Z(L)$$

$$\mathcal{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \qquad L = N$$

Twisting Prescription (cont.)

$d = 4$ twisteaters (simplest twist):

$$Z_{\mu\nu} = e^{2\pi i n_{\mu\nu}/N} \in Z(N)$$

$$n_{\mu\nu} = \begin{pmatrix} 0 & +n & & \\ -n & 0 & & \\ & & 0 & +n \\ & & -n & 0 \end{pmatrix} \qquad n \in \mathbb{Z}_N$$

Direct product of $L \times L$ clock and shift matrices

$$\left.\begin{array}{ll} \Gamma_1 = \mathcal{P} \otimes \mathbb{I} & \Gamma_2 = \mathcal{Q} \otimes \mathbb{I} \\ \Gamma_3 = \mathbb{I} \otimes \mathcal{P} & \Gamma_4 = \mathbb{I} \otimes \mathcal{Q} \end{array}\right\}$$

which is possible only if $N = L^2$ and $n = L$

More sophisticated twists are needed for tori

(boxes with p.b.c.)

Twisted Reduced Model (scalars)

Eguchi, Nakayama (1983)

Twisted reduced action

$$S_{\mathrm{TR}} = -N \sum_{\mu} \operatorname{tr} \Gamma_\mu \varphi \Gamma_\mu^\dagger \varphi + N \operatorname{tr} V(\varphi)$$

The averages

$$\langle F[\varphi(x)] \rangle \overset{\text{red.}}{=} \langle F\left[D^\dagger(x) \varphi D(x)\right] \rangle_{\mathrm{TR}}$$

How to prove the equivalence?

Matrices versus fields

Mapping matrices into fields on a N^2 lattice:

$$\varphi_{ij} \iff \varphi(ai, aj)$$

Explicitly

$$\varphi = \sum_x \Delta(x)\varphi(x)$$
$$\varphi(x) = N\,\mathrm{tr}\,[\varphi\,\Delta(x)]$$

by using the matrix function (in $d=2$)

$$\Delta_{ij}(x) = \frac{1}{N^2}\sum_{m\in\mathbb{Z}_L^2} \mathrm{e}^{-\mathrm{i}\frac{2\pi m}{aN}\times x}\left(\mathcal{P}^{m_1}\mathcal{Q}^{m_2}\right)_{ij}\omega^{-\frac{m_1 m_2}{2}}$$

Same coefficients of Fourier expansions

$$\varphi_{ij} = \frac{1}{N^2}\sum_{m\in\mathbb{Z}_L^2}\left(\mathcal{P}^{m_1}\mathcal{Q}^{m_2}\right)_{ij}\omega^{-\frac{m_1 m_2}{2}}\varphi(m)$$
$$\varphi(x) = \sum_{m\in\mathbb{Z}_L^2} \mathrm{e}^{\mathrm{i}\frac{2\pi m}{aN}\times x}\varphi(m)$$

Matrices versus fields (cont.)

Kinetic parts of the actions *coincide*

$$N\,\mathrm{tr}\left(\frac{M}{2}\varphi^2 - \sum_\mu \Gamma_\mu\varphi\Gamma_\mu^\dagger\varphi\right)$$
$$= \sum_x\left(\frac{M}{2}\varphi^2(x) - \sum_\mu \varphi(x)\varphi(x+a\hat{\mu})\right)$$

Cubic interaction has an extra phase

$$N\,\mathrm{tr}\,\varphi^3 = \frac{1}{N^4}\sum_{m,n}\varphi(-m-n)\varphi(m)\varphi(n) \times \mathrm{e}^{\pi\mathrm{i}\sum_{\mu,\nu} m_\mu n_{\mu\nu} n_\nu/N}$$

The planar limit $N\to\infty$ is the same

(because of this extra phase)

Important Theorem

Eguchi, Nakayama (1983)

González-Arroyo, Okawa (1983)

The presence of the phase is crucial to reproduce planar diagrams at $N = \infty$

This happens because of the *theorem* stating:

- the phases cancel out in planar graphs
- the phases remain in nonplanar graphs suppressing them as $N \to \infty$

It can be proven by rewriting

$$\mathrm{e}^{\pi \mathrm{i} \sum_{\mu,\nu} m_\mu n_{\mu\nu} n_\nu / N} = \mathrm{e}^{-\mathrm{i} p^\mu \theta_{\mu\nu} q^\nu / 2} = \mathrm{e}^{-\mathrm{i}(p_i \theta p_j + p_j \theta p_k + p_k \theta p_i)/2}$$

$$\theta_{\mu\nu} = \frac{a^2 N}{2\pi} n_{\mu\nu}^{-1} \qquad p_\mu \equiv \frac{2\pi \sum_\nu n_{\mu\nu} m_\nu}{aN}$$

