Twisting vs. Staggering: is twisted-mass lattice QCD a viable alternative to improved staggered fermions?

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Outline

- Introduction: status of lattice fermion actions
- What is twisted-mass lattice QCD (tmLQCD) and why is it interesting?
- Effective chiral Lagrangian for tmLQCD including discretization errors
- Results for “generic small mass" regime: $m_q \sim a\Lambda^2_{\rm QCD}$
- Results for “Aoki" regime: $m_q \sim a^2\Lambda^3_{\rm QCD}$
- Summary and outlook

Introduction

- Lattice QCD is entering an exciting era
  - Terascale computers (e.g. UKQCD's QCDOC)
  - Unquenched simulations with $m_\pi \to 250$ MeV and below
  - Potential for few percent control over all systematics
- Choices for light fermion action:
  - Improved staggered fermions
    - Fastest, but not unitary, and possibly not local, at $O(a^2)$
  - Chirally symmetric fermions
    - The ultimate choice, but slowest
  - Improved Wilson fermions
    - Straightforward, but slow
  - Twisted-mass
    - Maybe as fast as staggered, but potential not yet clear
- **Are twisted-mass fermions a viable alternative to staggered fermions?**

tmLQCD

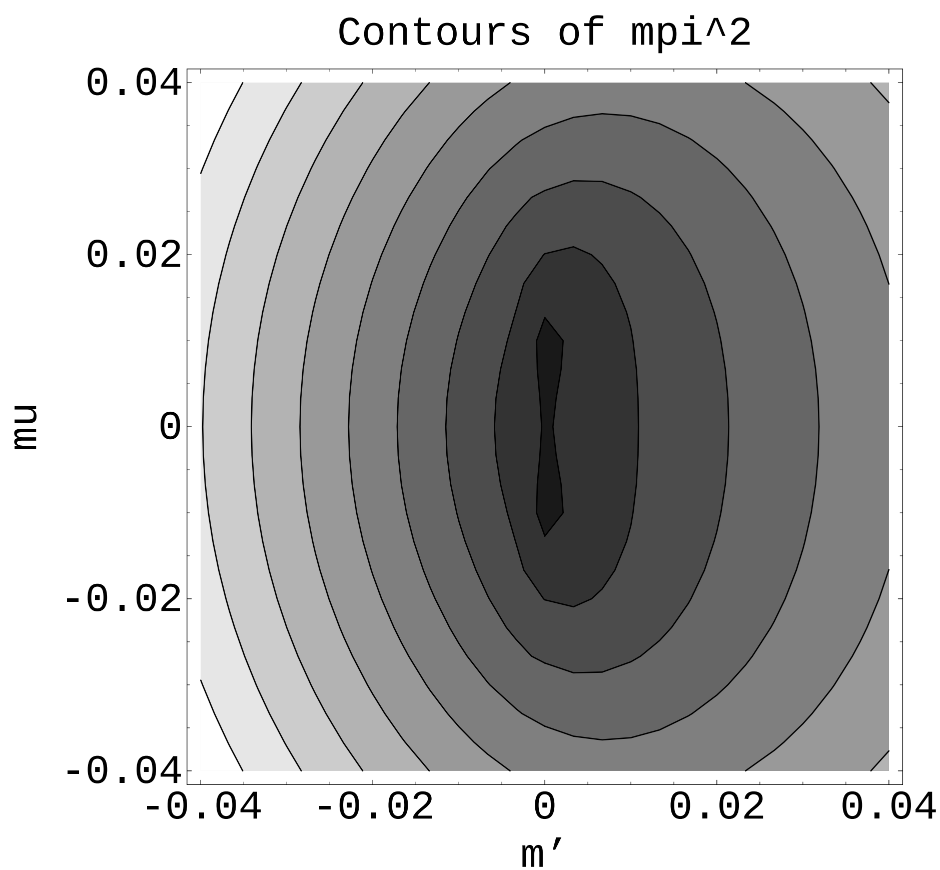
Variant of unimproved Wilson fermions with twisted mass
[Frezzotti, Grassi, Sint & Weisz, 2000]

- Advantages
  - $+$ WYSIWYG: no roots of determinant
  - $+$ speed comparable to staggered
  - $+$ at “maximal twist"
    - errors $\sim a^2$ automatically [Frezzotti & Rossi, 2003]
    - operator mixing as in continuum [Frezzotti & Rossi, 2004]
- Disadvantages
  - $-$ flavor is broken for $a \neq 0$: $SU(2) \to U(1)$

Detailed numerical studies underway [DESY-Zeuthen]

We use analytical methods to study its properties

What happens to Istvan's circles?



Contour plot of charged m_π^2 from tmχPT with parameters roughly tuned to match those of hep-lat/0410031, Farchioni *et al.*

What is tmLQCD?

Begin in the continuum.

- Action in "twisted basis" for two degenerate flavors ($\tau_3^2 = 1$):

$$\mathcal{L}_{tm} = \bar{\psi}(\not{D} + m_q e^{i\gamma_5\tau_3\omega})\psi = \bar{\psi}(\not{D} + m + i\gamma_5\tau_3\mu)\psi$$

$$\cos\omega = m/m_q\,, \quad \sin\omega = \mu/m_q\,, \quad m_q = \sqrt{m^2+\mu^2}$$

  - Maximal twist is $m = 0$, $\omega = \pi/2$.

- Flavor-breaking is fake: non-singlet axial transformation

$$\widehat{\psi} = \exp(i\gamma_5\tau_3\omega/2)\psi\,, \quad \widehat{\bar{\psi}} = \bar{\psi}\exp(i\gamma_5\tau_3\omega/2)$$

  brings $\mathcal{L}$ into usual form ("physical basis"):

$$\mathcal{L}_{tm} = \widehat{\bar{\psi}}(\not{D} + m_q)\widehat{\psi}$$

Kinematics of continuum tmQCD

Currents and densities:

$$A_\mu^j = \bar\psi \gamma_\mu \gamma_5 \tau_j \psi\,,\ \ V_\mu^j = \bar\psi \gamma_\mu \tau_j \psi\,,\ \ P^j = \bar\psi \gamma_5 (\tau_j/2) \psi\,,\ \ S^0 = \bar\psi \psi$$

Relation between operators in twisted and physical bases ($a = 1, 2$):

$$\begin{aligned}
\widehat{A}_\mu^a &= \cos\omega\, A_\mu^a + \epsilon^{3ab} \sin\omega\, V_\mu^b\,, \quad \widehat{A}_\mu^3 = A_\mu^3\,, \\
\widehat{V}_\mu^a &= \cos\omega\, V_\mu^a + \epsilon^{3ab} \sin\omega\, A_\mu^b\,, \quad \widehat{V}_\mu^3 = V_\mu^3\,, \\
\widehat{P}^3 &= \cos\omega\, P^3 + i \sin\omega\, S^0/2\,, \quad \widehat{P}^a = P^a\,, \\
\widehat{S}^0 &= \cos\omega\, S^0 + 2i \sin\omega\, P^3
\end{aligned}$$

Note: A_μ^3 and P^a create physical pions for all ω

What is tmLQCD?

