't Hooft-Polyakov Monopoles on the Lattice

Arttu Rajantie

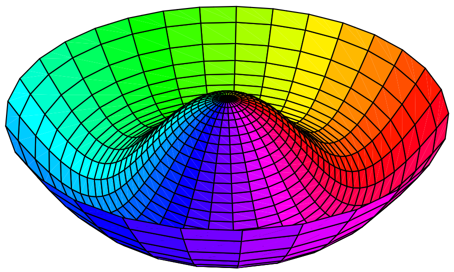
DAMTP and Churchill College
University of Cambridge

Davis,Kibble,Rajantie&Shanahan, JHEP11(2000)
Davis,Hart,Kibble&Rajantie, PRD65(2002)
Rajantie, in progress

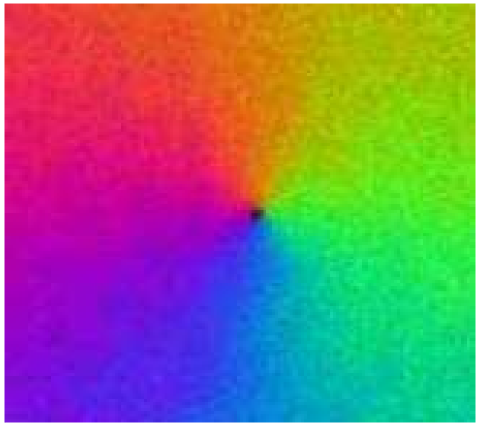
Introduction

- ’t Hooft-Polyakov monopoles
  - Pointlike magnetic charges
  - Georgi-Glashow model: SU(2)+adjoint Higgs
- Confinement in QCD and Yang-Mills
  - Monopole condensation?
  - Abelian projection?
- Predicted by all GUTs
  - Produced in the early universe
  - Greatly diluted by inflation
  - Constantly searched, none found yet
    (or possibly one on Valentine’s Day 1982 (Cabrera 1982))
- Theoretical interest
  - SUSY models
  - Dualities

Topological Solitons



- Localized, topologically stable field configurations
- Order parameter ϕ at spatial infinity $|\vec{r}| \to \infty$:
  - Finite energy $\Rightarrow$ Must approach vacuum
  - Possibly different vacuum in different directions
  - Defines a map from S^{d-1} to the vacuum manifold $\mathcal{M} \cong G/H$
- Solitons exist if $\pi_n(G/H) \neq 0$ for $n < d$
  - $n = 0$: Domain walls (kinks)
  - $n = 1$: Vortices (strings)
  - $n = 2$: Monopoles
  - Winding number $N_W \in \pi_n(G/H)$



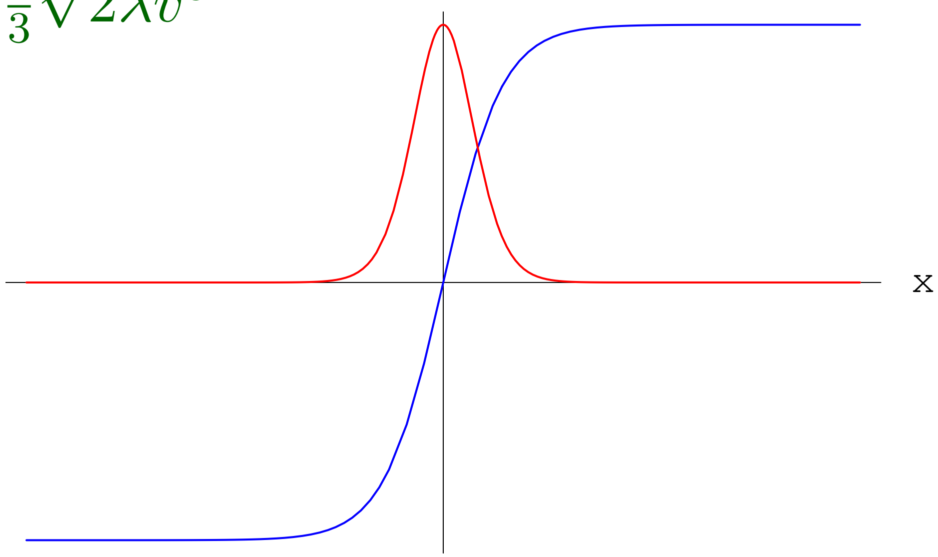
- Convenient theoretical probes of phase properties
  - Dualities
  - Confinement $\leftrightarrow$ Monopole condensation? ('t Hooft, Mandelstam)

Classical Kink

- 1+1D real scalar field

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi'^2 - \frac{\lambda}{4}(\phi^2 - v^2)^2$$

- Two vacua $\phi = \pm v \Rightarrow \pi_0 = \mathbb{Z}_2$, winding number 0 or 1
- Kink: Choose $\phi(\pm\infty) = \pm v$
- Exact stationary solution: $\phi(x) = v \tanh(\lambda v^2/2)^{1/2} x$
  Energy $M_{\mathrm{kink}} = \frac{2}{3}\sqrt{2\lambda} v^3$



Georgi-Glashow model

- Continuum:

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + \mathrm{Tr}\,[D_\mu,\Phi][D^\nu,\Phi] - m^2\mathrm{Tr}\,\Phi^2 - \lambda(\mathrm{Tr}\,\Phi^2)^2$$

  - SU(2) gauge field $A_\mu = A_\mu^a\sigma^a/2$, where $a \in \{1,2,3\}$
  - Adjoint Higgs field $\Phi = \Phi^a\sigma^a/2$

- Euclidean lattice action (lattice spacing$=1$)

$$\begin{aligned}\mathcal{L}_E &= 2\sum_\mu\left[\mathrm{Tr}\Phi(\vec{x})^2 - \mathrm{Tr}\Phi(\vec{x})U_\mu(\vec{x})\Phi(\vec{x}+\hat{\mu})U_\mu^\dagger(\vec{x})\right] \\ &\quad + \frac{2}{g^2}\sum_{\mu<\nu}[2 - \mathrm{Tr}U_{\mu\nu}(\vec{x})] + m^2\mathrm{Tr}\,\Phi^2 + \lambda(\mathrm{Tr}\,\Phi^2)^2\end{aligned}$$

  - Link variables $U_\mu \in \mathrm{SU}(2)$, $U_\mu \sim \exp(igA_\mu)$
  - Plaquette $U_{\mu\nu} = U_\mu(x)U_\nu(x+\hat{\mu})U_\mu^\dagger(x+\hat{\nu})U_\nu^\dagger(x)$

’t Hooft-Polyakov Monopole

- $m^2 < 0$: Symmetry breaking SU(2)⟶U(1)
  - Vacuum manifold $\{\mathrm{Tr}\Phi^2 = v^2 = |m^2|/\lambda\} \cong S^2$
  - $\pi_2(S^2) = \mathbb{Z} \Rightarrow$ Monopoles ('t Hooft, Polyakov)

$$
\begin{aligned}
\Phi^a(\vec{r}) &= \frac{r_a}{gr^2} H(gvr) \\
A_i^a(\vec{r}) &= -\epsilon_{aij} \frac{r_j}{gr^2}[1 - K(gvr)]
\end{aligned}
$$

