The Role of Large Field Configurations in Perturbation Theory

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Overview

- I. Motivations
- II. Simple Model Calculations
- III. 4D Gluodynamics
- IV. Projects

I. Motivations

Main Goal: construct perturbative series which are converging and accurate.

For a generic observable $Obs.$ in a $\lambda\phi^4$ theory

$$Obs.(\lambda) \simeq \sum_{k=0}^{K} a_k(\phi_{max})\lambda^k$$

where ϕ_{max} is a large field cutoff. The simplest implementation is a restricted range of integration in the measure of a lattice formulation

$$\prod_x \int_{-\phi_{max}}^{\phi_{max}} d\phi_x .$$

Practical Considerations

The calculation of the modified coefficients $a_k(\phi_{max})$ fall in three categories:

- Low k (the usual ones with exponentially small corrections; semi classical)
- Intermediate k (crossover; complicated but with universal features)
- Large k (power suppressed; no $k!$ behavior)

The method works well in nontrivial cases (the anharmonic oscillator and $D = 3$ Dyson hierarchical model; see YM PRL 88 141601)

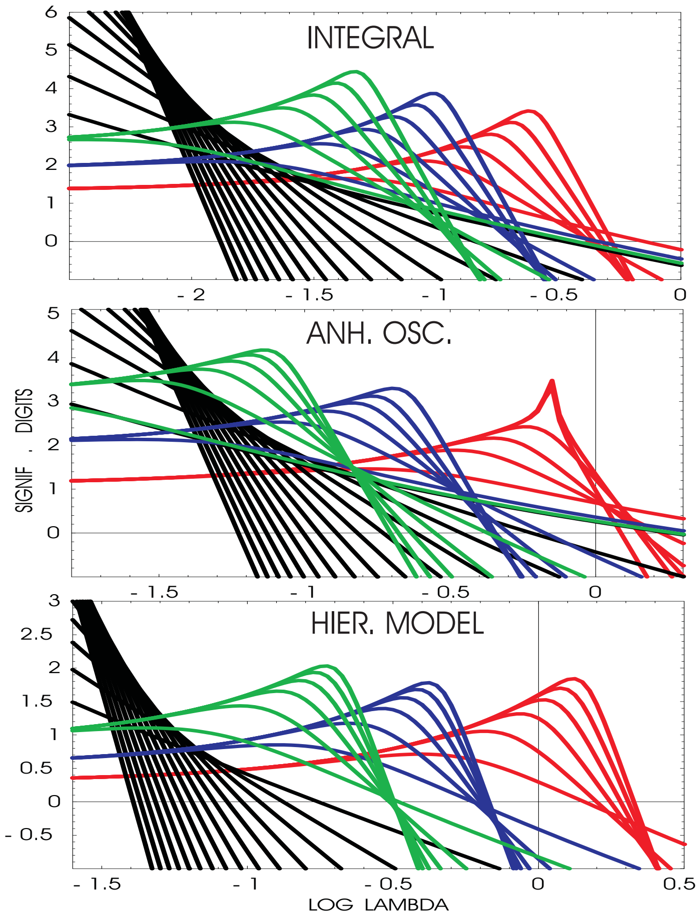


Figure 1: Number of significant digits obtained with regular perturbation theory at order 1, 3, 5, ..., 15 (black) and with $\phi_{max} = 3$ (green), 2.5 (blue) and 2 (red), at order 1, 3, ..., 11 , as a function of λ, for the ground state energy of the anh. osc and the renormalized mass of Dyson's model.

Optimization

- We can adjust $\phi_{max}(\lambda, K)$ in order to minimize or eliminate the discrepancy with the (usually unknown) correct value.

- The strong coupling can be used to calculate approximately this optimal $\phi_{max}(\lambda, K)$ (suitable to approach a weak/strong coupling crossover)

- The perturbative "coefficients" $a_k(\phi_{max}(\lambda, K))$ now depend on λ.

The large field cutoff/UV cutoff connection

Semi-classical examples of relations between these two cutoffs

1. Spherical charge density held together by some "strong" force within a radius R. The total charge is e.

Electrostatic Energy: $\mathcal{E} \propto \frac{e^2}{R}$

Maximum Electric field: $\left|\mathbf{E}\right|_{max} \propto \frac{e}{R^2} \propto \frac{\mathcal{E}^2}{e^3}$

2.Resonant Cavity

Maxwell equations without sources and the boundary conditions are invariant under a rescaling of the electric and magnetic fields.

The cavity explodes if $e\left|\mathbf{E}\right| \sim \dfrac{e^2}{4\pi\epsilon_0 r_B^2}$

$\mathcal{E}_{Ryd.} = \frac{e^2}{8\pi\epsilon_0 r_B}$; $r_B = \frac{\hbar}{m_e c\alpha}$

Maximum Electric field: $\left|\mathbf{E}\right|_{max} \propto \dfrac{\mathcal{E}^2_{Ryd.}}{e^3} \propto e m_e^2$

Note: the bound corresponds to $10^{18} Watt/cm^2$ and is extreme; eye protection materials can only sustain $10^{15} Watt/cm^2$

3.Instantons

The classical Yang-Mills equations are non-linear and a given solution cannot be rescaled arbitrarily, however translations and conformal transformation generate new solutions by changing the size (ρ) and the location $(\mathbf{x}_0)$. The magnitude of the field strength is maximized at $\mathbf{x} = \mathbf{x}_0$ and

$$|F^{\mu\nu}|_{max} \propto \rho^{-2}$$

If instantons of size $\rho < a \sim \Lambda$ are excluded then

$$|F^{\mu\nu}|_{max} \propto \Lambda^2$$

4. Classical Gravity

Spherical mass density supported by radiative pressure or Fermi pressure. The radius is R and the total mass M.

If we restrict our attention to the cases where $R > R_S = \frac{2GM}{c^2}$

Gravitational Energy: $\mathcal{E}_g \sim \frac{GM^2}{R} < Mc^2/2$

Gravitational Field $< \frac{GM}{R_S^2} = \frac{c^2}{2R_S}$

(no extra couplings here)

5. Effective potential for a scalar field

Examples: Higgs, inflaton (glueball?)

Suppose that there is some new physics at a scale Λ (e. g. $\Lambda = 1TeV$ for the Higgs or $\Lambda = M_P$ for the inflaton).

Does the new physics show up at large field fluctuations ($\phi > \Lambda^{(D-2)/2}$) or at large energy values ($V_{eff} > \Lambda^4$)?

Linde advocates the second possibility for the corrections of quantum gravity to the inflaton potential

I believe that the general form of the answer is that the corrections appear if $\phi > A\Lambda^{(D-2)/2}$ with A dimensionless and depending on the relevant couplings.

Remarks

Effective theories usually involve a UV cutoff but not a field cutoff

$D = 3$ critical potentials have finite radius of convergence (YM, PRD D67 025006 (large N); YM and Oktay, PRD 69 125016, $N = 1$). Interpretation?

A large field cutoff regulates some UV divergent quantities 2D and 3D scalar field theory (Li and YM, Lattice 2003)

Popular lattice models (Ising and $O(N)$ models, Wilson LGT with compact groups) have both a UV and a field cutoff (but this may also be a source of "artifacts")

II. Simple Model Calculations

- 1 site scalar field theory
- 1 plaquette gauge theory
- the anharmonic oscillator

1 site SFT (B. Kessler, Li and YM, PRD 69 045014)

$$\int_{-\infty}^{+\infty} d\phi e^{-\frac{1}{2}\phi^2-\lambda\phi^4} \neq \sum_{0}^{\infty} \frac{(-\lambda)^l}{l!} \int_{-\infty}^{+\infty} d\phi e^{-\frac{1}{2}\phi^2} \phi^{4l}$$

The peak of the integrand of the r.h.s. moves too fast when the order increases. On the other hand, if we introduce a field cutoff, the peak moves outside of the integration range and

$$\int_{-\phi_{max}}^{+\phi_{max}} d\phi e^{-\frac{1}{2}\phi^2-\lambda\phi^4} = \sum_{0}^{\infty} \frac{(-\lambda)^l}{l!} \int_{-\phi_{max}}^{+\phi_{max}} d\phi e^{-\frac{1}{2}\phi^2} \phi^{4l}$$