Lattice action [Frezzotti, Grassi, Sint & Weisz] :

$$S_F^L = a^4 \sum_x \bar\psi_l(x)\Big[\frac{1}{2}\sum_\mu \gamma_\mu(\nabla^\star_\mu + \nabla_\mu) + m_0 + i\gamma_5\tau_3\mu_0 - \frac{r}{2}\sum_\mu \nabla^\star_\mu\nabla_\mu\Big]\psi_l(x)$$

- cannot rotate away twist in mass
  - $\Rightarrow$ parity and flavor broken, though breaking vanishes in naive continuum limit
- Wilson term $\nabla^\star_\mu\nabla_\mu$ mixes with identity
  - $\Rightarrow$ usual additive renormalization of m_0: $m = Z_m(m_0 - m_c)/a$
- μ_0 is multiplicatively renormalized, like m_q in continuum:

  $\mu = Z_\mu \mu_0/a$
- renormalized twist angle and quark mass:

  $\tan\omega = \mu/m + O(a)\,,\ m_q = \sqrt{m^2+\mu^2} + O(a)$

Computational advantage of tmLQCD

Rewrite action:

$$S_F^L = a^4 \sum_x \bar{\psi}_l(x) \Big[D_W + m_0 + i\gamma_5 \tau_3 \mu_0 \Big] \psi_l(x)$$

$$= a^4 \sum_x \bar{\psi}_l(x) \gamma_5 \Big[H_W + i\tau_3 \mu_0 \Big] \psi_l(x)$$

with H_W the Hermitian Wilson-Dirac operator:

$$H_W = \gamma_5 (D_W + m_0) = H_W^\dagger$$

- computational problem: zero eigenvalues of H_W

  $\Rightarrow$ fermion determinant vanishes, slows algorithms

- solved by twisting: μ_0 provides IR cut-off

$$\det(H_W + i\tau_3 \mu_0) = \prod_\lambda (\lambda + i\mu_0)(\lambda - i\mu_0) = \prod_\lambda (\lambda^2 + \mu_0^2)$$

- Simulations comparable in speed to staggered fermions
  [Kennedy, Lattice 04]

Automatic $O(a)$ improvement

Key property of tmLQCD at maximal twist [Frezzotti & Rossi] :

$$Q_{lat} = Q_{cont}\left[1 + ca^2\Lambda_{QCD}^2 + c'a^2m_q^2 + O(a^4)\right]$$

Why does this work? We will study for $a \sim m_q$ so
$a\Lambda_{QCD} \gg am_q \sim a^2$

- At maximal twist, have $m_0 = m_c$, so

$$\begin{aligned} m_0 - \frac{r}{2}\sum_\mu \nabla_\mu^\star \nabla_\mu + i\mu\gamma_5\tau_3 &= i\mu\gamma_5\tau_3 + ac'(D_\mu^2)|_{cont} + O(a^2) \\ &\longrightarrow \mu - ac' i\gamma_5\tau_3(D_\mu^2)|_{cont} + O(a^2) \end{aligned}$$

  when rotate to physical basis

$\Rightarrow$ $O(a)$ corrections necessarily violate parity and flavor

$\Rightarrow$ physical (parity-flavor conserving) quantities corrected only at $O(a^2)$

Range of validity of automatic improvement

- Possibilities are: ($a^{-1} = 2$ GeV, $\Lambda_{QCD} = 300$ MeV)
  (A) $m_q \gg a\Lambda_{QCD}^2 \sim 45$ MeV
  (B) $m_q \sim a\Lambda_{QCD}^2 \gg a^2\Lambda_{QCD}^3 \sim 7$ MeV
  (C) $m_q \geq ca^2\Lambda_{QCD}^3 \sim 7$ MeV, with $c = O(1)$
- Since $(m_u + m_d)/2 \approx 3$ MeV, want (B) or, better, (C)
- [Frezzotti & Rossi] argue that (A) is needed:
  - Vacuum should be determined by $O(m_q)$ and not $O(a)$ effects
- We claim that can relax to (B) with appropriate definition of ω
- We agree with [Aoki & Bär] that (C) can also hold, although with $c \neq 0$ in general.

Our method: tmχPT

- We study long-distance physics (vacuum & pion properties) using the chiral Lagrangian extended to include discretization errors
  - understand, analytically, competition between $O(m_q)$ and $O(a)$ effects
  - interpret and guide simulations (which are in new territory)
- consider 2 light degenerate flavors
  - results valid for 2+1 flavors if $p^2 \sim m_\pi^2 \ll m_K^2$
- use two step method of [SRS + Singleton, 1998]

  (1) construct continuum $\mathcal{L}_{\rm eff}$ describing lattice theory [Symanzik]

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{QCD} + a\mathcal{L}_1 + a^2\mathcal{L}_2 + \ldots$$

  (2) construct chiral effective theory for $\mathcal{L}_{\rm eff}$
- power counting: work to NLO using $p^2 \sim m_q \sim a$

Step 1: effective continuum Lagrangian

Need most general Lagrangian consistent with lattice symmetries, which include

- Parity + discrete flavor ($\psi_l \to i\tau_1\psi_l$)
- Parity $+ \mu_0 \to -\mu_0$

Result is simple [Münster & Schmidt, SRS & Wu] :

$$\begin{aligned}\mathcal{L}_0 &= \mathcal{L}_{\rm glue} + \bar\psi(\not{D} + m + i\gamma_5\tau_3\mu)\psi \\ \mathcal{L}_1 &= b_1[g^2(a)]\bar\psi i\sigma_{\mu\nu}F_{\mu\nu}\psi\end{aligned}$$