- Broken phase: U(1) symmetry ⇒ Electrodynamics
  - Field strength $\mathcal{F}_{\mu\nu} = \mathrm{Tr}\hat{\Phi}F_{\mu\nu} + (2ig)^{-1}\mathrm{Tr}\hat{\Phi}[D_\mu, \hat{\Phi}][D_\mu, \hat{\Phi}]$
    – Unitary gauge $\hat{\Phi} = \sigma_3$: Reduces to $\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
  - Magnetic field $\mathcal{B}_i = \frac{1}{2}\epsilon_{ijk}\mathcal{F}_{jk}$:
    – If $\Phi \neq 0$, then $\vec{\nabla} \cdot \vec{\mathcal{B}} = 0$
    – For a smooth configuration $\vec{\nabla} \cdot \vec{\mathcal{B}}(\vec{x}) = (4\pi/g) \sum_i \pm\delta(\vec{x} - \vec{x}_i)$
    ⇒ Magnetic monopoles with charge $\pm 4\pi/g$

Magnetic Field on the Lattice

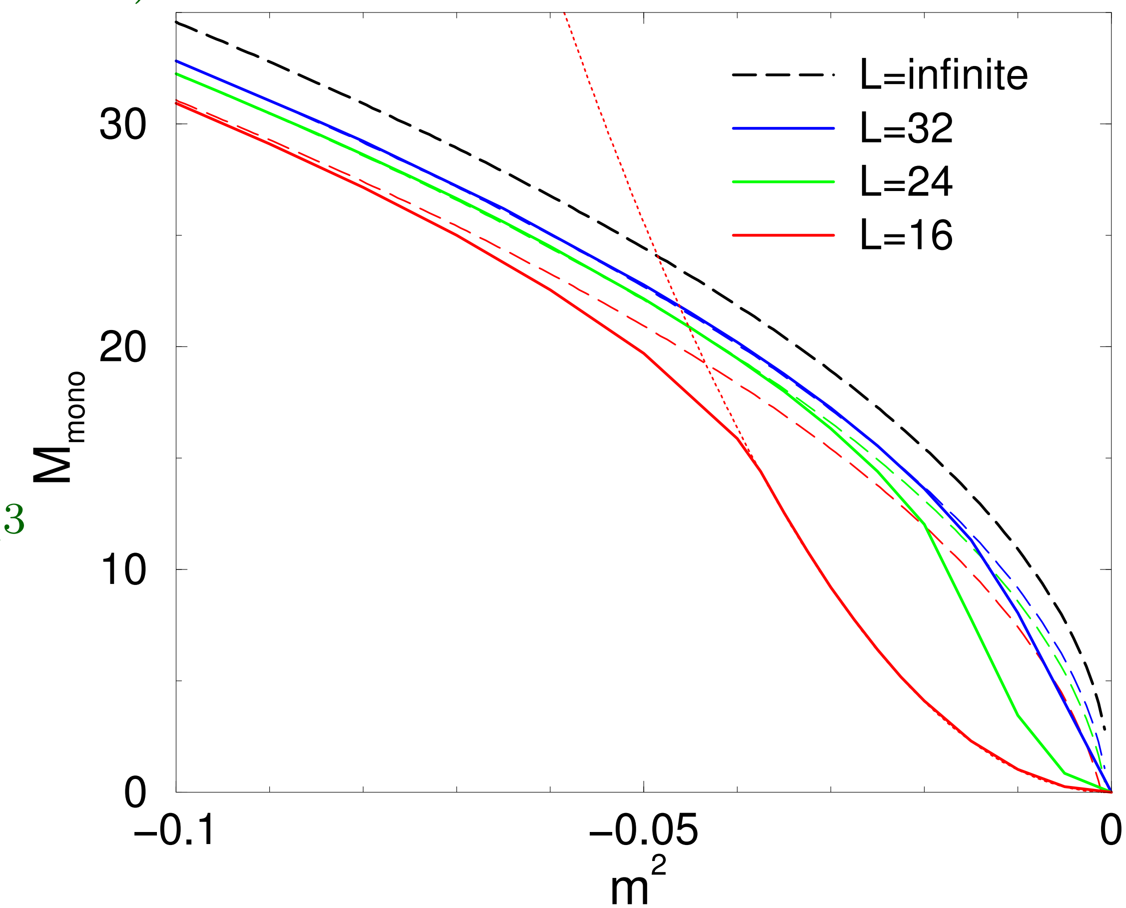
- Discretized version of $\mathcal{F}_{\mu\nu}$:
  - Define projection $\Pi_+ = \frac{1}{2}(1+\hat{\Phi})$ $\left[= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]$
  - Projected link $u_\mu(x) = \Pi_+(x) U_\mu(x) \Pi_+(x+\hat{\mu})$ $\left[\propto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]$
  - U(1) field strength tensor

$$\alpha_{\mu\nu} = (2/g) \arg \mathrm{Tr}\, u_\mu(x) u_\nu(x+\hat{\mu}) u_\mu^\dagger(x+\hat{\nu}) u_\nu^\dagger(x)$$

- Magnetic field $\hat{B}_i = \frac{1}{2}\epsilon_{ijk}\alpha_{jk}$
  - Magnetic charge in a lattice cell
    $\hat{\rho}_M = \sum_i \left[\hat{B}_i(x+\hat{\imath}) - \hat{B}_i(x)\right] \in (4\pi/g)\mathbb{Z}$
    $\Rightarrow$ Stable monopoles

Classical Monopole Mass

- Continuum result
  $M = (4\pi m_W/g^2) f(m_H/m_W)$
  - $f(x) \approx 1 + x/2 + (x^2/2)(\ln x + \sqrt{2})$
    (Kirkman&Zachos 1981)
- Example: $\lambda = 0.1, g = 1/\sqrt{5}$
- Finite size effects
  - Coulomb force $|m^2| \gg 1/L^2$:
    $\Delta E(L) \approx 11.0/g^2 L$
  - Symmetry restoration
    $\Delta E(L) \approx V(0)L^3 = (\lambda v^4/4)L^3$
- Infinite-volume extrapolation:
  $f(x) \approx 1.10$



Perturbative Quantum Corrections

- Find lowest energy eigenvalue $E(N_W)$ with a given winding number N_W
  - Soliton mass $M = E(1) - E(0)$
- Perturbative approach: (Dashen et al. 1974)
  - Loop expansion around classical solution $\varphi_0(x)$
    - Write $\varphi(t,x) = \varphi_0(x) + \delta(t,x)$
    - Quantize $\delta(t,x)$: Field in a x-dependent potential
    - Order δ^2: Harmonic potential $U(\delta) = \frac{1}{2}V''(\varphi_0(x))\delta^2$
    - Diagonalize:
      $$\left[-\vec{\nabla}^2 + V''(\varphi_0(x))\right]\delta_k(x) = \omega_k^2\delta_k(x)$$
      $\Rightarrow$ Frequencies ω_k
  - One-loop level: $\Delta E = \sum_k(\omega_k^1 - \omega_k^0)/2$
  - Higher-order corrections: Difficult

One-loop Kink Mass

- Equation for ω_k:

$$\left[-\frac{\partial^2}{\partial x^2} + \lambda v^2 \left(3 \tanh^2 \sqrt{\lambda v^2/2} x - 1\right)\right] \delta_k(x) = \omega_k^2 \delta_k(x)$$

  - Can be solved exactly:
    $\omega_0^2 = 0$, $\omega_1^2 = 3\lambda v^2/2$ and a continuum $\omega_q^2 = (q^2/2 + 2)\lambda v^2$
  - Caveats: Zero mode, measure for q, UV regularisation

- Result: (Dashen et al. 1974)

$$M_{\rm kink} \approx \frac{2}{3}\sqrt{2\lambda} v^3 + \left(\frac{1}{2\sqrt{6}} - \frac{3}{\sqrt{2}\pi}\right)\sqrt{\lambda} v$$