Counting Topologies

Filk (1996)

Minwalla, van Raamsdonk, Seiberg (2000)

Nonplanar diagram of genus h is suppressed

$$\left(p^{2d} \det_{\mu\nu} \theta_{\mu\nu}\right)^{-h} \sim N^{-2h}$$

in perfect agreement with 't Hooft expansion

This has immediate important consequences for gauge fields

Twisted EK Model

González-Arroyo, Okawa (1983)

Same reduction of gauge field but now

$$D_\mu \equiv D(x + a\hat{\mu})\, D^\dagger(x) = \Gamma_\mu$$

do *not commute*

TEK action

$$S_{\text{TEK}} = \frac{1}{2} \sum_{\mu \neq \nu} \left(1 - Z_{\mu\nu} \frac{1}{N} \operatorname{tr} U_\nu^\dagger U_\mu^\dagger U_\nu U_\mu \right)$$

TEK is nothing but LGT on a unit hypercube with twisted b.c.

$$U_\mu(x + \ell_\nu) = \Omega_\nu(x + a\hat{\mu})\, U_\mu(x)\, \Omega_\nu^\dagger(x)$$

$$\Omega_\mu(x + \ell_\nu)\, \Omega_\nu(x) = Z_{\mu\nu}\, \Omega_\nu(x + \ell_\mu)\, \Omega_\mu(x)$$

$Z_{\mu\nu} \in Z(N)$ represents the 't Hooft flux

Twisted EK Model (cont.)

$U(1)^d$ symmetry is *not* broken for all couplings owing to the twisting factor

Vacuum state is *not* diagonal

$U_\mu^{\text{cl}} = \Gamma_\mu$ (modulo gauge transformation)

Wilson loops

$$W_{\text{TEK}}(C) = \left\langle \frac{1}{N} \operatorname{tr} D^\dagger(C)\, \frac{1}{N} \operatorname{tr} \boldsymbol{P} \prod_{i \in C} U_{\mu_i} \right\rangle_{\text{TEK}}$$

where $D(C) = \boldsymbol{P} \prod_{i \in C} \Gamma_{\mu_i}$

is traceless for open loops

$$W_{\text{TEK}}(C_{xy}) = \delta_{xy}\, W_{\text{TEK}}(C_{xx})$$

except for passing throu the lattice: $\Gamma_\mu^L = 1$

This guarantees the equivalence at $N = \infty$

Continuum TEK

González-Arroyo, Korthals Altes (1983)

Usual continuum limit $a \to 0$:

$$U_\mu = e^{iaA_\mu} \quad \Gamma_\mu = e^{-iaP_\mu} \quad Z_{\mu\nu} = e^{ia^2 B_{\mu\nu}}$$

$$[P_\mu, P_\nu] = -iB_{\mu\nu}\,\mathbf{1} \qquad B_{\mu\nu} = \frac{2\pi n_{\mu\nu}}{N a^2} = \theta^{-1}_{\mu\nu}$$

Continuum TEK action

$$S_{\text{TEK}} = -\frac{v}{4g^2}\,\mathrm{tr}\,\left([A_\mu, A_\nu] + iB_{\mu\nu}\right)^2$$

remember that $\quad \mathrm{tr}\,[A_\mu, A_\nu] \neq 0$

for infinite-dimensional matrices (operators)

Vacuum configuration

$A^{\text{cl}}_\mu = -P_\mu$ (modulo gauge transformation)

Continuum TEK (cont.)

Wilson loops

$$W_{\text{TEK}}(C_{yx}) = \left\langle \frac{1}{N}\,\mathrm{tr}\,\boldsymbol{P}\,e^{i\int_{C_{yx}} d\xi^\mu P_\mu}\,\frac{1}{N}\,\mathrm{tr}\,\boldsymbol{P}\,e^{i\int_{C_{yx}} d\xi^\mu A_\mu}\right\rangle_{\text{TEK}}$$

vanish for open loops

$\Rightarrow$ the R^d symmetry is *not broken* and

planar diagrams are *reproduced*

Same theorem provides the equivalence at $N = \infty$

It can be proved also using loop equation where

$$\left(\frac{\delta^{(d)}_\Lambda(x-y)}{\delta^{(d)}_\Lambda(0)}\right)^2 = \frac{\delta^{(d)}_{\sqrt{2}\Lambda}(x-y)}{\delta^{(d)}_{\sqrt{2}\Lambda}(0)}$$

which was $\quad (\delta_{xy})^2 = \delta_{xy} \quad$ on the lattice

TEK with matter

Fundamental matter in the Veneziano limit

$N_f = n_f N \to \infty$ Veneziano (1976)