- same as for Wilson fermions aside from twisted mass in $\mathcal{L}_0$
- other potential operators in $\mathcal{L}_1$ vanish by LO equations of motion, or are NNLO in our power counting, e.g. $\bar\psi\sigma_{\mu\nu}F_{\mu\nu}\gamma_5\tau_3\psi$ requires factor of $a\mu$
- $\mathcal{L}_2$ is same as for Wilson theory [Bär, Rupak, Shoresh] , but can ignore as introduces no additional symmetry breaking

Step 2: map onto chiral Lagrangian

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{glue} + \bar{\psi}(\not{D} + m + i\gamma_5\tau_3\mu)\psi + ab_1\bar{\psi}i\sigma_{\mu\nu}F_{\mu\nu}\psi$$

- For $m = \mu = a = 0$ have $SU(2)_L \times SU(2)_R$ chiral symmetry:

$$\psi_{L,R} \longrightarrow U_{L,R}\psi_{L,R}\,, \quad U_{L,R} \in SU(2)_{L,R}$$

- Symmetry broken by mass and a terms in same way
  - $\bar{\psi}(m + i\gamma_5\tau_3\mu)\psi = \bar{\psi}_L\mathcal{M}\psi_R + \bar{\psi}_R\mathcal{M}^\dagger\psi_L$ with
    $\mathcal{M} = m + i\tau_3\mu = m_q \exp(i\omega\tau_3)$
  - $a\bar{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi \propto \bar{\psi}_L\hat{A}\sigma_{\mu\nu}F_{\mu\nu}\psi_R + \bar{\psi}_R\hat{A}^\dagger\sigma_{\mu\nu}F_{\mu\nu}\psi_L$
- $\mathcal{L}_{\text{eff}}$ invariant if treat $\mathcal{M}$, $\hat{A}$ as spurions:
  $\mathcal{M} \to U_L\mathcal{M}U_R^\dagger$ and $\hat{A} \to U_L\hat{A}U_R^\dagger$

$\Rightarrow$ standard χPT analysis can be used for $\mathcal{M}$ and $\hat{A}$
[SRS & Singleton; Bär, Rupak & Shoresh]

Resulting chiral Lagrangian

$$
\begin{aligned}
\mathcal{L}_\chi &= \frac{f^2}{4}\mathrm{Tr}(D_\mu\Sigma D_\mu\Sigma^\dagger) - \frac{f^2}{4}\mathrm{Tr}(\chi^\dagger\Sigma + \Sigma^\dagger\chi) - \frac{f^2}{4}\mathrm{Tr}(\hat{A}^\dagger\Sigma + \Sigma^\dagger\hat{A}) \\
&-L_1\mathrm{Tr}(D_\mu\Sigma D_\mu\Sigma^\dagger)^2 - L_2\mathrm{Tr}(D_\mu\Sigma D_\nu\Sigma^\dagger)\mathrm{Tr}(D_\mu\Sigma D_\nu\Sigma^\dagger) \\
&+L_{45}\mathrm{Tr}(D_\mu\Sigma^\dagger D_\mu\Sigma)\mathrm{Tr}(\chi^\dagger\Sigma + \Sigma^\dagger\chi) - L_{68}\left[\mathrm{Tr}(\chi^\dagger\Sigma + \Sigma^\dagger\chi)\right]^2 \\
&+W_{45}\mathrm{Tr}(D_\mu\Sigma^\dagger D_\mu\Sigma)\mathrm{Tr}(\hat{A}^\dagger\Sigma + \Sigma^\dagger\hat{A}) - W_{68}\mathrm{Tr}(\chi^\dagger\Sigma + \Sigma^\dagger\chi)\mathrm{Tr}(\hat{A}^\dagger\Sigma + \Sigma^\dagger\hat{A}) \\
&-W'_{68}\left[\mathrm{Tr}(\hat{A}^\dagger\Sigma + \Sigma^\dagger\hat{A})\right]^2 + W_{10}\mathrm{Tr}(D_\mu\hat{A}^\dagger D_\mu\Sigma + D_\mu\Sigma^\dagger D_\mu\hat{A})
\end{aligned}
$$

where

$$
\begin{aligned}
\Sigma &\sim \langle\psi_L\bar{\psi}_R\rangle \longrightarrow U_L\Sigma U_R^\dagger\,,\ D_\mu\Sigma = \partial_\mu\Sigma - i\ell_\mu\Sigma + i\Sigma r_\mu\,, \\
\chi &= 2B_0(s+ip) \rightarrow 2B_0\mathcal{M}\,, \quad \hat{A} \rightarrow \hat{a} = 2W_0 a
\end{aligned}
$$

and constants are not determined by symmetries

- f, B_0 and L_i are from continuum χPT
- W_i, W'_i are introduced by discretization errors

tmχPT at leading order

Lagrangian takes continuum form if use variable $\chi' = \chi + \hat{A}$:

$$\mathcal{L}_{\chi,LO} = \frac{f^2}{4}\mathrm{Tr}(D_\mu \Sigma D_\mu \Sigma^\dagger) - \frac{f^2}{4}\mathrm{Tr}(\chi^\dagger \Sigma + \Sigma^\dagger \chi) - \frac{f^2}{4}\mathrm{Tr}(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A})$$
$$= \frac{f^2}{4}\mathrm{Tr}(D_\mu \Sigma D_\mu \Sigma^\dagger) - \frac{f^2}{4}\mathrm{Tr}(\chi'^\dagger \Sigma + \Sigma^\dagger \chi')$$

- corresponds to $O(a)$ shift in m and m_c:

$$m \to m' = m + aW_0/B_0\,,\ m_q \to \sqrt{m'^2 + \mu^2}\,,\ \tan\omega_0 \equiv \mu/m'\,,\ \chi' = 2B_0 m_q e^{i\omega_0\tau_3}$$

- condensate aligns with χ':

$$\Sigma = \exp(i\omega_0\tau_3/2)\Sigma_{ph}\exp(i\omega_0\tau_3/2)\,, \quad \Sigma_{ph} = \exp(i\vec{\pi}\cdot\vec{\tau}/f)$$

$\Rightarrow$ LO pion interactions have no $O(a)$ corrections for any ω_0

$$\mathcal{L}_{\chi,LO} = \frac{f^2}{4}\mathrm{Tr}(D_\mu \Sigma_{ph} D_\mu \Sigma_{ph}^\dagger) - \frac{f^2}{4}|\chi'|\mathrm{Tr}(\Sigma_{ph} + \Sigma_{ph}^\dagger)$$