Leading-log Monopole Mass

- Same principles, many extra complications
  - Gauge fixing
  - Two coupled fields
  - Higher dimensionality
  - Renormalisation issues
- Only leading log in the $m_H/m_W \to 0$ limit has been calculated (Kiselev&Selivanov 1988)

$$M = \frac{4\pi m_W}{g^2}\left(1 + \frac{g^2}{8\pi^2}\ln\frac{m_H^2}{m_W^2} + O(g^2)\right)$$

  - Infrared divergence as $m_H/m_W \to 0$
  - Related to Coleman-Weinberg effect:
    $m_H/m_W \gg g$ due to quantum fluctuations
  - Difficult to test: Need small $m_H/m_W \to 0$
    $\Rightarrow$ Small g $\Rightarrow$ Small quantum correction

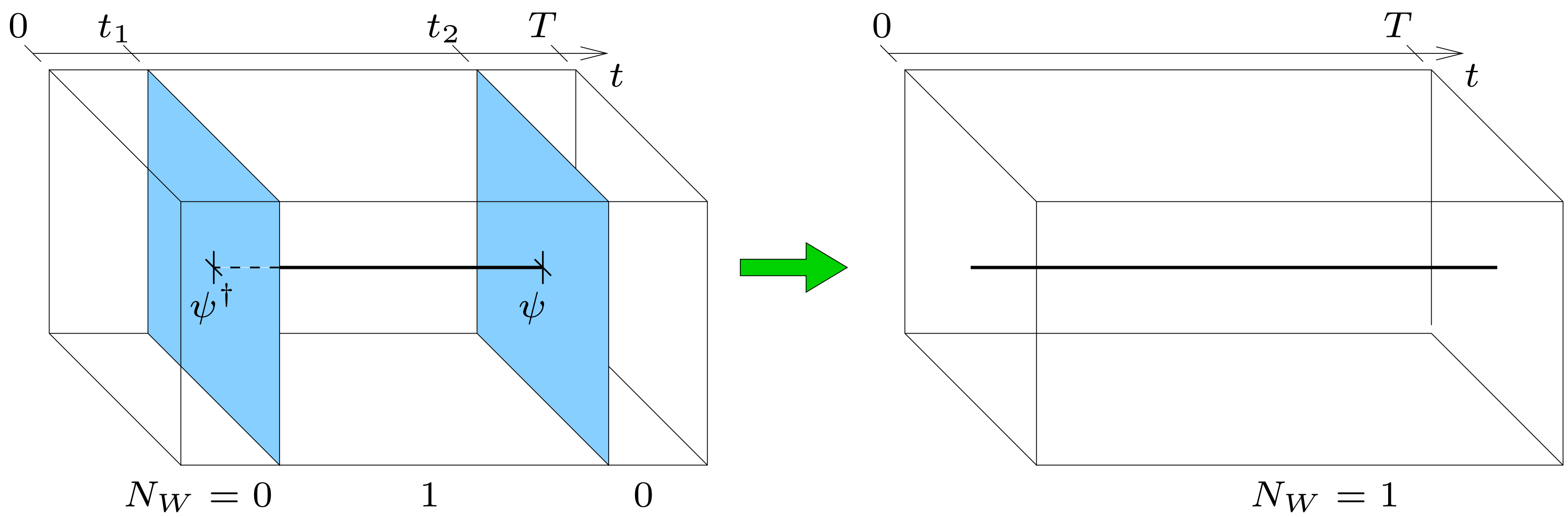
Non-perturbative Soliton Masses

- Soliton creation and annihilation operators $\psi^\dagger$ and ψ (Kadanoff&Ceva 1971)
  - $\langle 0|\psi^\dagger(t_1)\psi(t_2)|0\rangle \propto e^{iM(t_2-t_1)}$
- Path integral formulation (integrate over φ with $N_W = 0$)

$$e^{-M(t_2-t_1)} \propto Z_0^{-1} \int_0 D\varphi \psi^\dagger(t_1)\psi(t_2)e^{-S[\varphi]}$$

- Easy to do in simple cases: Kinks, vortices
- Less straightforward for monopoles:
  - Magnetic field $\Rightarrow$ ψ necessarily non-local
  - Compact QED: Duality maps to an integer-valued gauge theory (Polley&Wiese)
    $\Rightarrow$ Becomes much simpler
  - Non-Abelian theories: Several attempts (Frohlich&Marchetti, Di Giacomo et al.)
    – Idea: Add a classical monopole configuration between t and $t + \delta t$ (Dirac string with an endpoint, BPS monopole. . .)
    – Boundary conditions problematic

Removing Start and Endpoints



- Take $t_2 \to t_1 + T$, where T is temporal size
  - $\langle \psi^\dagger(t_1)\psi(t_2)\rangle \to Z_1/Z_0 = \exp(-MT)$

  $\Rightarrow M = -\ln(Z_1/Z_0)/T$
- Define Z_1 using appropriate boundary conditions
- Monte Carlo: Cannot calculate Z_1 or Z_0 directly
  - Only expectation values: Derivatives or differences

Mass Derivatives

- $M = -(\ln Z_1/Z_0)/T$, but cannot calculate Z_1 or Z_0 directly
- Calculate derivative with respect to some parameter λ:

$$\frac{\partial M}{\partial \lambda} = \frac{1}{T}\left(\frac{1}{Z_0}\frac{\partial Z_0}{\partial \lambda} - \frac{1}{Z_1}\frac{\partial Z_1}{\partial \lambda}\right)$$

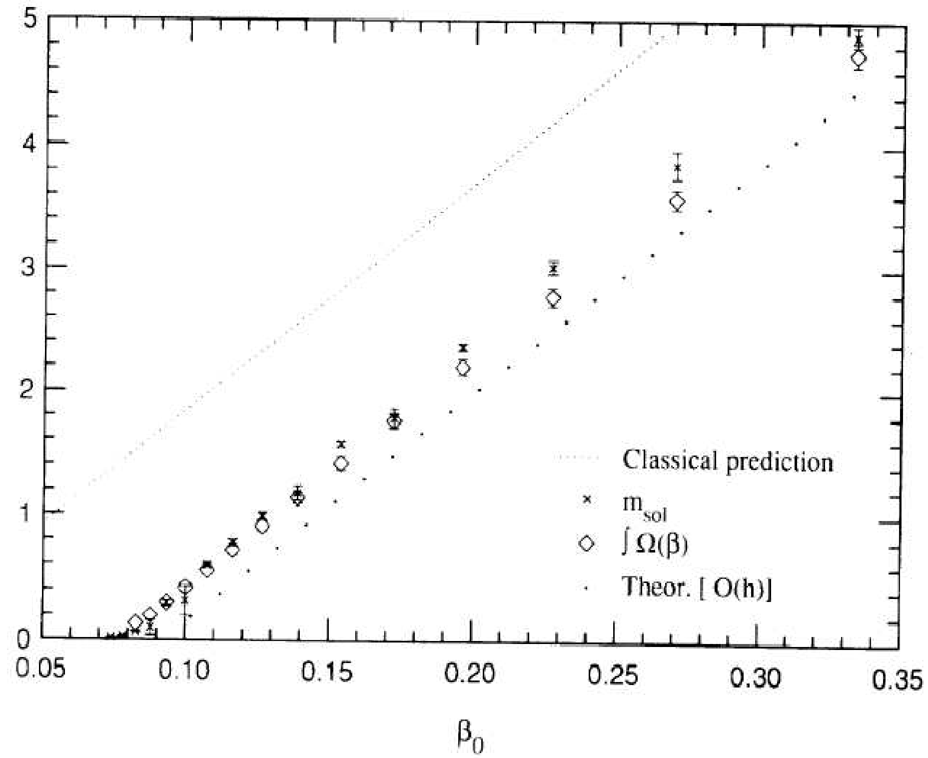
- Express in terms of expectation values:

$$\frac{1}{Z_{N_W}}\frac{\partial Z_{N_W}}{\partial \lambda} = -\frac{1}{Z_{N_W}}\int_{N_W} D\varphi \left(\frac{\partial S}{\partial \lambda}\right) e^{-S} = -\left\langle \frac{\partial S}{\partial \lambda} \right\rangle_{N_W}$$