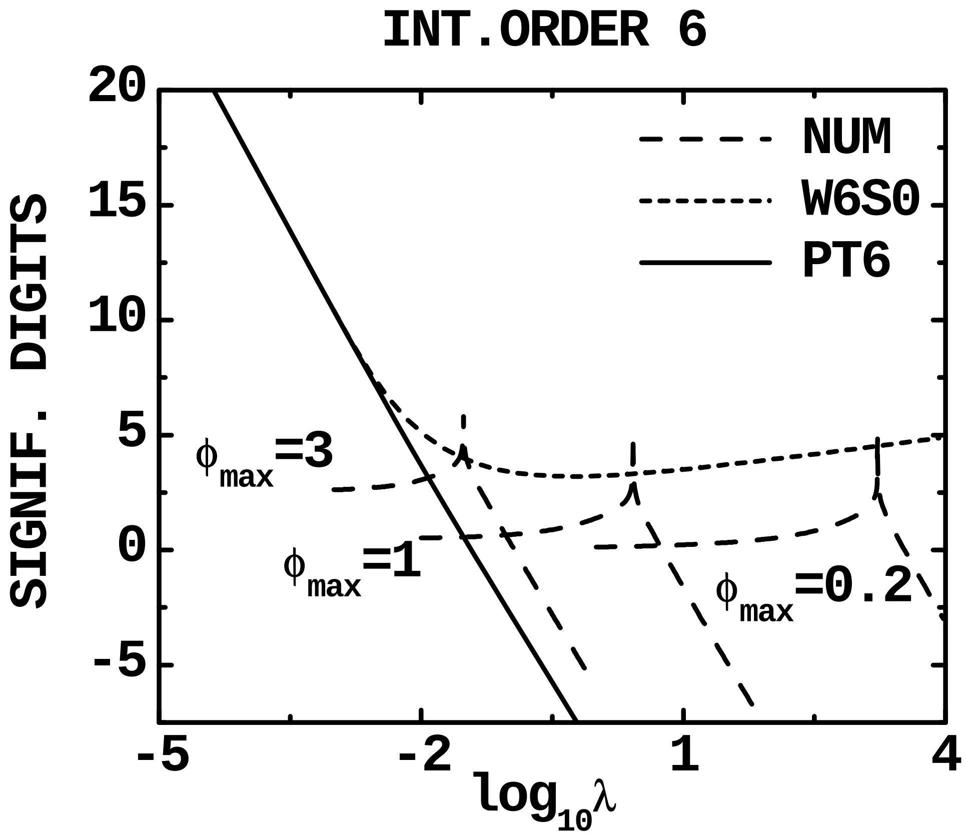


Figure 2: Significant digits obtained with the optimal cut $\phi_{max}(\lambda)$ estimated using a strong coupling expansion at lowest order (W6S0), compared to results at three fixed cuts and regular perturbation theory (PT6) at order 6.

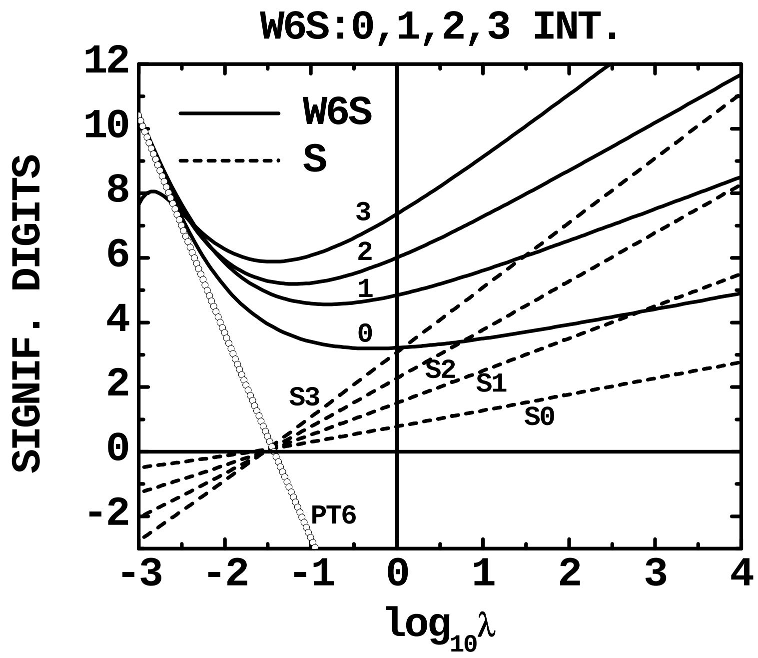


Figure 3: Significant digits obtained with the optimal cut $\phi_{max}(\lambda)$ (corresponding to a truncated expansion at order 6 in the weak coupling) estimated using the strong coupling expansion at orders 0, 1, 2 and 3 (solid lines), compared to significant digits using only the strong coupling expansion of the integral at the same orders in the strong coupling (dashed lines) and regular perturbation theory at order 6 (PT6).

1 plaquette LGT, (Li, YM hep-lat/0501023 PRD in pr.)

$$Z(\beta, N) = \int \prod_{l \in p} dU_l \mathrm{e}^{-\beta(1-\frac{1}{N}ReTrU_p)} ,$$

$$Z(\beta, 2) = (2/\beta)^{3/2}\frac{1}{\pi}\int_0^{2\beta} dt t^{1/2}\mathrm{e}^{-t}\sqrt{1-(t/2\beta)}$$

$$Z(\beta, 2, t_{max}) = (2/\beta)^{3/2}\frac{1}{\pi}\int_0^{t_{max}} dt t^{1/2}\mathrm{e}^{-t}\sqrt{1-(t/2\beta)}$$

$$Z(\beta, 2, t_{max}) = (\beta\pi)^{-3/2}2^{1/2}\sum_{l=0}^{\infty} A_l(t_{max})(2\beta)^{-l} ,$$

with

$$A_l(t_{max}) \equiv \frac{\Gamma(l+1/2)}{l!(1/2-l)}\int_0^{t_{max}} dt \mathrm{e}^{-t} t^{l+1/2} ,$$

$$Z(\beta, t_{max}) = (\beta\pi)^{-3/2} 2^{1/2} \sum_{l=0}^{\infty} A_l(t_{max})(2\beta)^{-l} \ ,$$

$$A_l(t_{max}) \equiv \frac{\Gamma(l+1/2)}{l!(1/2-l)} \int_0^{t_{max}} dt e^{-t} t^{l+1/2} \ ,$$

When $t_{max} \rightarrow \infty$ the integral becomes the (complete) Γ function and the coefficients grow factorially. In lattice perturbation theory, we "add the tails".

When t_{max} is finite, the integral is bounded by a power of t_{max}. When $t_{max} \leq 2\beta$, the sum converges.

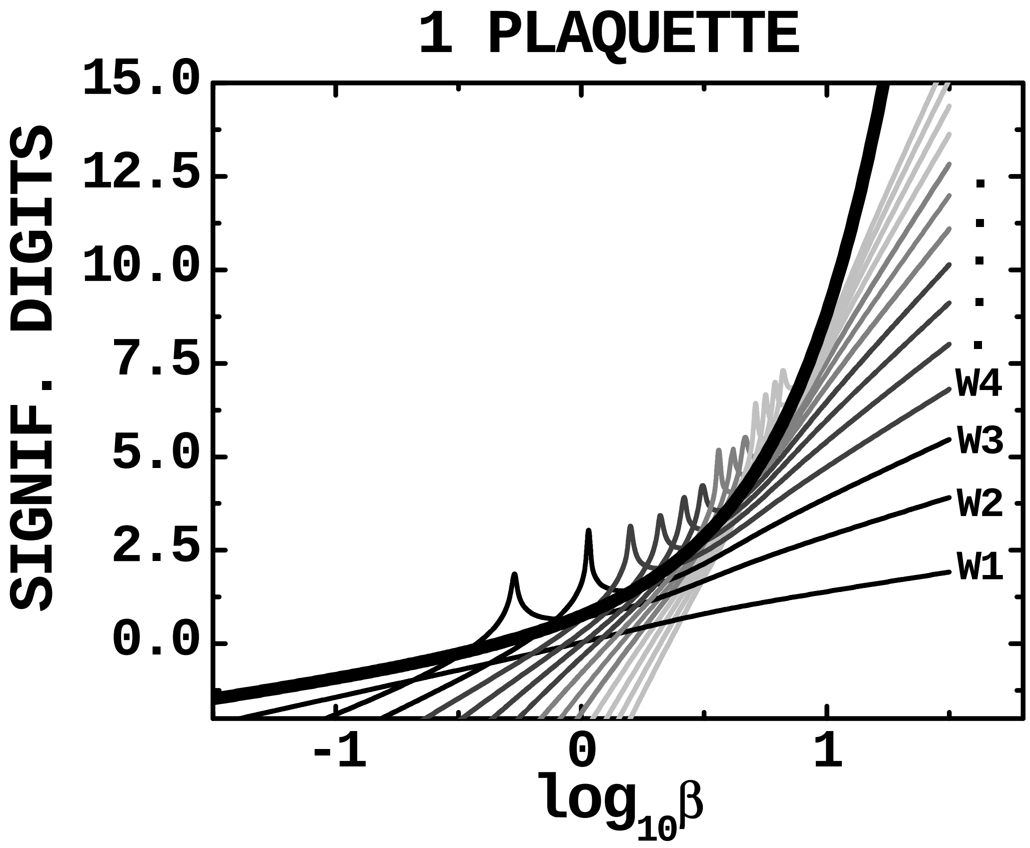


Figure 4: Number of correct significant digits as a function of β at successive orders of the regular perturbative series for $Z(\beta)$. As the order increases from 1 to 15, the curves ($W1$, $W2, \dots$) get lighter. The thick solid line is $\log_{10}(\beta^{-1}e^{-2\beta}/Z)$ ("instanton effect")