NLO result: $O(a)$ improvement at $\omega_0 = \pi/2$

- express chiral Lagrangian in terms of χ':

$$\begin{aligned}\mathcal{L}_{\chi,\mathrm{NLO}} &= \text{continuum terms} + \tilde{W}\ \mathrm{Tr}(D_\mu\Sigma^\dagger D_\mu\Sigma)\mathrm{Tr}(\hat{A}^\dagger\Sigma + \Sigma^\dagger\hat{A}) \\ &\quad -W\ \mathrm{Tr}(\chi'^\dagger\Sigma + \Sigma^\dagger\chi')\mathrm{Tr}(\hat{A}^\dagger\Sigma + \Sigma^\dagger\hat{A}) + O(a^2)\end{aligned}$$

- Lagrangian is invariant under symmetry:
  - $\pi \to -\pi,\ \omega_0 \to -\omega_0 \Rightarrow \Sigma \leftrightarrow \Sigma^\dagger,\ \chi' \leftrightarrow \chi'^\dagger$

  which implies:
  - $\mathrm{Tr}(D_\mu\Sigma^\dagger D_\mu\Sigma)$ and $\mathrm{Tr}(\chi'^\dagger\Sigma + \Sigma^\dagger\chi')$ are even in π
  - $\mathrm{Tr}(\hat{A}^\dagger\Sigma + \Sigma^\dagger\hat{A}) = \cos\omega_0 \times (\text{even in } \pi) + \sin\omega_0 \times (\text{odd in } \pi)$

- for $\omega_0 = \pi/2$, physical vertices, with an even number of pions, receive no $O(a)$ contributions
- unphysical vertices with an odd number of pions are $O(a)$

$\Rightarrow$ **automatic $O(a)$ improvement valid even if $m_q \sim a\Lambda^2_{QCD}$ as long as use ω_0 and not ω**

Recall: Kinematics of continuum tmQCD

Currents and densities:

$$A^j_\mu = \bar\psi\gamma_\mu\gamma_5\tau_j\psi\,,\;\; V^j_\mu = \bar\psi\gamma_\mu\tau_j\psi\,,\;\; P^j = \bar\psi\gamma_5(\tau_j/2)\psi\,,\;\; S^0 = \bar\psi\psi$$

Relation between operators in twisted and physical bases ($a = 1, 2$):

$$\begin{aligned}
\widehat{A}^a_\mu &= \cos\omega\, A^a_\mu + \epsilon^{3ab}\sin\omega\, V^b_\mu\,, \quad \widehat{A}^3_\mu = A^3_\mu\,, \\
\widehat{V}^a_\mu &= \cos\omega\, V^a_\mu + \epsilon^{3ab}\sin\omega\, A^b_\mu\,, \quad \widehat{V}^3_\mu = V^3_\mu\,, \\
\widehat{P}^3 &= \cos\omega\, P^3 + i\sin\omega\, S^0/2\,, \quad \widehat{P}^a = P^a\,, \\
\widehat{S}^0 &= \cos\omega\, S^0 + 2i\sin\omega\, P^3
\end{aligned}$$

Specific NLO results: twist angle

- How determine m' and thus ω_0?

  (1) enforce $\langle \hat{V}_\mu^2(x)\hat{P}^1(y)\rangle = 0$ implying

$$\tan\omega_A \equiv \frac{\langle V_\mu^2(x)P^1(y)\rangle}{\langle A_\mu^1(x)P^1(y)\rangle}$$

  (2) enforce $\langle \hat{S}^0(x)\hat{A}_\mu^3(y)\rangle = 0$ implying

$$\tan\omega_P \equiv \frac{i\langle S^0(x)A_\mu^3(y)\rangle}{2\langle P^3(x)A_\mu^3(y)\rangle}$$

- result: ω_0 determined to $O(a)$ accuracy

$$\omega_0 = \omega_A + \frac{16\hat{a}s}{f^2}\left(W + W_{10}/4 + 2\hat{a}cW'/(2B_0 m_q)\right)$$

$$\omega_P = \omega_A + \frac{4\hat{a}s(4W + W_{10})}{f^2}$$

  with $c = \cos\omega_A$ and $s = \sin\omega_A$

Required accuracy for twist angle?

- $O(a)$ ambiguity in ω is inevitable due to discretization errors
- ambiguity does not impact automatic $O(a)$ improvement

$$\begin{aligned} \mathcal{Q}_{\rm lat} &= \mathcal{Q}_{\rm cont}\left[1 + a\cos\omega + O(a^2)\right] \\ \delta Q_{\rm lat} &= \mathcal{Q}_{\rm cont}\, a\, \delta(\cos\omega) + O(a^2) \\ &= -\mathcal{Q}_{\rm cont}\, a\, \sin\omega\, \delta\omega + O(a^2) \\ &= O(a^2) \end{aligned}$$

$\Rightarrow$ can set either $\omega_A = \pi/2$ or $\omega_P = \pi/2$

- we propose $\omega_A = \pi/2$ as canonical choice since easier to implement in simulations

Specific NLO results: pion masses

- Charged pion mass is automatically improved:
  result agrees with [Scorzato]

$$
\begin{aligned}
m^2_{\pi_\pm} &= |\chi'| + \frac{16}{f^2}\Big[|\chi'|^2(2L_{68} - L_{45}) \\
&\quad + |\chi'|\hat{a}c(2W - \widetilde{W}) + 2\hat{a}^2c^2W'\Big] + \text{cont. 1-loop} + \ldots
\end{aligned}
$$

- Pion isospin splitting is $O(a^2)$ and maximal for $\omega = \pi/2$:

$$
\begin{aligned}
m^2_{\pi_3} - m^2_{\pi_{1,2}} &= -\frac{32}{f^2}\hat{a}^2s^2W' + O(a^3) \\
&= -W'\frac{32}{f^2}\frac{\hat{a}^2\mu^2}{m'^2 + \mu^2} + O(a^3)
\end{aligned}
$$

  - Harder to calculate numerically as requires quark-disconnected contractions
  - Measures constant W'

Specific NLO results: matrix elements

- Pion decay constant agrees with [Münster & Schmidt]

$$f_A \quad = \quad f\left\{1+\frac{4}{f^2}\left[2|\chi'|L_{45}+\hat{a}c(2\tilde{W}+W_{10})\right]+\text{cont. 1-loop}\right\}$$