  - Can be calculated with Monte Carlo simulations
- Integrate to obtain $M(\lambda)$
  - Start in symmetric phase: No integration constant

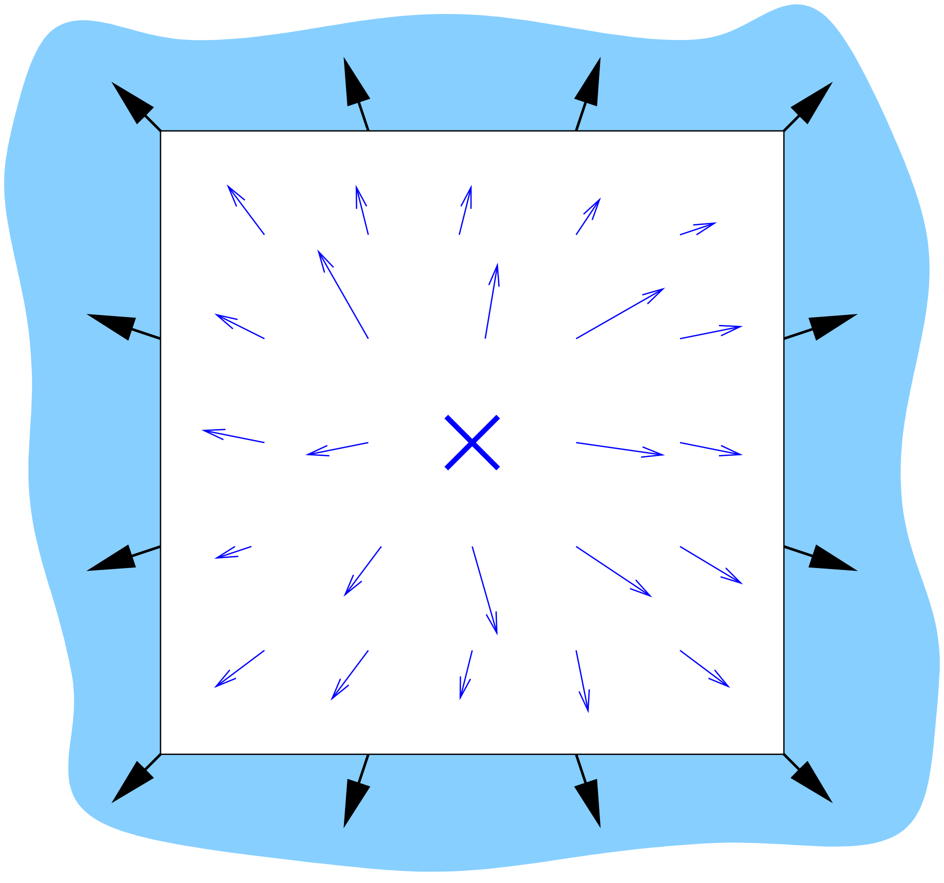
Non-perturbative Kink Mass

- Comparison of one-loop, operator and twist results (Ciria&Tarancon 1994)
  - Twist: Simply antiperiodic b.c. $\phi(L) = -\phi(0)$



- Non-perturbative results agree with each other
- Twist has much smaller errors
  - Also true for monopoles in compact QED (Vettorazzo&de Forcrand 2004)
- Slightly above one-loop result

Fixed Boundary Conditions



- Fix the field to the classical solution at the boundary (Smit&van der Sijs 1994, Cea&Cosmai 2000)
- Boundary effects?

Twisted Boundary Conditions

- Most common choice: Periodic boundary conditions
  - No boundary effects: Consequence of translation invariance
  - Magnetic Gauss law $\vec{\nabla} \cdot \vec{B} = \rho_M \Rightarrow$ Magnetic charge $Q_M = 0$
- Translation invariance only requires periodicity up to symmetries
  - C-periodic: (Kronfeld&Wiese 1991)

    $$U_\mu(x + N\hat{\jmath}) = U^*_\mu(x) = \sigma_2 U_\mu(x)\sigma_2$$
    $$\Phi(x + N\hat{\jmath}) = \Phi^*(x) = -\sigma_2\Phi(x)\sigma_2$$

    – Charge conjugation: Avoid Gauss law problem

    – Restricts Q_M to even values $\Rightarrow$ Use this to define Z_0
  - Twisted b.c.:

    $$U_\mu(x + N\hat{\jmath}) = \sigma_j U_\mu(x)\sigma_j$$
    $$\Phi(x + N\hat{\jmath}) = -\sigma_j\Phi(x)\sigma_j$$

    – Locally gauge equivalent to C-periodic - but not globally!

    – Always gives odd $Q_M \Rightarrow$ Use this to define Z_1 (JHEP 2000)