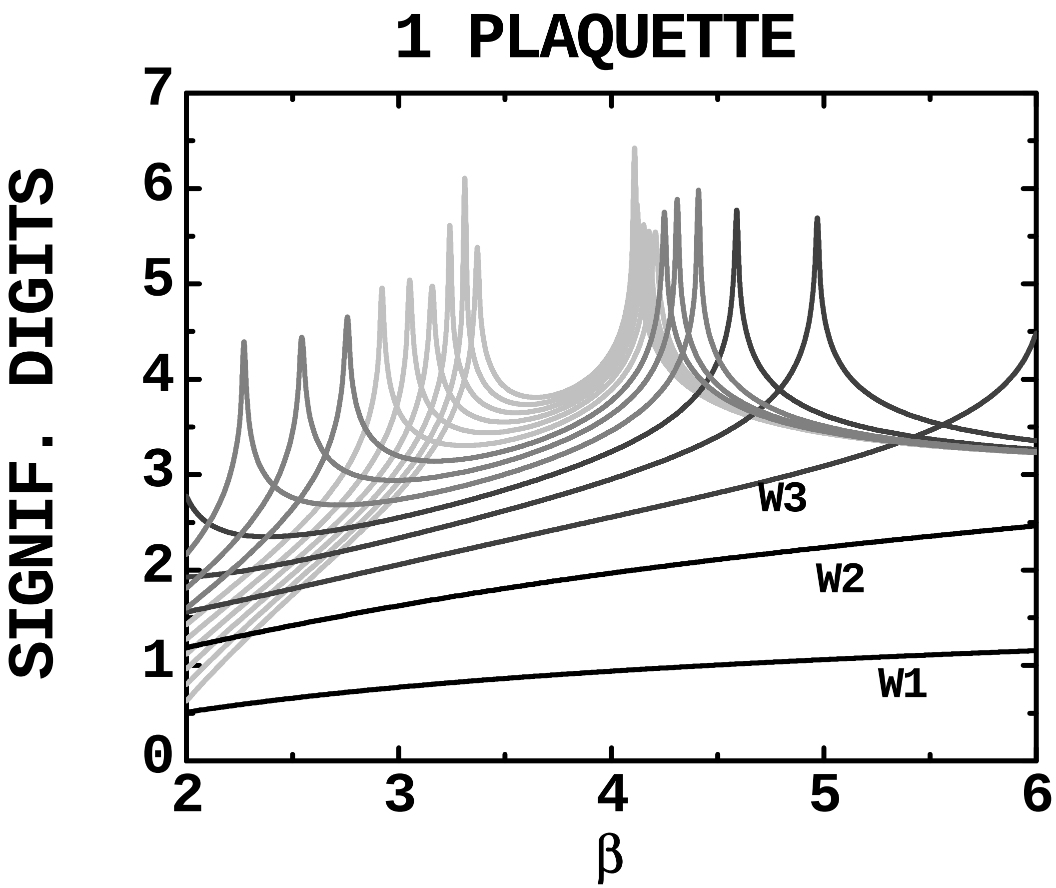


Figure 5: Number of correct significant digits as a function of β for a fixed cut $t_{max} = 8$. As the order increases from 1 to 15 ($W1,\ W2,\ldots$), the curves become lighter.

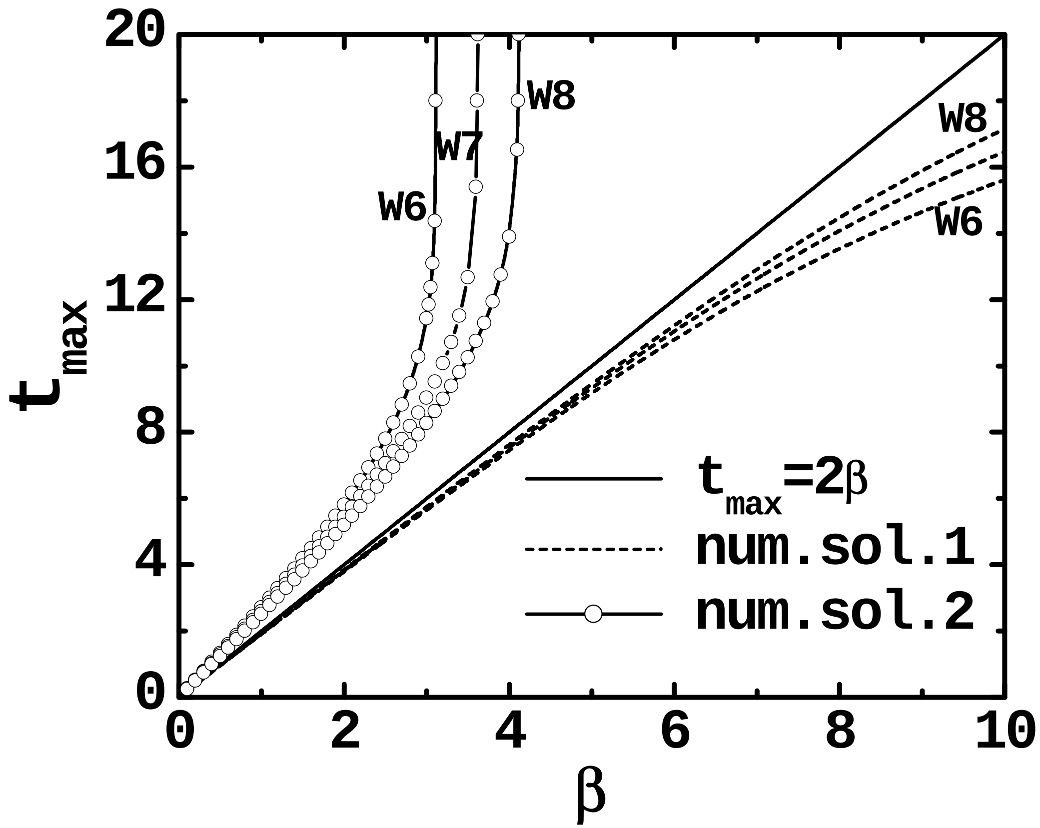


Figure 6: Location of the exact matching between the series at order 6, 7 and 8 and $Z(\beta)$ in the β-t_{max} plane. The dashed lines represent the solution within the radius of convergence and the empty circles the other solution.

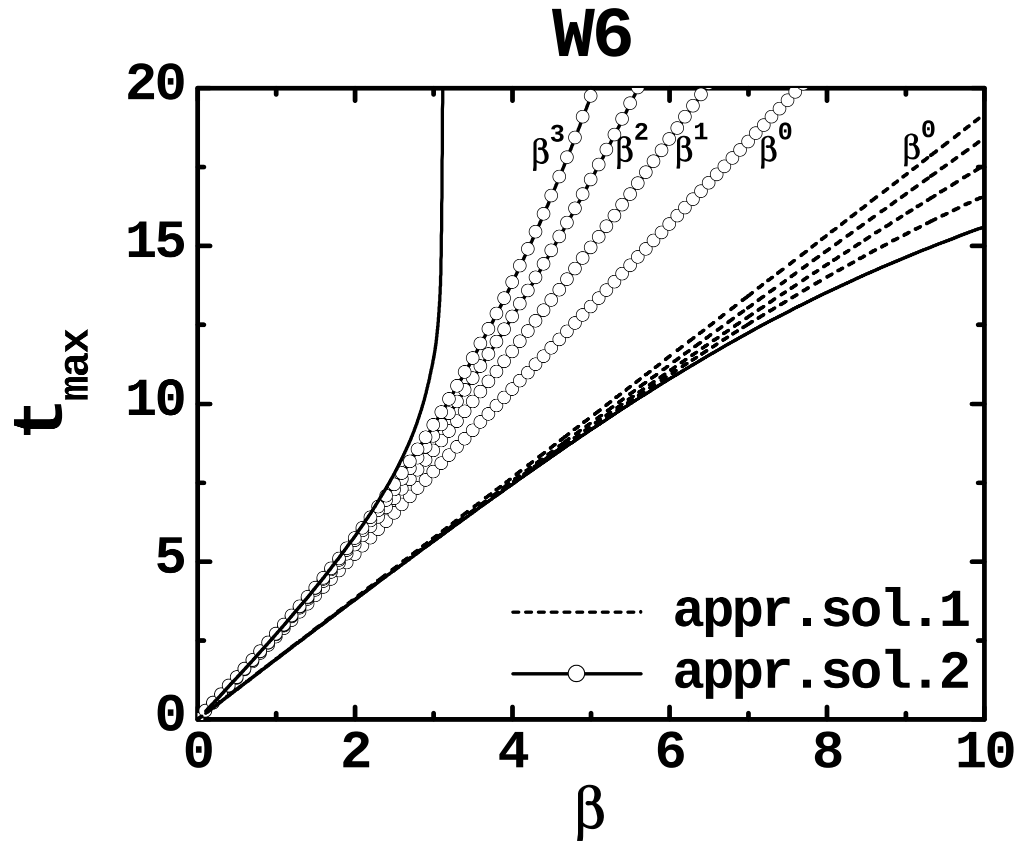


Figure 7: Approximate locations in the (β, t_{max}) plane of the matching between the order 6 weak coupling expansion and $Z(\beta)$. The two solid lines are the two numerical solutions at that order (as in Fig. 6). The dash line (empty circles) represent the first (second) approximate solutions at order $0, \ldots, 4$ in β.