TEK action with matter Das (1983)

$$S = S_{\rm TEK} + N\,{\rm tr}\left[M\varphi^\dagger\varphi - \sum_\mu \left(\Gamma_\mu \varphi^\dagger U_\mu^\dagger \varphi + \Gamma_\mu^\dagger \varphi^\dagger U_\mu \varphi\right)\right]$$

describes n_f flavors

Continuum TEK action with matter

$$S = S_{\rm TEK} + vN\,{\rm tr}\left[m^2\varphi^\dagger\varphi + \sum_\mu |A_\mu\varphi + \varphi P_\mu|^2\right]$$

remember that $A_\mu = -P_\mu$ for free theory

$\Rightarrow$ commutator

Similar formulas exist for fermions

Noncommutative theories from TEK

Connes, Douglas, Schwarz (1998)

To reproduce planar diagrams

$$\theta_{\mu\nu} = \frac{a^2 N}{2\pi} n_{\mu\nu}^{-1} \to \infty \quad \text{as} \quad N \to \infty$$

$\theta_{\mu\nu}$ may be kept finite if

$a \sim N^{-1/2} \quad (d=2)$ or $a \sim N^{-1/4} \quad (d=4)$

— the double scaling limit

Aoki, Ishibashi, Iso, Kawai, Kitazawa, Tada (2000)

Noncommutative QFT's can be obtained

Star product

Mapping matrices into fields

$$N\,\mathrm{tr}\,\varphi^3 = \sum_x \varphi(x) \star \varphi(x) \star \varphi(x)$$

with the noncommutative star product

$$f_1(x)\star f_2(x) = \frac{1}{N^2}\sum_{y,z} \mathrm{e}^{-2\mathrm{i} y_\mu \theta^{-1}_{\mu\nu} z_\nu} f_1(x+y) f_2(x+z)$$

In the continuum

$$f_1(x) \star f_2(x) = f_1(x)\,\exp\left(\frac{\mathrm{i}}{2}\,\overleftarrow{\partial}_\mu\,\theta_{\mu\nu}\,\overrightarrow{\partial}_\nu\right) f_2(x)$$

$\Leftarrow$ noncommutative product of matrices

Noncommutative space

$$x_\mu \star x_\nu - x_\nu \star x_\mu = \mathrm{i}\,\theta_{\mu\nu}$$

Noncommutative QFT's can be constructed

NC Yang–Mills

$U_\theta(1)$ gauge theory

$$S = \frac{1}{4\lambda}\int \mathrm{d}^d x\,\mathcal{F}^2 \qquad \lambda = g^2_{\mathrm{TEK}} N$$

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - \mathrm{i}\,(\mathcal{A}_\mu \star \mathcal{A}_\nu - \mathcal{A}_\nu \star \mathcal{A}_\mu)$$

cubic + quartic interactions like in Yang–Mills

$\Rightarrow$ usual $U(1)$ = Maxwell theory as $\theta \to 0$

$\Rightarrow$ usual $U(\infty)$ Yang–Mills theory as $\theta \to \infty$

is invariant under star gauge transformation

$$\mathcal{A}_\mu \stackrel{\text{g.t.}}{\longrightarrow} \Omega \star \mathcal{A}_\mu \star \Omega^* + \mathrm{i}\,\Omega \star \partial_\mu \Omega^*$$

$$\Omega \star \Omega^* = 1 = \Omega^* \star \Omega \qquad \text{(star unitary)}$$

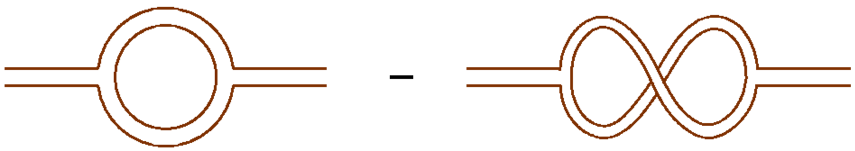
includes several coordinate transformations:

translations, rotations, parity reflection, . . .

One-loop renormalization

One-loop diagrams:

planar — nonplanar



UV divergent — UV convergent

$$\text{pl.} = \frac{20}{3}\lambda \int^{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2(p-k)^2} \approx \frac{5}{12\pi^2}\lambda \ln \frac{\Lambda^2}{p^2}$$

$$\text{np.} = \frac{20}{3}\lambda \int^{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{e^{ip\theta k}}{k^2(p-k)^2} \approx \frac{5}{12\pi^2}\lambda \ln \frac{\Lambda^2_{\text{eff}}}{p^2}$$

$$\Lambda^{-2}_{\text{eff}} = |\,\theta p\,|^2 + \Lambda^{-2} \qquad |\,p\,||\,\theta p\,| \ll 1$$

$$\text{as } \theta \to \infty : \quad \int^{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{e^{ip\theta k}}{k^2(p-k)^2} \sim \frac{1}{p^{2d}\det(\theta_{\mu\nu})}$$

One-loop renormalization (cont.)