  - flavor breaking only at NNLO

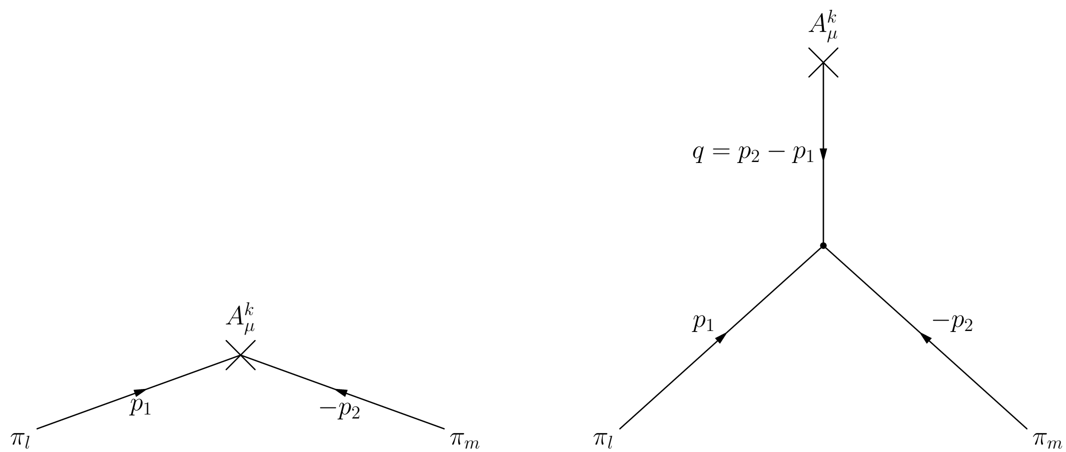
- Results for $\langle 0|P|\pi\rangle$, scalar and vector form factors, have similar form showing automatic $O(a)$ improvement and no flavor breaking
- can measure physical condensate using P^3

$$\langle 2iP^3\rangle \quad = \quad -2f^2B_0\; s\left\{1+\frac{4}{f^2}\left[|\chi'|(8L_{68}+H_2)+\hat{a}c(4W+W_{10})\right]+\text{1-loop}\right\}$$

Parity violating matrix elements

Axial and pseudoscalar form factors of pion non-vanishing, e.g.

$$\langle \pi_a | \hat{A}^a_\mu | \pi_3 \rangle \,, \ \langle \pi_a | \hat{A}^3_\mu | \pi_a \rangle \,, \ \langle \pi_3 | \hat{A}^3_\mu | \pi_3 \rangle \,,$$



Example of results:

$$\langle \pi_a(p_2) | \hat{P}^3 | \pi_a(p_1) \rangle \quad = \quad \frac{16 \hat{a} s i B_0}{f^2} \left[\frac{-W_{10}}{4} + W - \widetilde{W} + \frac{2 \hat{a} c W'}{q^2 + m^2_{\pi_3}} + \frac{(\widetilde{W}/2 - W) q^2}{q^2 + m^2_{\pi_3}} \right]$$

- present at maximal twisting $s = 1$, $c = 0$
- use to determine all W's and then to test tmχPT at NLO

Summary for $m_q \sim a\Lambda_{QCD}^2$

- Predicted functional forms for pionic quantities for all μ, m'
- Can determine maximal twisting non-perturbatively using $\omega_A = \pi/2$ or $\omega_P = \pi/2$ (or by maximizing pion mass splitting)
  - automatically includes $O(a)$ shift in m_c
  - $O(a)$ ambiguity in ω cannot be avoided
- Automatic $O(a)$ improvement at maximal twist holds in GSM regime
- Parity-flavor violating quantities ($\omega_A - \omega_P$, axial and pseudoscalar form factors) are $O(a)$
  - provide measure of discretization errors, i.e. size of W's
  - provide tests of tmχPT at NLO
  - can correct *a posteriori* $O(a)$ errors in untwisted simulations
- Flavor breaking in physical quantities occurs at $O(a^2)$
  - only example in NLO calculation is pion mass splitting

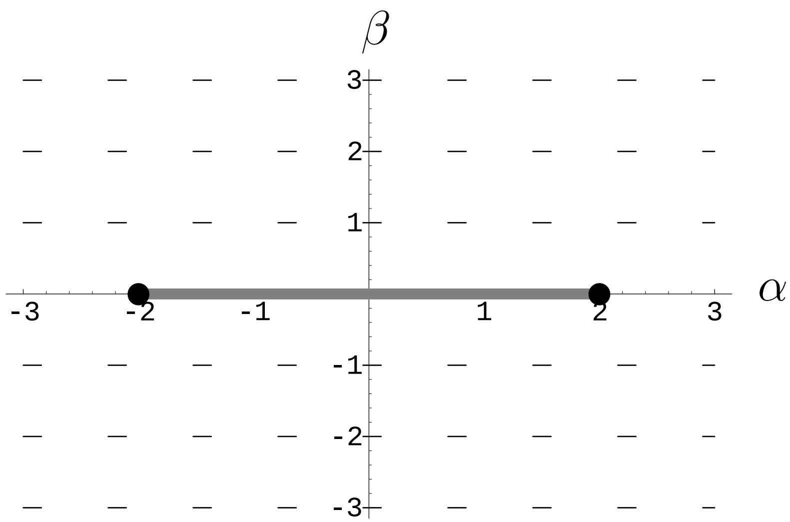
Results for Aoki regime

- Work to LO in power counting $m_q \sim a^2 \Lambda^3_{QCD}$
- Lagrangian collapses to:

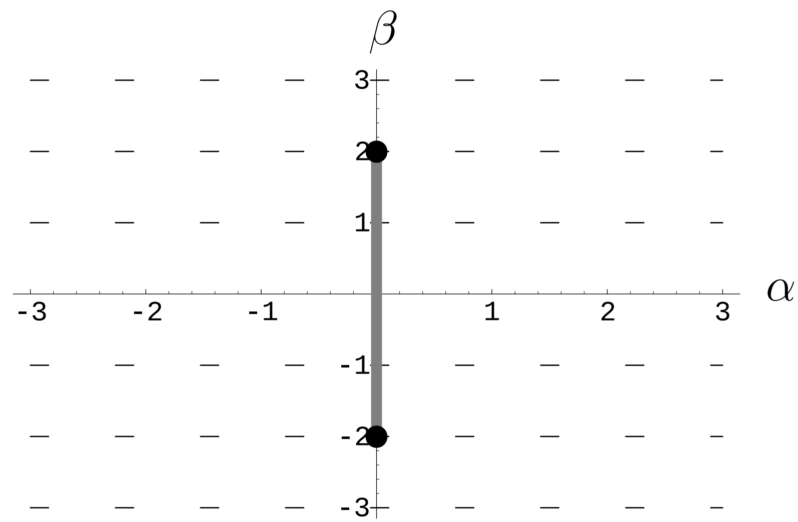
$$\begin{aligned}\mathcal{L}_\chi &= \frac{f^2}{4}\mathrm{Tr}(D_\mu \Sigma D_\mu \Sigma^\dagger) - \frac{f^2}{4}\mathrm{Tr}(\chi'^\dagger \Sigma + \Sigma^\dagger \chi') - W' \left[\mathrm{Tr}(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A})\right]^2 \\ &+ \tilde{W}\ \mathrm{Tr}(D_\mu \Sigma^\dagger D_\mu \Sigma)\mathrm{Tr}(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A}) - W\ \mathrm{Tr}(\chi'^\dagger \Sigma + \Sigma^\dagger \chi')\mathrm{Tr}(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A}) \\ &+ W_{10}\ \mathrm{Tr}(D_\mu \hat{A}^\dagger D_\mu \Sigma + D_\mu \Sigma^\dagger D_\mu \hat{A}) + O(a^3)\end{aligned}$$

- Competition between m_q and a^2 terms leads to non-trivial phase structure [Aoki, Creutz, SRS & Singleton, Münster, Scorzato, SRS & Wu]
- Phase structure depends on sign of W' (which also determines the sign of pion mass splitting [Scorzato])
- Can extend calculations from GSM regime into Aoki regime

Possible phase diagrams in Aoki regime



(a) Phase diagram for $c_2 > 0$



(b) Phase diagram for $c_2 < 0$

$$c_2 = -16W'\hat{a}^2\,, \quad \alpha = 2B_0 f^2 m'/|c_2|\,, \quad \beta = 2B_0 f^2 \mu/|c_2|$$

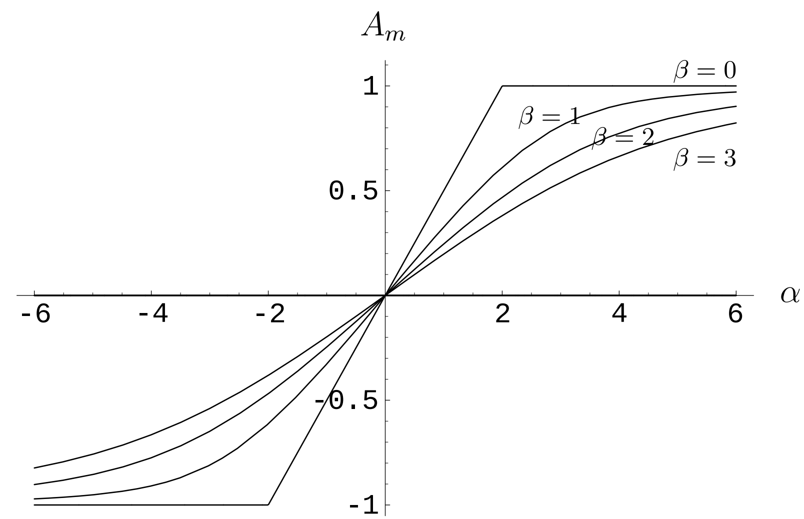
- Solid lines are first-order phase transistions with second-order endpoints

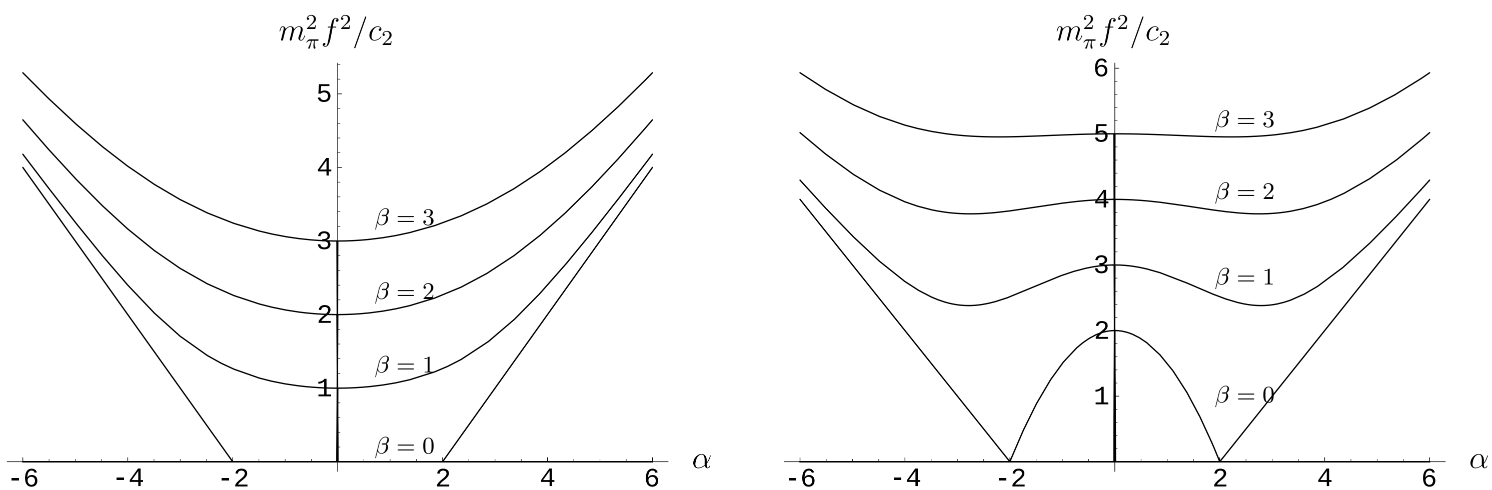
$W' < 0$: Aoki phase

- Condensate:

  $\langle 0|\Sigma|0\rangle = A_m + iB_m\tau_3$

- Aoki phase washed out for $\mu \propto \beta \neq 0$
- Note $A_m = 0$ for $\alpha = m' = 0$