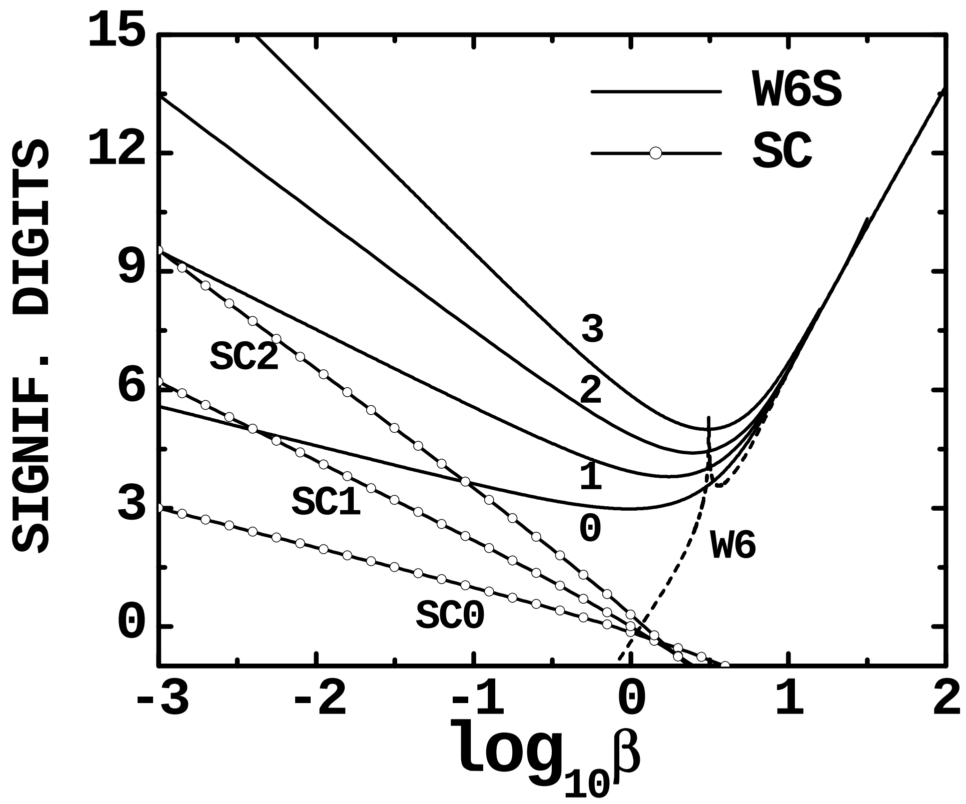


Figure 8: Significant digits obtained from the weak series truncated at order 6 using the first solution for t_{max}/β at order 0 to 3 compared to the weak coupling expansion at order 6 (dotted line W6) and the strong coupling expansion at order 0 to 2 (empty circles SC)

The anharmonic oscillator (L. Li and YM hep-th/0503047)

$$H = \frac{p^2}{2} + V(x) \ ,$$

with

$$V(x) = \begin{cases} \frac{1}{2}\omega^2 x^2 + \lambda x^4 & \text{if} \quad |x| < x_{\max} \\ \infty & \text{if} \quad |x| \geq x_{\max} \end{cases}$$

$$E_0(x_{max}) = \omega \sum_{k=0}^{\infty} E_0^{(k)}(x_{max})(\lambda/\omega^3)^k \ ,$$

$$R_k(x_{max}) \equiv E_0^{(k)}(x_{max})/E_0^{(k)}(\infty) \ ,$$

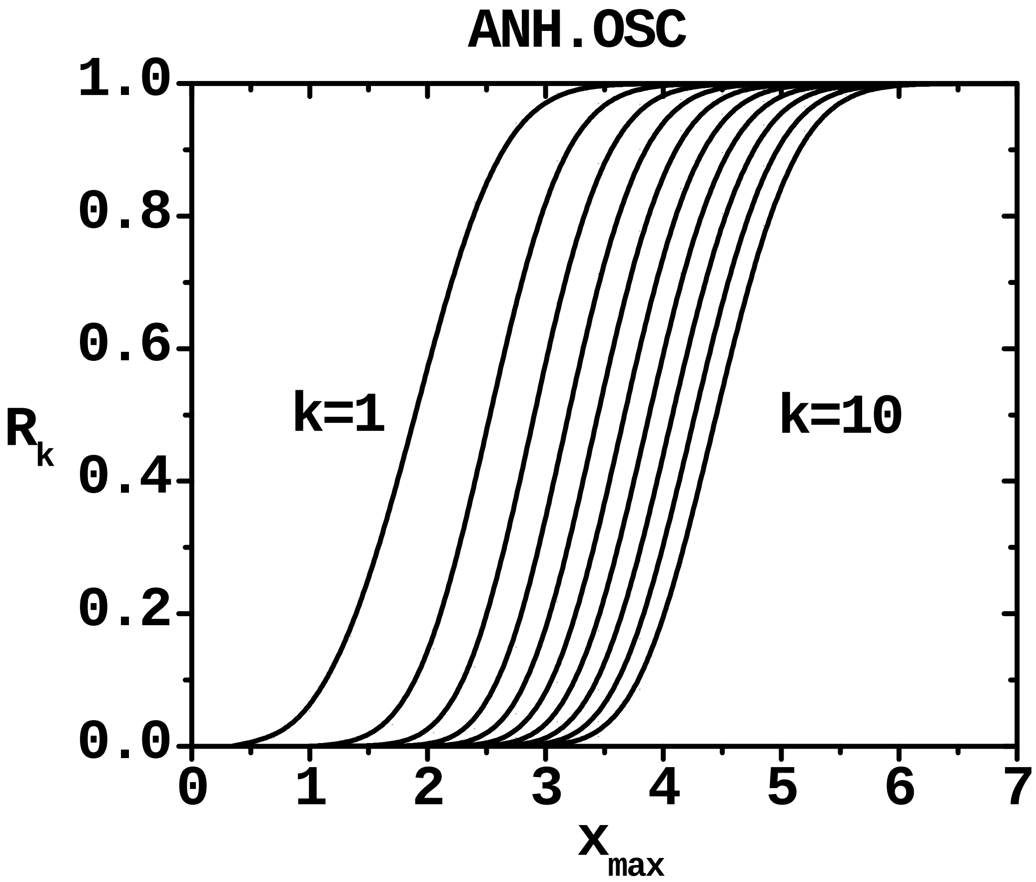


Figure 9: $R_k(x_{\max}) = E_0^{(k)}(x_{max})/E_0^{(k)}(\infty)$ for k going from 1 to 10.

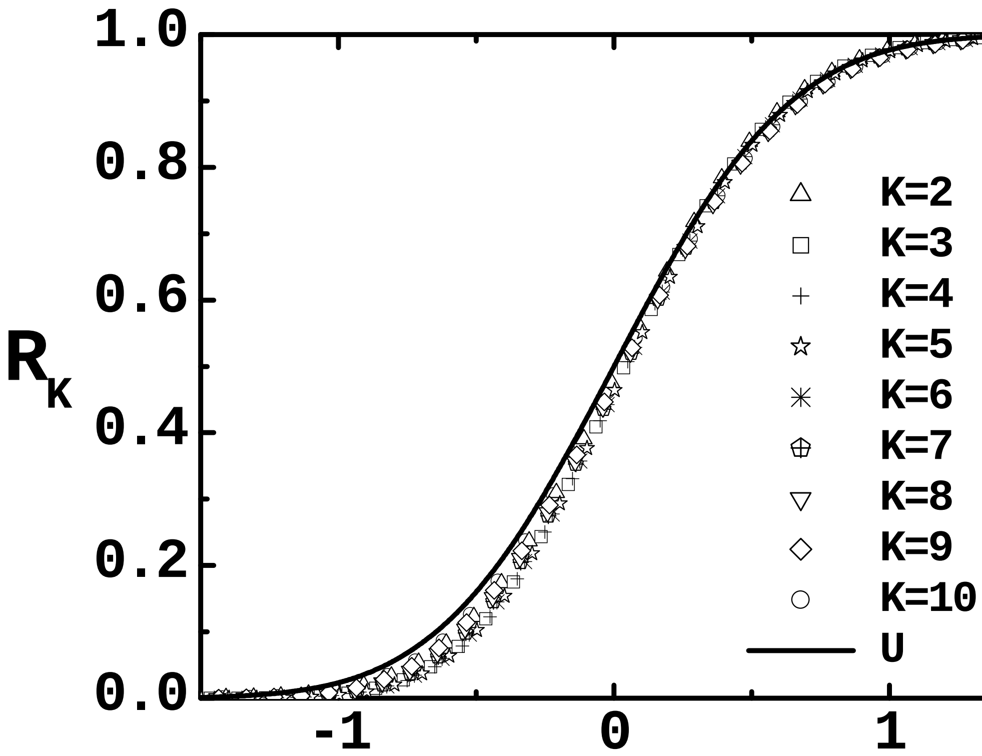


Figure 10: $R_k(x_+x_0(k))$ for $k = 2, \ldots 10$ and the universal function $U(x)$

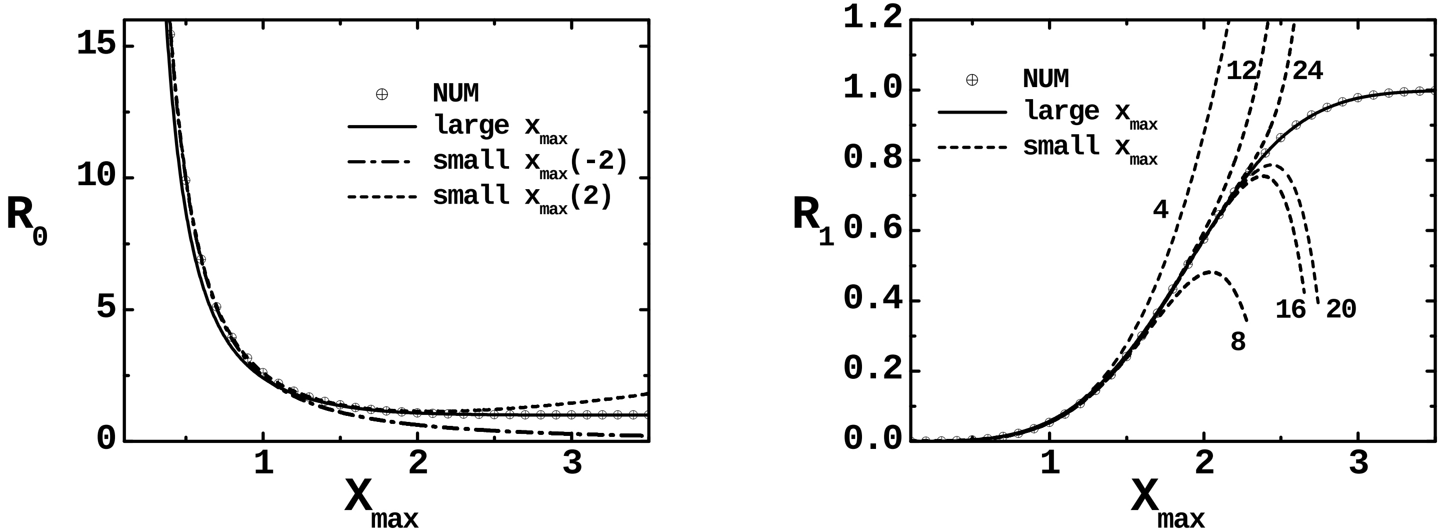


Figure 11: Numerical values of $R_0(x_{max})$ and $R_1(x_{max})$. The solid lines represent the large x_{max} expressions. The broken lines represent lowest orders in the small x_{max} approximation.