$U_\theta(1)$ gauge theory is *asymptotically free*

— described only by the planar diagram

Same one-loop Gell-Mann–Low function as for $N=\infty$ Yang–Mills

If $\theta \to 0$, to subtract the nonplanar diagram — two contributions are canceled for usual $U(1)$

$U(1)$ is recovered for very large distances $\sim \Lambda\theta$ rather than $\sim \sqrt{\theta}$: UV/IR mixing

If $\quad \Delta x_1 \gtrsim \frac{1}{\Lambda}$ — UV cutoff

then $\quad \Delta x_2 \sim \frac{\theta}{\Delta x_1} \lesssim \theta\Lambda$ — IR cutoff

since $\quad \Delta x_1 \Delta x_2 \sim \theta$

UV and IR regularizations should be consistent

NC Quantum Electrodynamics

Noncommutative QED

$$S = \int \mathrm{d}^d x \left[\frac{1}{4\lambda} \mathcal{F}^2 + \bar{\psi}\gamma_\mu(\partial_\mu - \mathrm{i}\mathcal{A}_\mu \star)\psi + m\bar{\psi}\psi \right]$$

Fundamental matter:

$$\psi \xrightarrow{\text{g.t.}} \Omega \star \psi \qquad \bar{\psi} \xrightarrow{\text{g.t.}} \bar{\psi} \star \Omega^*$$

One-loop Gell-Mann–Low function

Hayakawa (2000)

$$\mathcal{B}(\lambda) = \frac{\lambda^2}{12\pi^2}(-11 + 2n_\mathrm{f})$$

same as for $N = \infty$ QCD in the Veneziano limit

(no nonplanar diagram with fermionic loop)

Limits of NC $U_\theta(1)$ gauge theory

Distances:	Theories:
$r \ll \sqrt{\theta}$	Veneziano limit of QCD
$\sqrt{\theta} \lesssim r \ll \theta\Lambda$	NC $U_\theta(1)$ QED
$\theta\Lambda \lesssim r$	QED

TEK for finite volume

Quotient condition imposed on A_μ:

Connes, Douglas, Schwarz (1998)

$$A_\mu + 2\pi R_\mu \delta_{\mu\nu} = \Omega_\nu A_\mu \Omega_\nu^\dagger$$

Finite N approximation

Ambjørn, Nishimura, Szabo, Y. M. (1999)

$$\mathrm{e}^{2\pi \mathrm{i}\, a\, \delta_{\mu\nu} R_\mu} U_\mu = \Omega_\nu U_\mu \Omega_\nu^\dagger$$

TEK for finite volume (cont.)

For $\Omega_\mu = \prod_\nu \Gamma_\nu^{m\varepsilon_{\mu\nu}}$ with integer

$$n = \frac{L}{m} \qquad (n = 1 \text{ for original TEK})$$

TEK with the quotient condition is equivalent to NCLGT on a finite lattice $\ell^d = (am)^d$:

$$S = \frac{1}{2} \sum_{x \in \mathbb{T}_m^d} \sum_{\mu \neq \nu} \left(1 - U_\nu^*(x) \star U_\mu^*(x + a\hat{\nu}) \star U_\nu(x + a\hat{\mu}) \star U_\mu(x)\right)$$

invariant under the star gauge transformation

$$U_\mu(x) \xrightarrow{\text{g.t.}} \Omega(x + a\hat{\mu}) \star U_\mu(x) \star \Omega^*(x)$$

Lattice star product is as before with

$$\theta_{\mu\nu} = -\frac{\ell^2}{\pi}\frac{n}{m}\varepsilon_{\mu\nu}$$

Morita Equivalence

A.Schwarz (1998)

For integer ratio $m/n \in \mathbb{Z}$ the two models

1) TEK with quotient condition

2) NCLGT with integer m/n

are equivalent to the third one:

3) usual LGT on the lattice $(an)^d$ with twisted b.c. and gauge group $SU(p)$

$$p = \left(\frac{m}{n}\right)^{d/2}$$

3) also has $n^d p^2 = m^d$ degrees of freedom

(1) and 3) coincide for $n = 1$)

1) is a EK reduction of 3)

Continuum torus can be obtained when both $m \to \infty$, $n \to \infty$ at fixed ratio p

Conclusions

- $N = \infty$ QCD = Reduced Matrix Models

  (space $\Longleftrightarrow$ gauge group)

- Finite-N approximation of TEK = NCLGT

  (finite lattice)

- NCQFT's at $\theta = \infty$ describe planar diagrams

  (except for 2D NCYM!!!)

- Gauge Theories on twisted torus

  = TEK with quotient condition

  (not known in 1980's)

- Help in solving $N = \infty$ QCD ?

  (probably not enough investigated)

- SUSY M(atrix) models of Superstrings

  (most interesting recent development)