Derivative of Monopole Mass

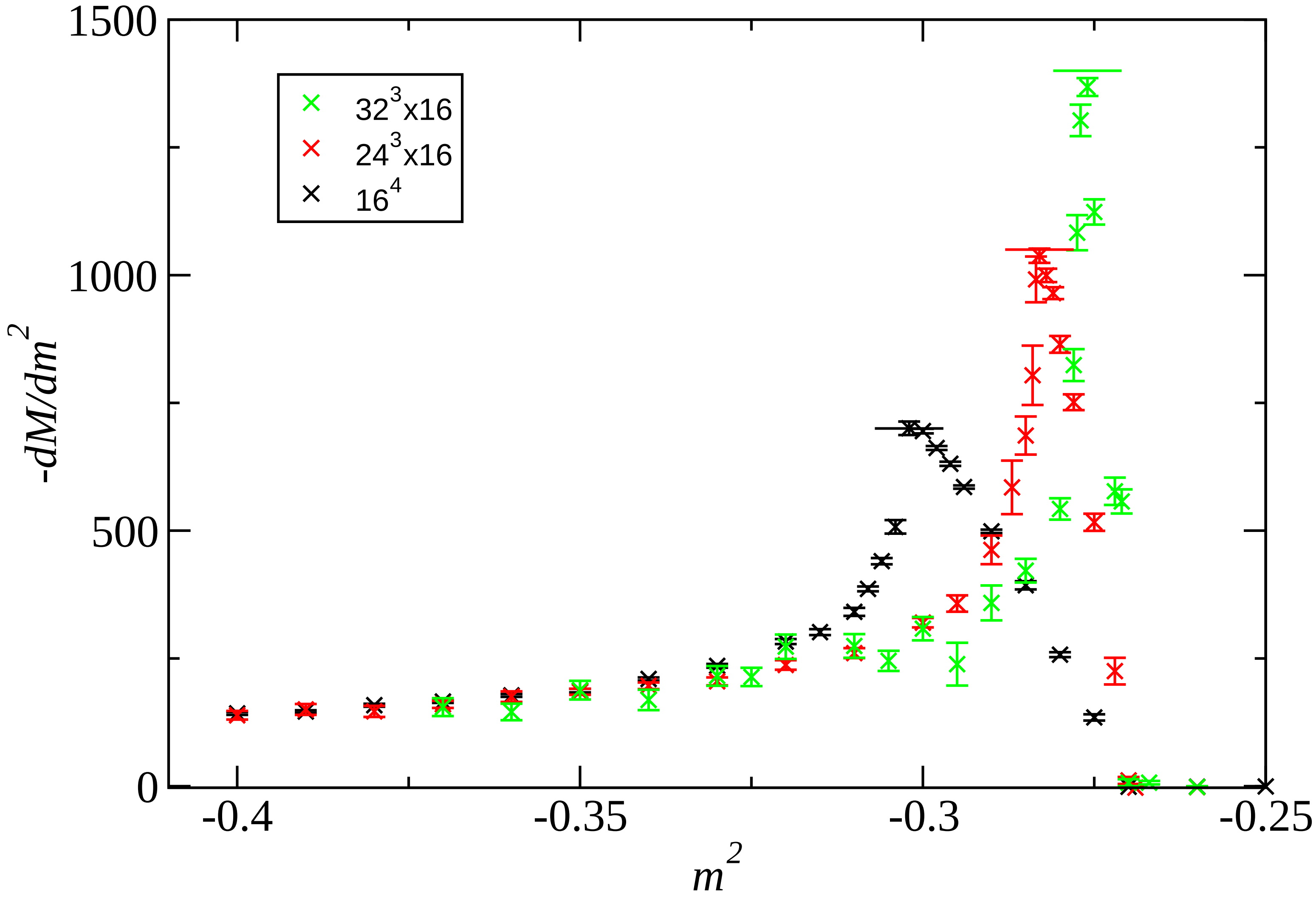
- Choose m^2 as the integration variable
  - Start at high enough $m^2 \Rightarrow$ Symmetric phase
  - Measure $\langle \mathrm{Tr}\Phi^2 \rangle_{N_W}$ at many values of m^2 using lattice Monte Carlo
  - Integrate:

$$M = L^3 \int_{m_0^2}^{m^2} dm^2 \left(\langle \mathrm{Tr}\Phi^2 \rangle_1 - \langle \mathrm{Tr}\Phi^2 \rangle_0 \right)$$

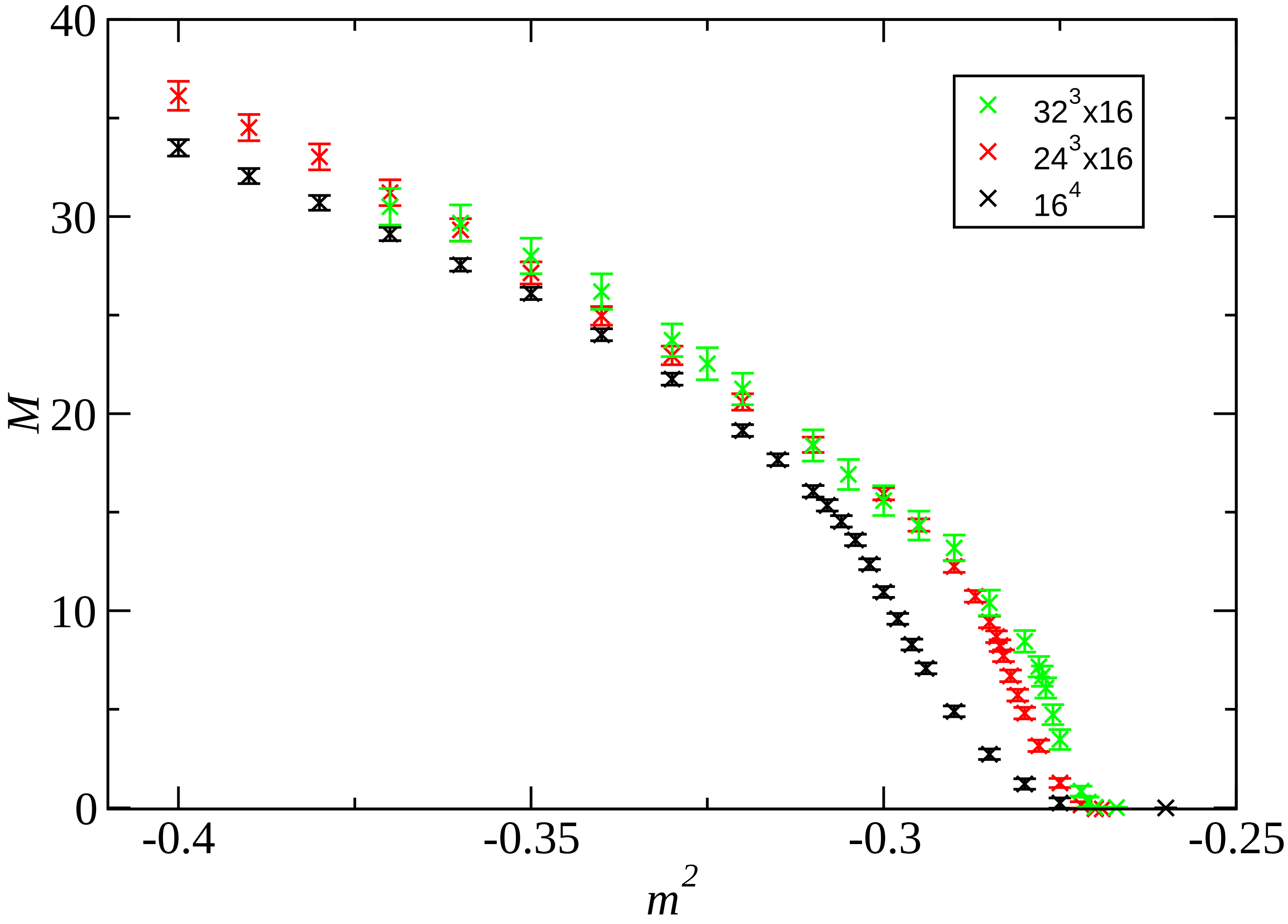
  - Better: Finite differences

$$M = \frac{1}{T} \sum_n \left(\langle e^{\Delta m^2 T L^3 \mathrm{Tr}\ \Phi^2} \rangle_{1,m_n^2} - \langle e^{\Delta m^2 T L^3 \mathrm{Tr}\ \Phi^2} \rangle_{0,m_n^2} \right)$$

Derivative of Monopole Mass: Results



Monopole Mass: Results



Direct Calculation

- Problems: – Must go through a phase transition
  – Errors accumulate
- Direct way of calculating M at given m^2
  - Gauge transformation $\longrightarrow$ C-periodic except

$$
\begin{aligned}
U_3(t,x,L,L-1) &= -U_3^*(t,x,0,L-1) \\
U_1(t,L-1,y,L) &= -U_1^*(t,L-1,y,0) \\
U_1(t,L-1,L,z) &= -U_1^*(t,L-1,0,z)
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  - Change of variables

$$
\begin{aligned}
U_3(t,x,L,L-1) &\rightarrow -U_3(t,x,L,L-1) \\
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Direct Calculation