III. 4D Gluodynamics

1. Gauge field cuts (gauge invariant or Landau gauge?)

2. Effects of a field cut (small except in the crossover region)

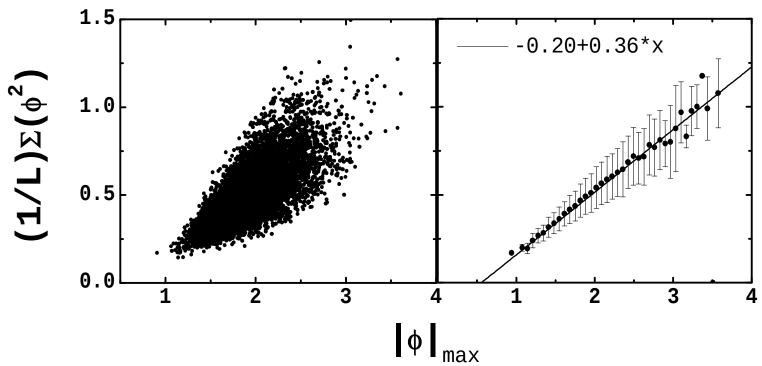
3. Large Field dynamics ($\beta < 0$; first order PT at $\beta \simeq -22$ for $SU(3)$)

4. Lattice Perturbation Theory (3rd order phase transition?)

5. What is the correct theory?

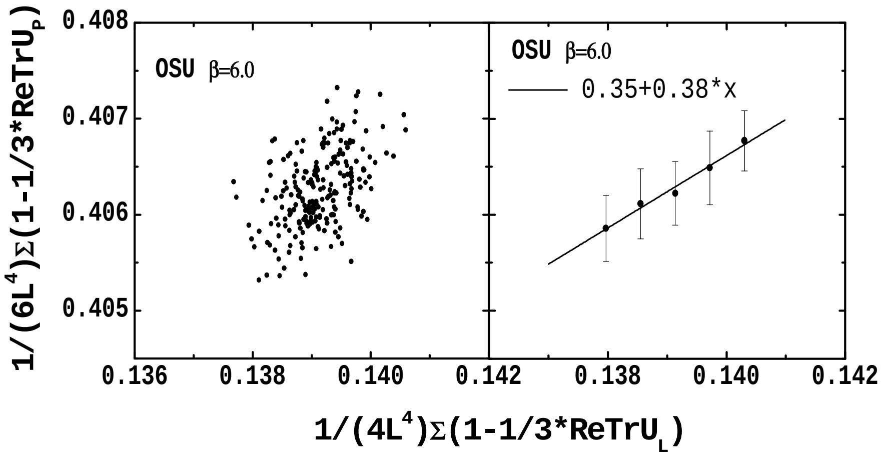
1. Gauge Field Cuts

Spheres versus Cubes (scalar case): for a scalar field configuration $\{\phi_x\}$ seen as L^D-dimensional vector, there are two obvious norms: $(\sum \phi_x^2)^{1/2}$ (defines spheres) and $Max_x|\phi_x|$ (defines cubes). There are strong correlations between these two quantities (shown for 10,000 configs. for $D = 1$ below)



Gauge invariant versus Landau gauge cuts

The volume average of $1 - (1/N)ReTrU_L$ in the Landau gauge is well correlated (the sample correlation is 0.46) with the gauge invariant P.



The maximum value of $1 - (1/N) ReTrU_{link}$ is well correlated with its average (when the Landau gauge algorithm is used long enough!)

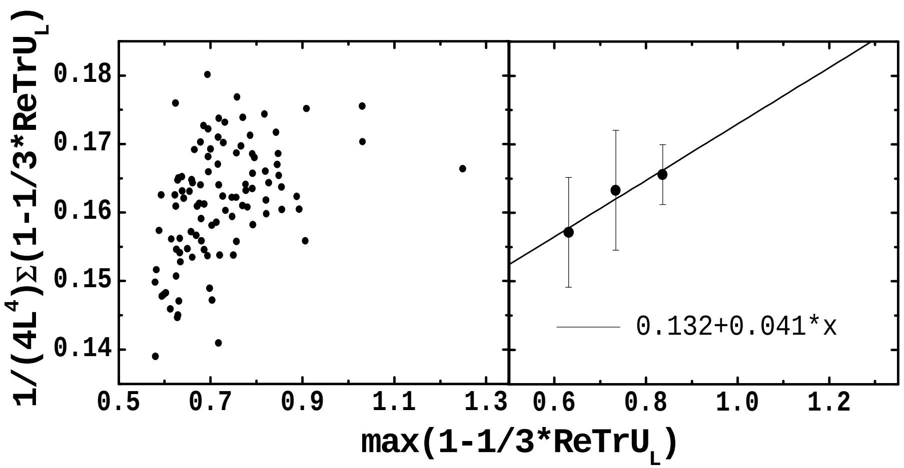
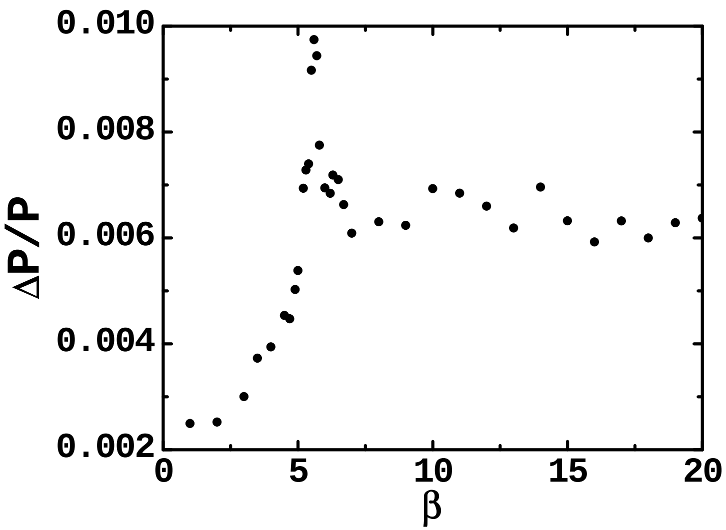


Figure 12: Largest absolute value of $1-(1/N)ReTrU_{link}$ versus its average; sample correlation: 0.426.

2. Effect of a gauge invariant cut on P

The effect of the cut is very small but of a different size below, near or above $\beta = 5.6$. The relative change of the configuration average of P when 80 percent of the large field configurations are discarded, for various values of β in a pure $SU(3)$ LGT on a 8^4 lattice is shown below



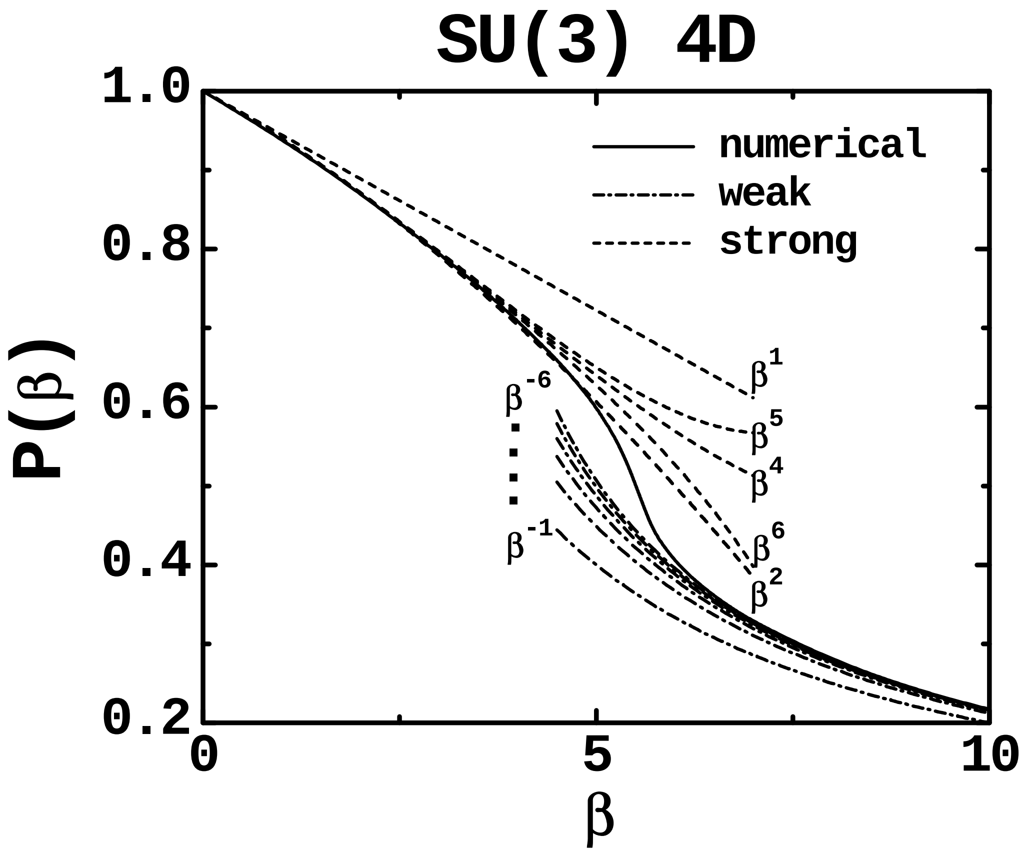


Figure 13: P versus β for $SU(3)$ in 4 dimensions. The solid line represents the numerical values; the dashed lines on the left, successive orders in the strong coupling expansion; the dot-dash lines on the right, successive orders in the weak coupling expansion.

This is in agreement with the idea that modifying the weight of the large field configurations affects the crossover behavior (where we usually get some kind of continuum scaling)

Methods to change the weight of the large field configurations:

- adding adjoint action (Bhanot, Creutz, Hasenbusch, Necco)
- adding a monopole chemical potential (Mack et al. , Brower et al.)
- removing center vortices (de Forcrand and D'Elia)

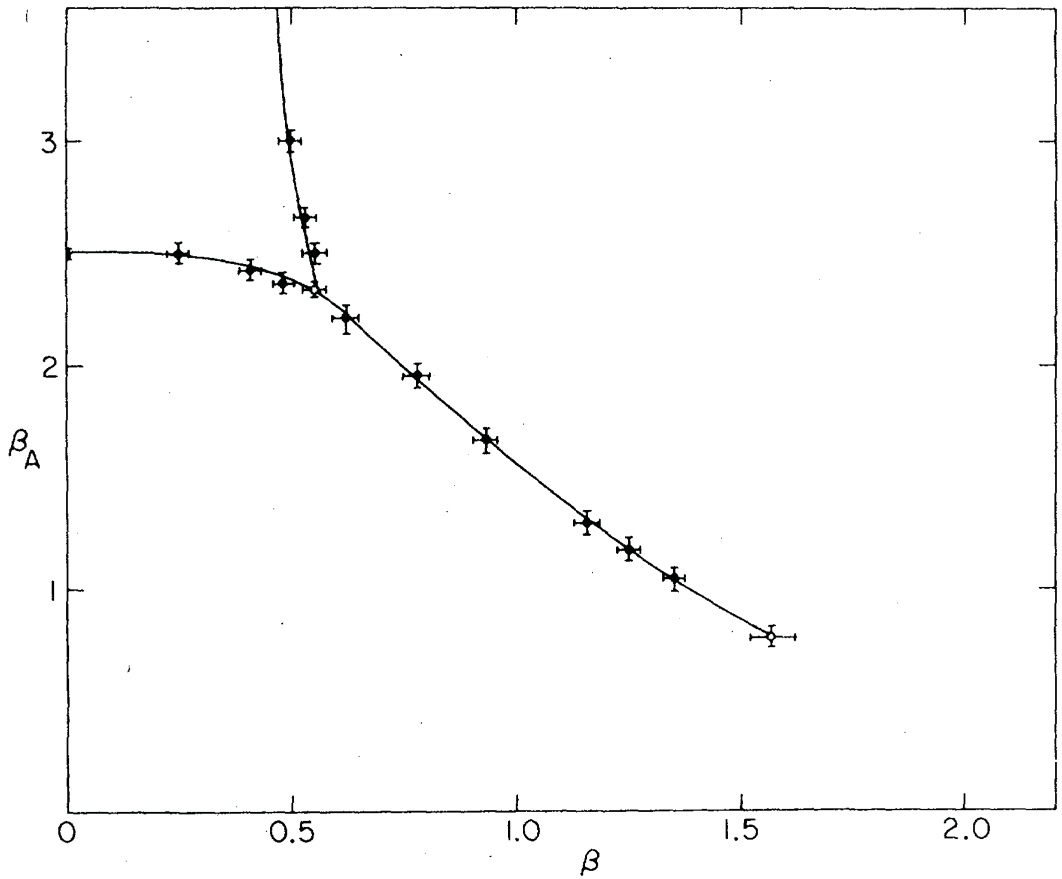


FIG. 3. The full phase diagram. The open circles represent the location of the triple point and the critical point. The solid circles trace out the first-order transition lines. The solid curves are drawn to guide the eye.

Figure 14: From Bhanot and Creutz PRD 24 3212

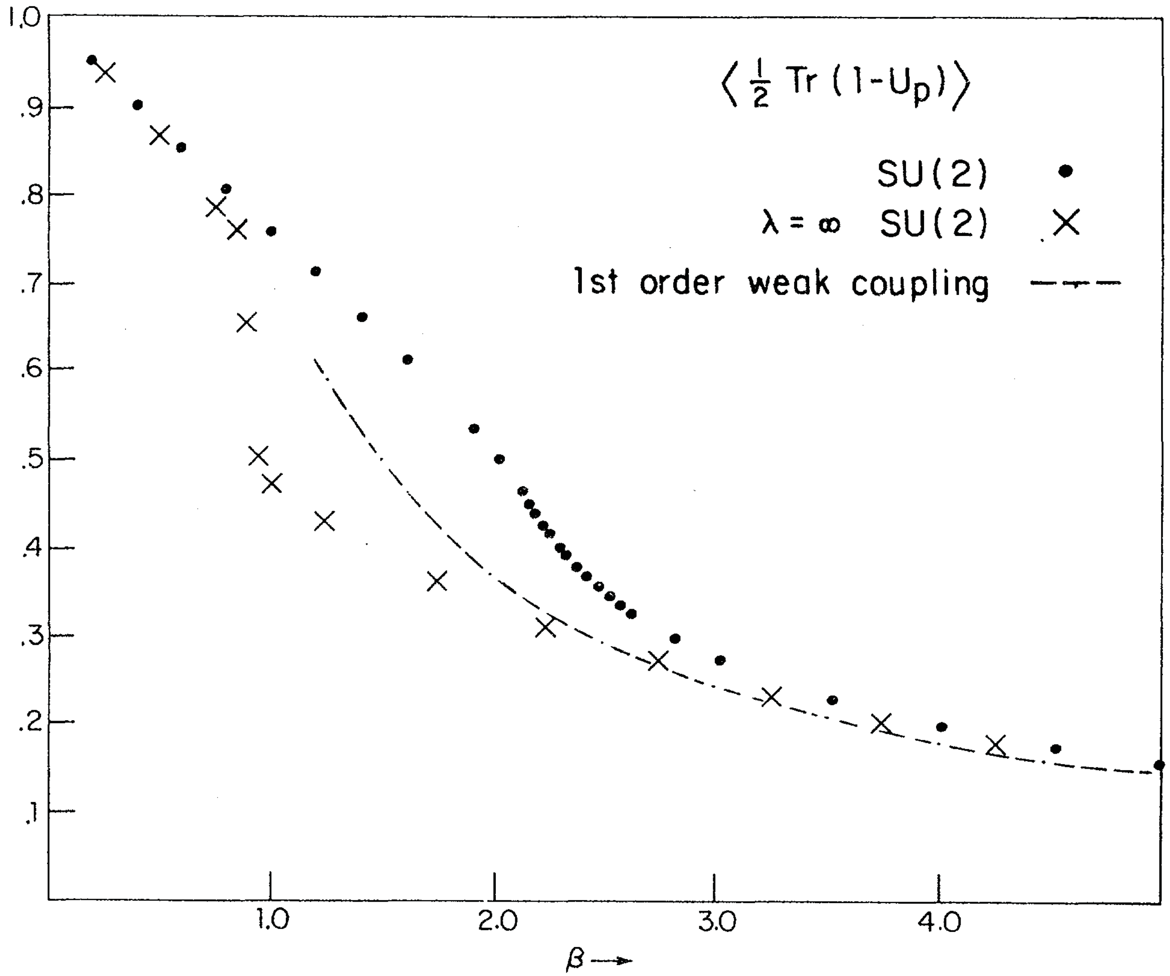


FIG. 2. Average plaquette for MP model (×).

Figure 15: From R. Brower et al. PRL 47 621

3. Gluodynamics at negative β

Continuum expectations: when $g \to ig$, $g^2 \to -g^2$

- The terms $ig\partial AAA$ become non-hermitian
- The terms g^2AAAA become unbounded from below

However, it is possible to obtain a positive spectrum for potentials such as ix^3 or $-x^4$ by changing the boundary conditions (Bender and Boettcher, PRL 80, 5243; see also Bender, Brody and Jones hep-th/0402011 for field theory)

In LGT, the large field problem occurs when we "decompactify".