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  - Change of variables

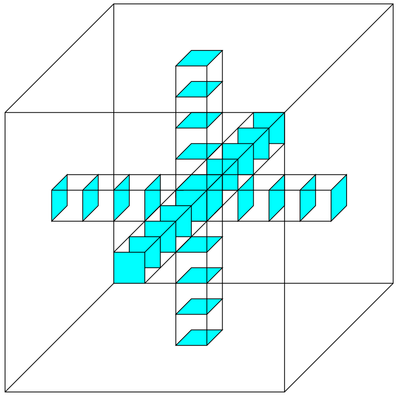
$$Z_1 = \int_{\mathrm{C-per}} DU_\mu D\Phi \exp(-S - \Delta S) = \langle \exp(-\Delta S) \rangle_0 Z_0$$

where

$$\Delta S = \beta \sum_{t,x=0}^{L-1} \left[\mathrm{Tr}\, U_{23}(x, y_0, z_0) + \mathrm{Tr}\, U_{13}(x_0, y, z_0) + \mathrm{Tr}\, U_{12}(x_0, y_0, z) \right]$$

- Three orthogonal 't Hooft lines crossing each other at (x_0, y_0, z_0)

Direct Calculation



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- Three orthogonal ’t Hooft lines crossing each other at (x_0, y_0, z_0)

Non-Integer Twists

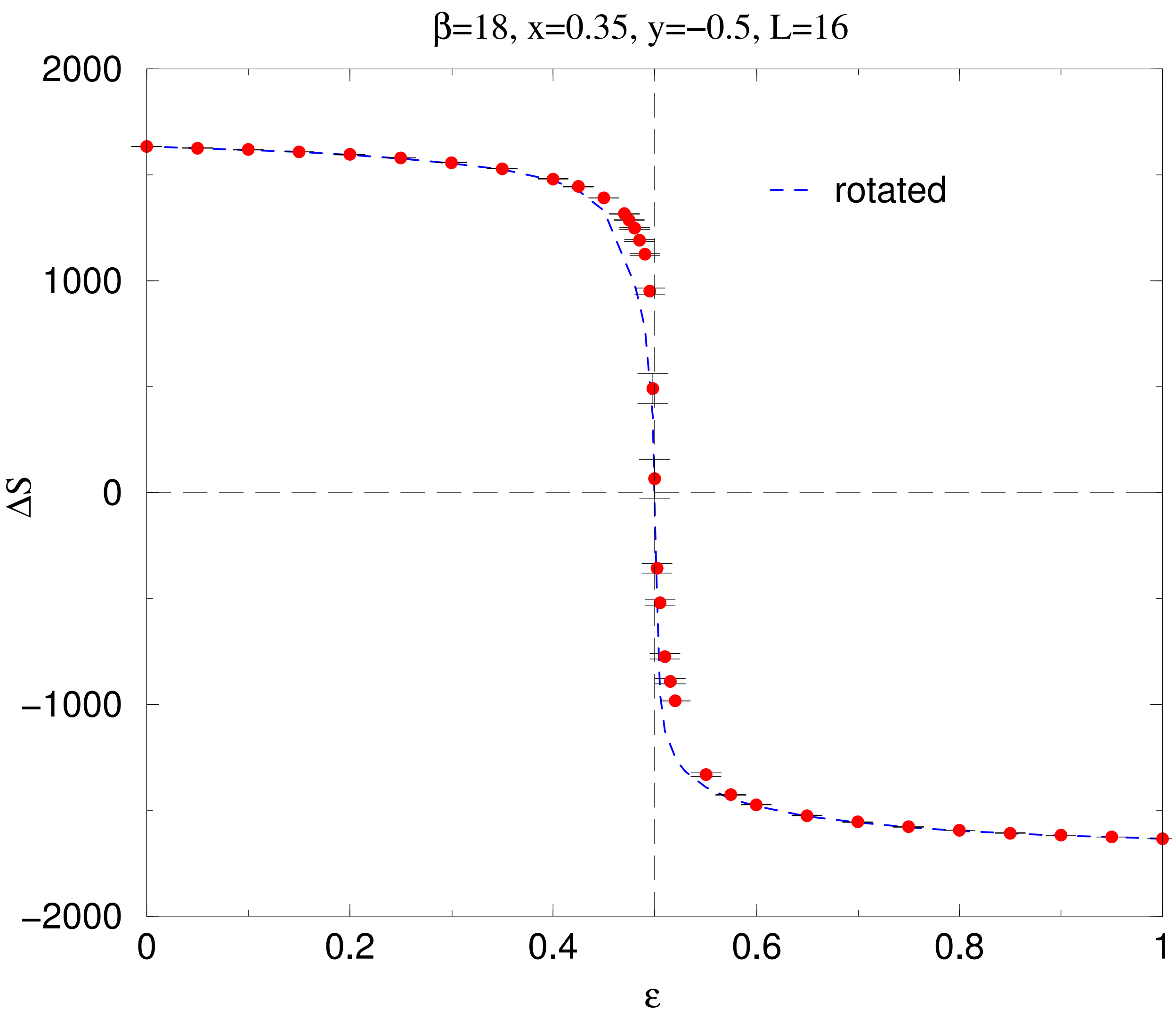
- Difficult to calculate $\langle \exp(-\Delta S) \rangle$: Poor overlap
- Define for $\epsilon \in [0, 1]$

$$Z_\epsilon = \int_{\mathrm{C-per}} DU_\mu D\Phi \exp(-S - \epsilon \Delta S)$$

  - Unphysical for non-integer ϵ
  - Still well-defined
- Differentiate with respect to ϵ

$$\frac{dM}{d\epsilon} = -\langle \Delta S \rangle_\epsilon$$

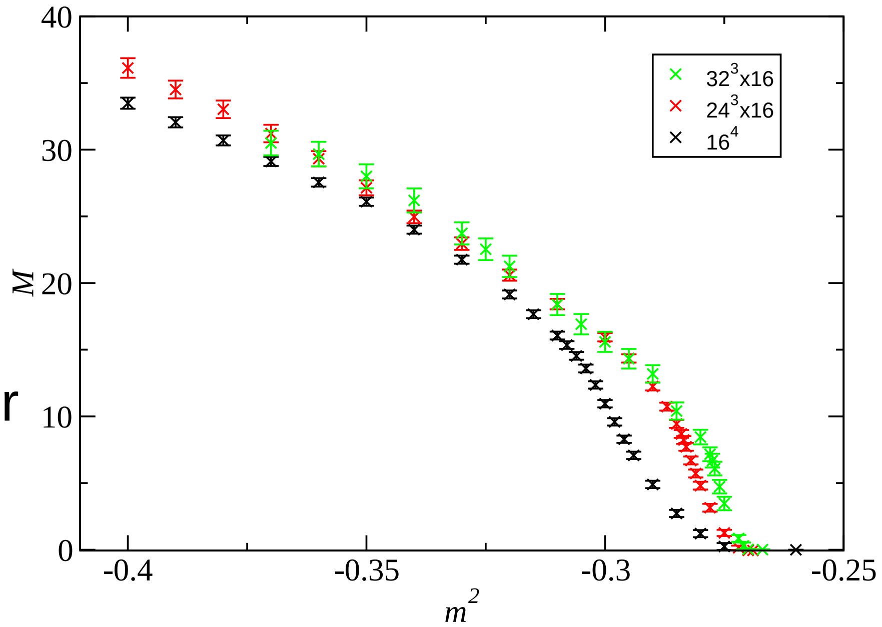
Non-Integer Twists



- From 3D simulation (PRD65(2002))