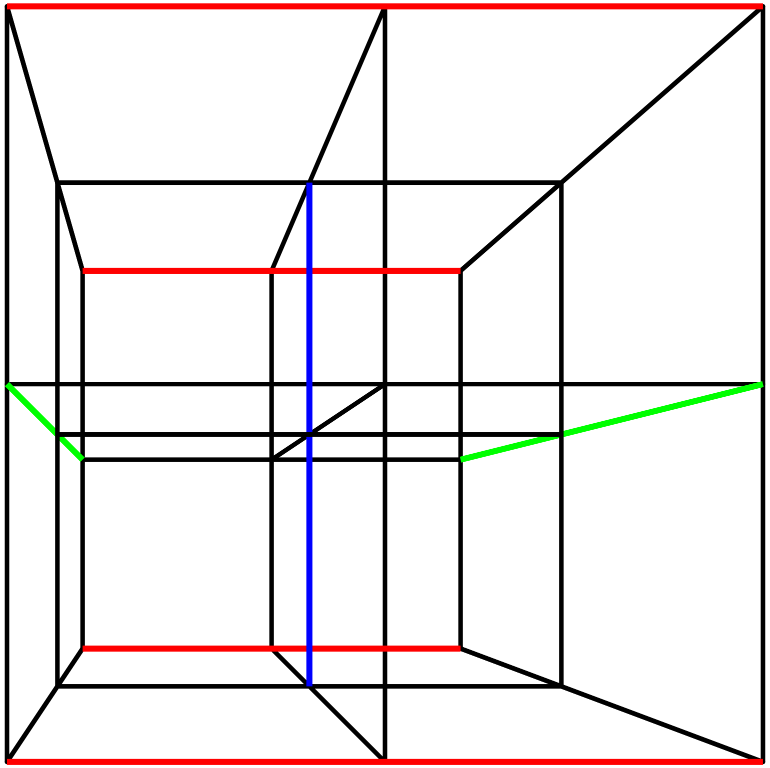
Minimum action configurations at negative β

for detail see L. Li and YM, hep-lat/0410929, PRD 71

For $SU(2)$ and $SU(3)$ with $\beta < 0$, an absolute minimum of $\beta \sum_p (1 - (1/N) ReTr(U_p))$ can be obtained if U_p is a non trivial element of the center for each plaquette.

We can construct a set of lines $\mathcal{L}$ such that every plaquette shares one and only one link with this set. These lines cannot intersect. These lines cannot be obtained from each other by exactly one translation of one lattice spacing in one single direction. For $D = 2$, parallel lines separated by two lattice spacing do the job. For $D > 2$, we require that any plane contains this type of solution (we thus reduce the problem to a 2^D lattice with periodic b.c.)

For $D = 3$, an example of $\mathcal{L}$ is $\{(A,0,0),(0,A,1),(1,1,A)\}$ with A arbitrary. It is not difficult to show that there are 8 distinct $\mathcal{L}$.



For $D = 4$, $\mathcal{L}$ contains 8 lines. An example of solution is

$$\begin{aligned} \{ \quad & (A,0,0,0),(0,A,0,1),(0,1,A,0),(0,0,1,A), \\ & (1,1,0,A),(1,0,A,1),(1,A,1,0),(A,1,1,1) \ \} \ . \end{aligned}$$

For $D = 2$ ($D = 3$), the 4 (8) $SU(2)$ configurations with $-\mathbb{1}$ on $\mathcal{L}$ and the identity on the other links, are gauge equivalent.

For $SU(2)$, there is only one set of U_p for a configuration of minimal action (all $-\mathbb{1}$). For $SU(3)$, we have two choices:$U_p = \mathrm{e}^{\pm i2\pi/3}\mathbb{1}$ (Ising like).

$SU(2)$

Using a *change of variables*, $U_l \to -U_l$ for $l \in \mathcal{L}$ and the invariance of the Haar measure ($-\mathbb{1} \in SU(2)$), we find

$$Z(-\beta) = \mathrm{e}^{2\beta\mathcal{N}_p} Z(\beta)$$

Taking the logarithmic derivative, we obtain

$$P(\beta) + P(-\beta) = 2 \; .$$

This identity can be seen in the symmetry of the curve $P(\beta)$

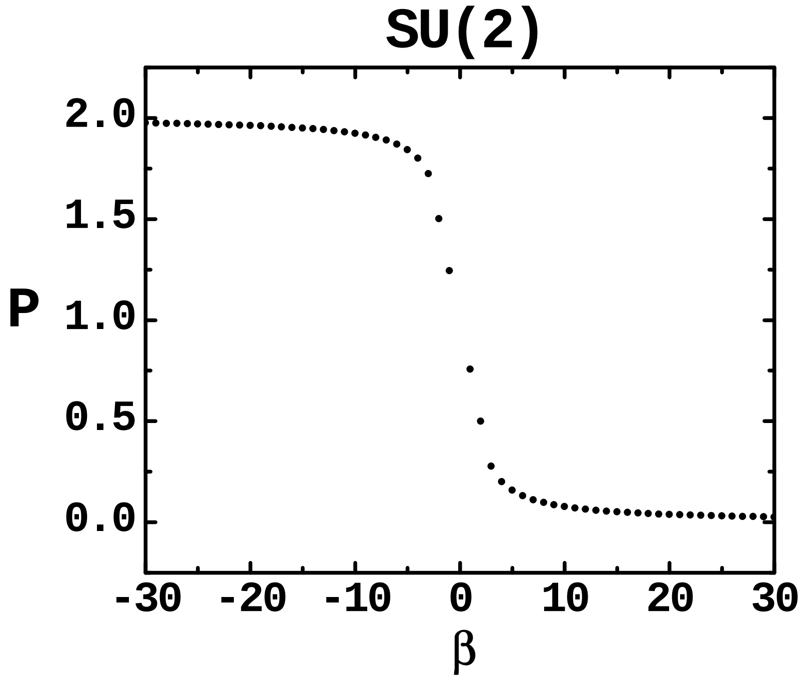


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Figure 16: The average action density $P(\beta)$ for $SU(2)$.

$P(+\infty) = 0$, implies that P seen as a function of $g^2 = 2N/\beta$, jumps discontinuously by 2 as g^2 becomes negative. This invalidates the idea that P could have a regular expansion about $g^2 = 0$ with a non-zero radius of convergence (essential singularity).

This relation can also be used in the opposite limit and expanded about $\beta = 0$. The odd terms cancel automatically. The even terms of order 2 and higher add and cannot cancel. Consequently, the even coefficients of the strong coupling expansion of $P(\beta)$ and the odd coefficients of the free energy should vanish, in agreement with explicit calculations (Balian, Drouffe and Itzykson).

$$\langle W(C)\rangle_{-\beta} = (-1)^{|A|}\,\langle W(C)\rangle_{\beta}$$

We can now try to interpret the change of the Wilson loop with the area in a term of a potential. We consider a rectangular $R \times T$ contour C and write

$$W(R,T,\beta) \equiv \langle W(C)\rangle_{\beta} \propto \mathrm{e}^{-E(R,\beta)T} \ .$$

From Eq. (1) this implies

$$E(R,-|\beta|) = E(R,|\beta|) + i\pi R \ .$$

This property can be related to the fact that the configurations of minimum action are invariant under translations by two lattice spacings but not under translations by one lattice spacing.

$SU(3)$

For $N = 3$, $-\mathbb{1}$ is *not* a group element and the closest thing to the change of variables used for $N = 2$ that we can invent is a multiplication by a nontrivial element of the center $\zeta\mathbb{1}$ for the links of a particular set $\mathcal{L}$. We then obtain

$$
\begin{aligned}
Z(\zeta\beta) &= \mathrm{e}^{(1-\zeta)\beta\mathcal{N}_p} Z(\beta) \\
&\times \left\langle \mathrm{e}^{(\beta/3)\quad {}_p(\zeta Re\zeta^{\star} TrU_p - ReTrU_p)} \right\rangle_{\beta} \quad . \qquad (1)
\end{aligned}
$$

In the case $N = 2$, ζ is replaced by -1, $\langle ... \rangle_{\beta}$ becomes 1

First Order PT at $\beta \simeq -22.$

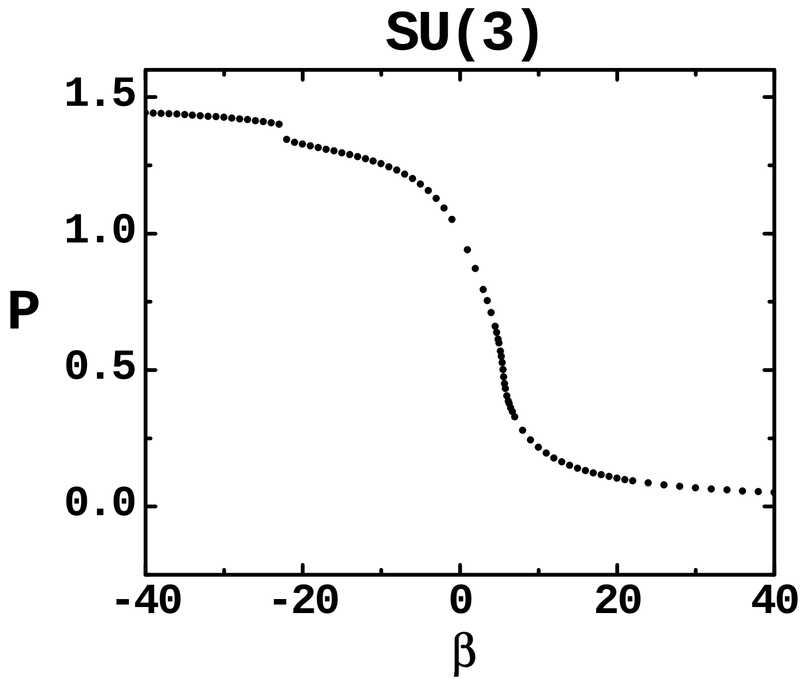


Figure 17: MC calculation of the average action density $P(\beta)$ for $SU(3)$

4. Lattice Perturbation Theory

Three steps (Heller and Karsch, NPB 251 254)

1. $\beta = 2N/g^2$; $U = e^{igA}$ with $A = A^a T^a$

2. Extend the range of integration of the A^a from $-\infty$ to $+\infty$

3. Expand in g

We used the series of Di Renzo et al. JHEP 10 038 hep-lat/0011067.

$$P(1/\beta) = \sum_{m=0}^{10} b_m \beta^{-m} + \ldots .$$

$r_m = b_m/b_{m-1}$, the ratio of two successive coefficients extrapolates near 6 when $m \to \infty$. On the other hand, we expect a linear growth for an asymptotic series.