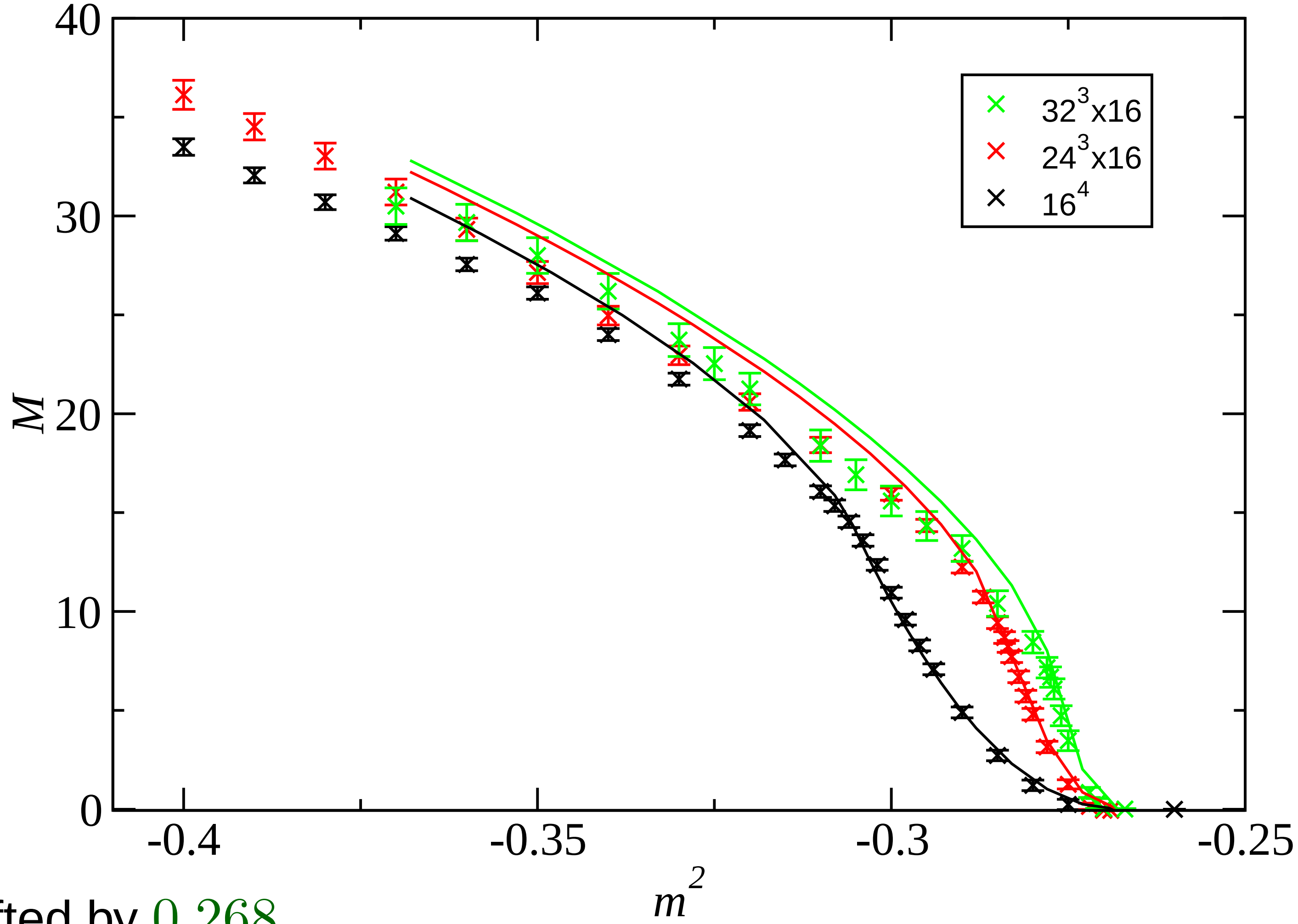
Renormalisation

- Comparison with classical results?
  - m^2, λ, g bare couplings
  - Must renormalise
- Scheme dependence
- Perturbative renormalisation
  - Monopole mass only to the same order in perturbative expansion



- Non-perturbative approach:
  - Measure three different quantities (say g, m_H, m_W)
  - Use them to fix the classical couplings
- For the moment, simply ignore logs and finite terms
  - Shift m^2 axis by a constant amount

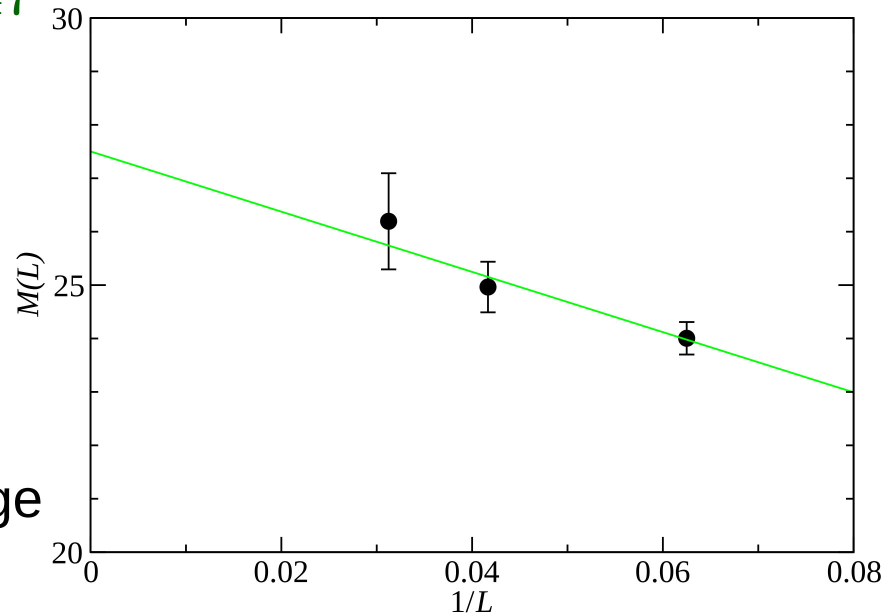
Comparison with Classical Mass



- m^2 shifted by 0.268
- Quantum masses generally lower (renormalisation?)

Effective Couplings

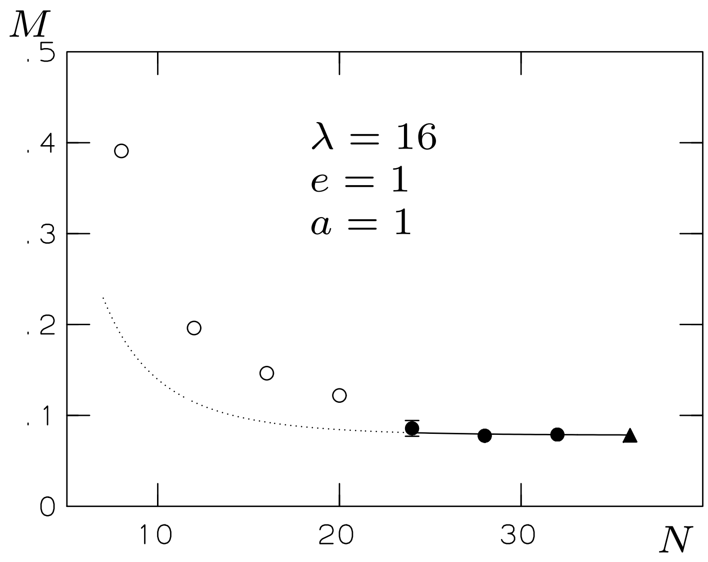
- Classical simulation ⇒ Finite size effect $\Delta E(L) = 11.0/g^2 L$
- Fit quantum finite size effect to determine g_R
  - Gives $g_R \approx 0.44(5)$ vs bare $g \approx 0.447$



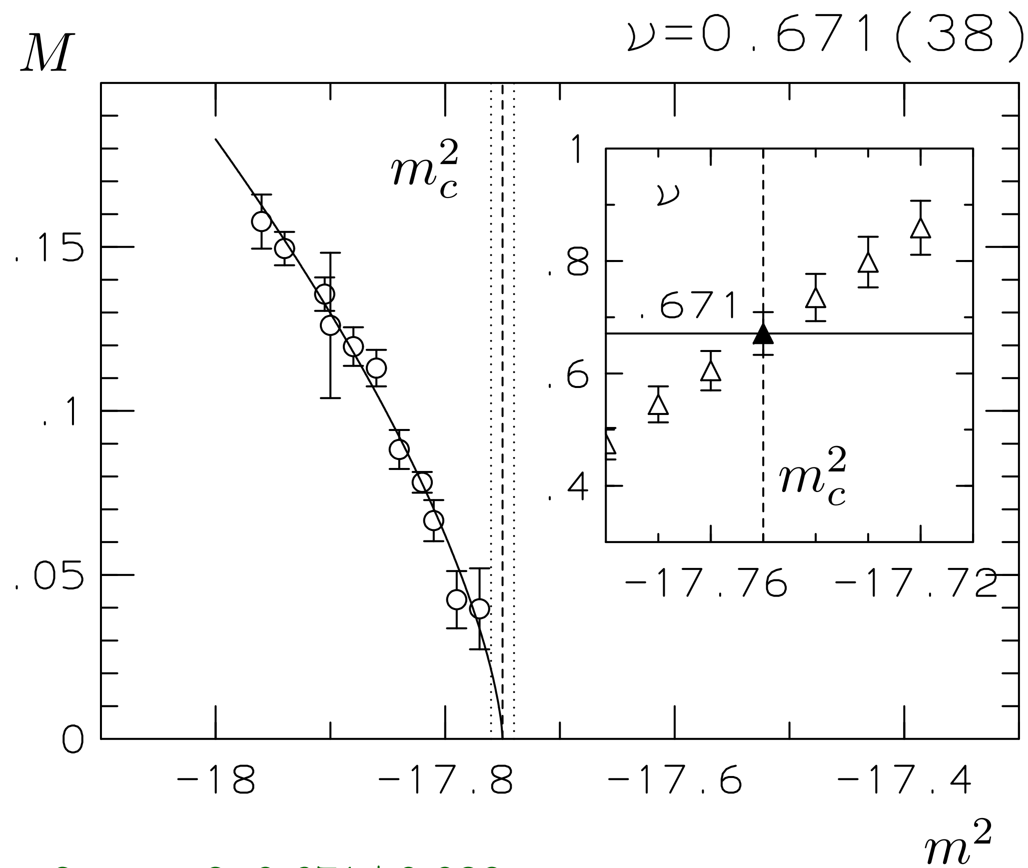
- Masses m_H and m_W from correlation functions
  - Difficult to measure m_W
- Expectations: As $m^2 \to m_c^2$
  - Triviality: $\lambda_R \to 0$
  - Asymptotic freedom: g_R becomes large
  - $m_H/m_W = \sqrt{\lambda_R}/g_R \to 0$
  - $M/m_W = (4\pi/g_R^2) f(m_H/m_W) \to 0$?
  - Will $W^{\pm}$ decouple?

    ⇒ Charged scalar + photon (+ neutral scalar)

Asymptotic Duality in 2+1D Abelian Higgs Model



(NPB2004)



- Near the critical point, $M_{\rm vort} \propto (m_c^2 - m^2)^{0.671\pm0.038}$
  - Vortex becomes the lightest particle: $m_\gamma, m_s \propto (m_c^2 - m^2)^{1/2}$
  - Dual to complex scalar field theory?
- Numerical evidence: XY model critical exponent

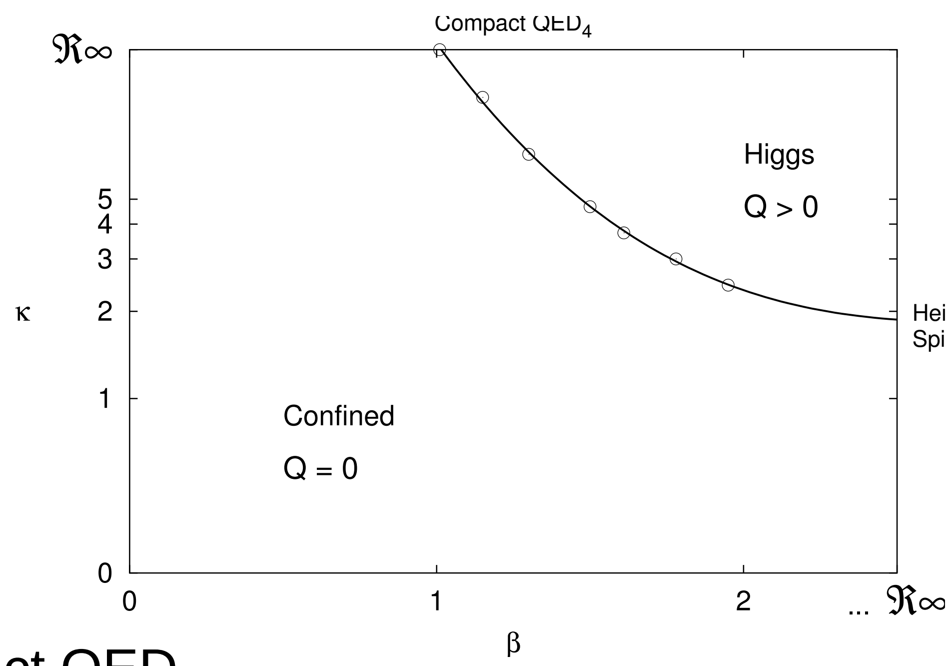
Speculation: Asymptotic Duality in Georgi-Glashow Model?

Georgi-Glashow model	Abelian Higgs model
Higgs phase	**Coulomb phase**
electric/magnetic field	magnetic/electric field
magnetic monopole	charged scalar
massless photon	massless photon
Confining phase	**Higgs phase**
confinement	superconductivity
confining string	vortex line

- Puts the ’t Hooft-Mandelstam dual superconductor idea on firm footing
- Same duality is known to exist in supersymmetric theories

Hints for Monopole Duality

- Phase diagram for $\lambda \to \infty$ (Greensite et al. 2004)



- Limit $\kappa \to \infty$ = compact QED
  - Exactly dual to 4D frozen superconductor (Peskin 1978)
  - Frozen superconductor = $\lambda, \kappa \to \infty$ limit of Abelian Higgs model
  - Duality maps electric and magnetic field to each other
- Will duality survive near critical point even for finite λ, κ?

Conclusions

- Monopole mass using twisted boundary conditions
  - Well defined even on the lattice
  - No cooling needed
  - No reference to any specific field configs
- Integrating the derivative
  - Derivative with respect to m^2
    - Straightforward
    - Growing errors
  - Derivative with respect to non-integer twist ϵ
    - Non-integer values unphysical
    - Direct measurement of M at given couplings
- Comparison with classical result
  - Significant correction in terms of bare couplings
  - Renormalisation: Perturbative/Non-perturbative
- Critical behaviour: Duality?