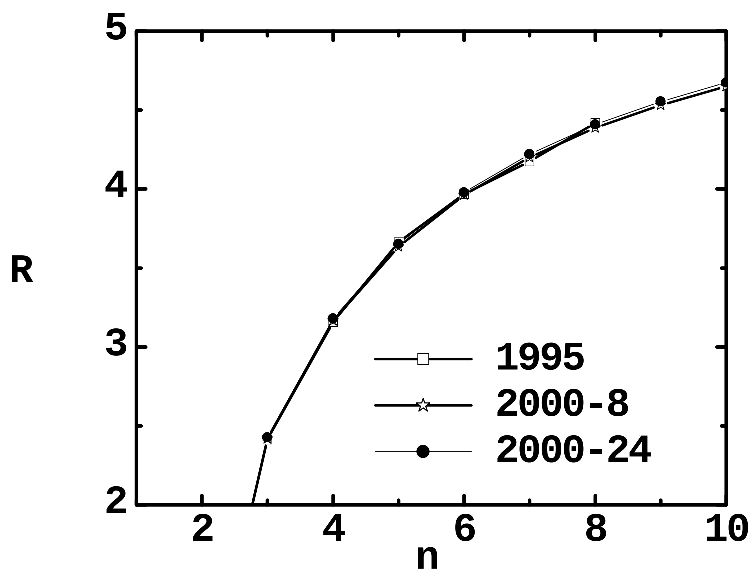


Figure 18: Ratios for the 95 and 2000 data (on 8^4 and 24^4 lattices).

$$P = (1/\beta_c - 1/\beta)^{-\gamma}(A_0 + A_1(\beta_c - \beta)^{\Delta} + ...) \ ,$$

We introduce quantities (Nickel) called the extrapolated ratio ($\widehat{R}_m$) and the extrapolated slope ($\widehat{S}_m$) in order to estimate β_c and γ. These quantities are defined as

$$\widehat{R}_m = m r_m - (m-1) r_{m-1} \ ,$$

and

$$\widehat{S}_m = m S_m - (m-1) S_{m-1} \ ,$$

where

$$S_m = -m(m-1)(r_m - r_{m-1})/(m r_m - (m-1) r_{m-1})$$

is called the normalized slope. When A_0 and A_1 are constant, one finds that the $1/m$ corrections disappear:

$$\widehat{S}_m = \gamma - 1 - B m^{-\Delta} + O(m^{-2}).$$

Series Analysis

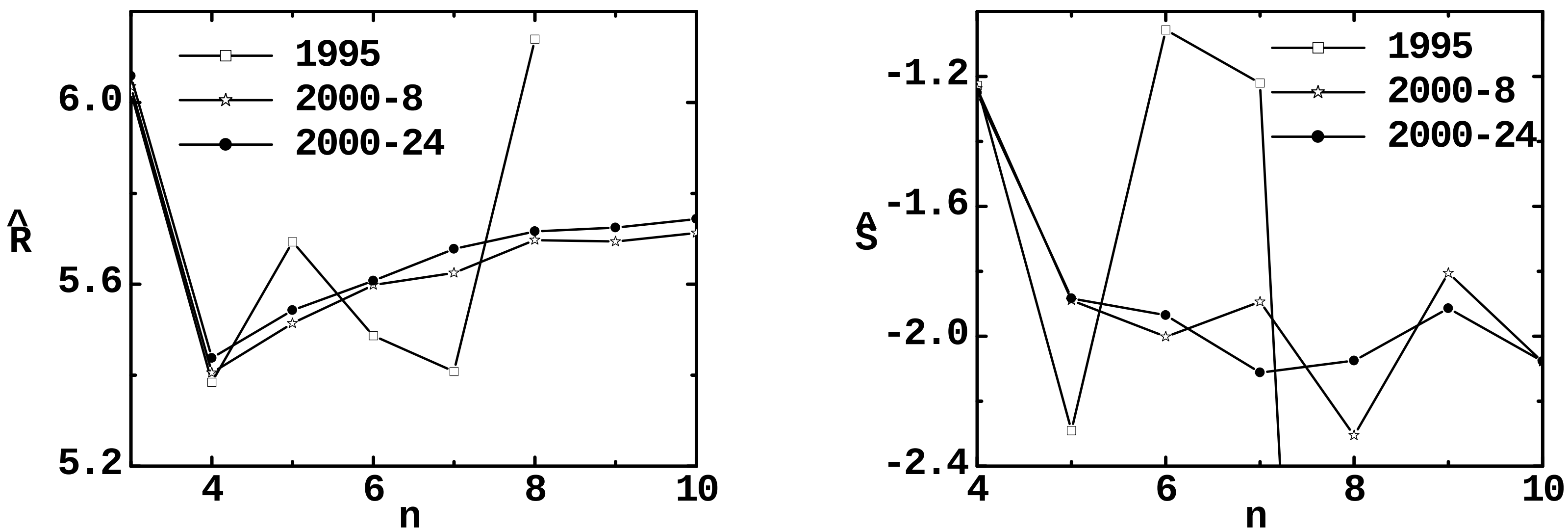


Figure 19: The extrapolated ratios (left) suggests $\beta_c \simeq 5.72$. The extrapolated slope (right)suggests $\gamma = -1.07$.

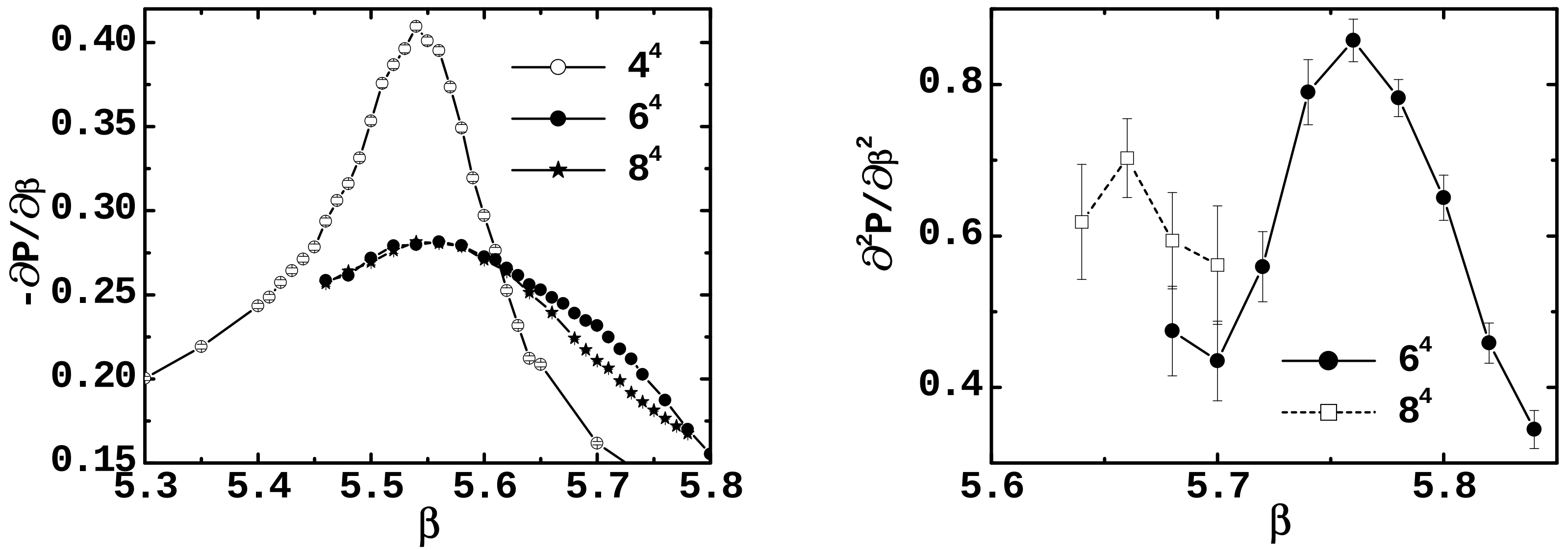
Conclusions of the series analysis

- $P \propto (1/5.7 - 1/\beta)^{1.07}$
  - not expected
  - could be visible in 2d derivative of P (statistical errors permitting)

- Not incompatible with asymptotic series (P could be a superposition of a function that has an asymptotic series and one that has a finite radius of convergence) provided that the factorial behavior shows up at a large enough order.

see also Horsley et al. Nucl.Phys.Proc.Suppl.106:870-872,2002 Also in *Berlin 2001, Lattice field theory* 870-872 e-Print Archive: hep-lat/0110210

Direct Search for Singularities in P' and P''



$\partial^2 P/\partial\beta^2$ is "twice subtracted" and the errors increase rapidly with the volume.

What is the right theory?

- Continuum behavior is sensitive to field cuts
- Does the new 1st order PT line terminate at a critical point?
- Is the 3rd order PT suggested by perturbation theory a feature relevant for the continuum?
- What kind of effective theory do we want? Massive glueball with M/Λ not so small?
- How do we decide about the non-universal features?

IV. Projects

- Effective theory for $ReTrU_p$
- Stochastic Perturbation Theory with boundaries
- Inverse and determinant of matrices in cut perturbation theory