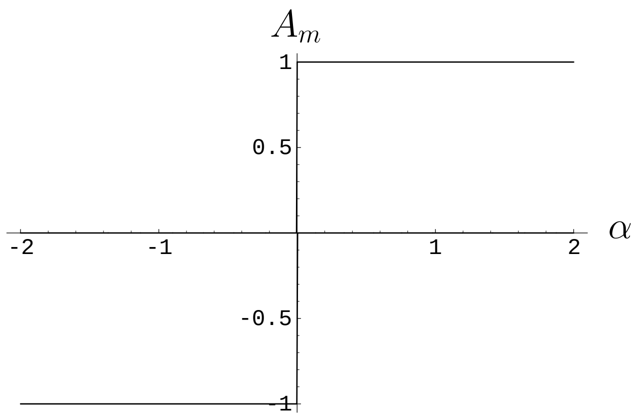


(a) Mass of π_1 and π_2

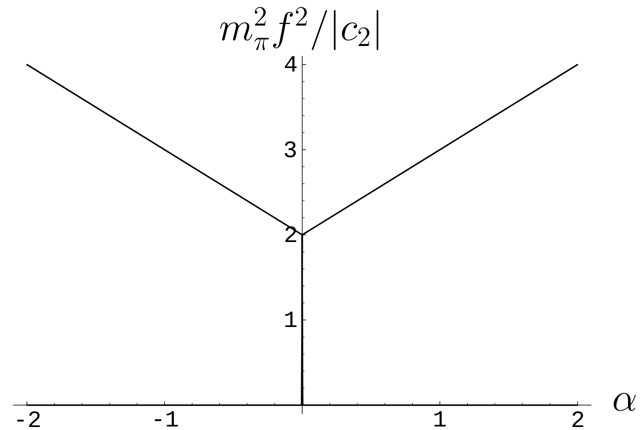
(b) Mass of π_3

$W' > 0$: no Aoki phase

Along Wilson axis:

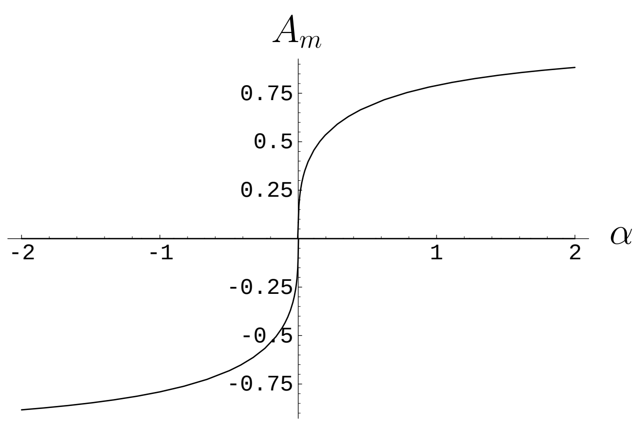


(a) Global minimum, $\beta = 0$

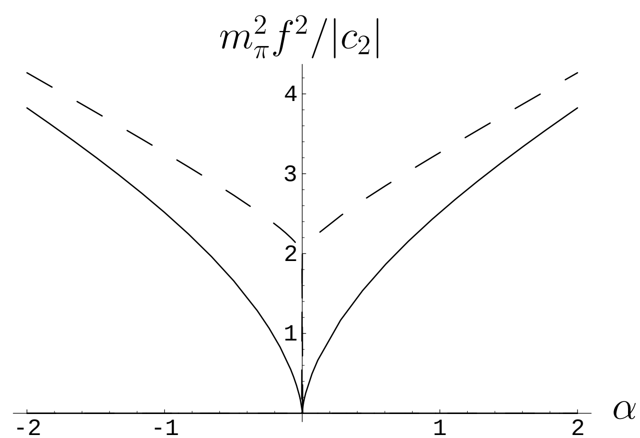


(b) Pion masses, $\beta = 0$

At top of phase transition: dashed: charged; solid: neutral



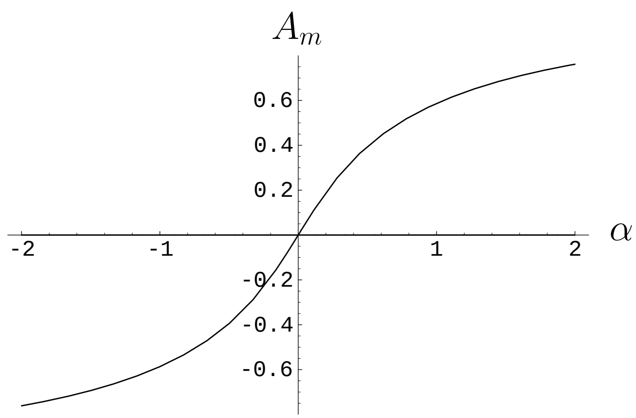
(e) Global minimum, $\beta = 2$



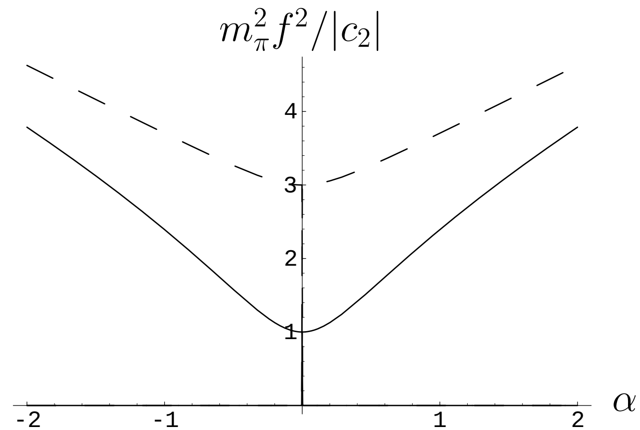
(f) Pion masses, $\beta = 2$

More on $W' > 0$

Above phase transition: dashed: charged; solid: neutral



(g) Global minimum, $\beta = 3$

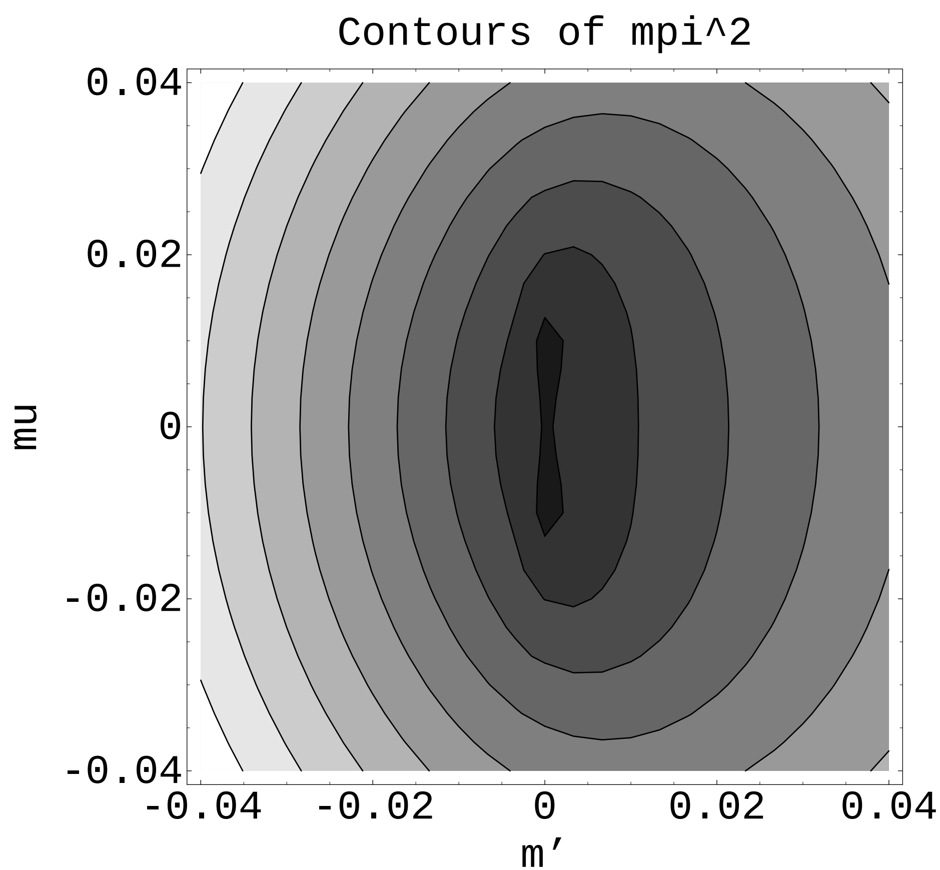


(h) Pion masses, $\beta = 3$

- Can use minimum of pion masses as an alternative for determining where $m' = 0$
- Away from transition, condensate has $A_m = 0$ for maximal twisting, i.e. lies along direction of quark mass as in continuum

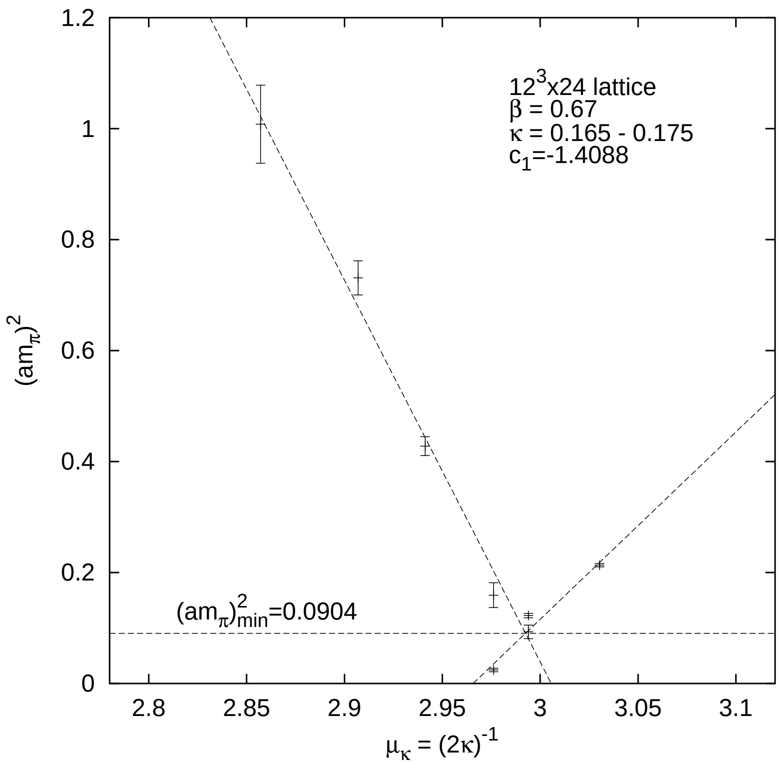
$\Rightarrow$ Automatic $O(a)$ improvement at maximal twist still holds in Aoki regime away from phase transitions [Aoki & Bär]

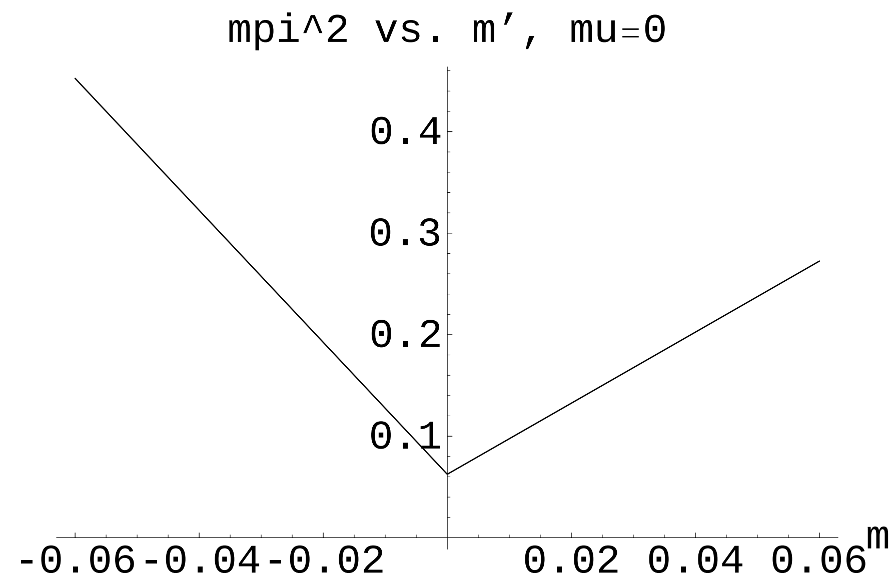
Predictions from tmχPT



Contour plot of charged m_π^2 from tmχPT with parameters roughly tuned to match those of hep-lat/0410031, Farchioni *et al.*

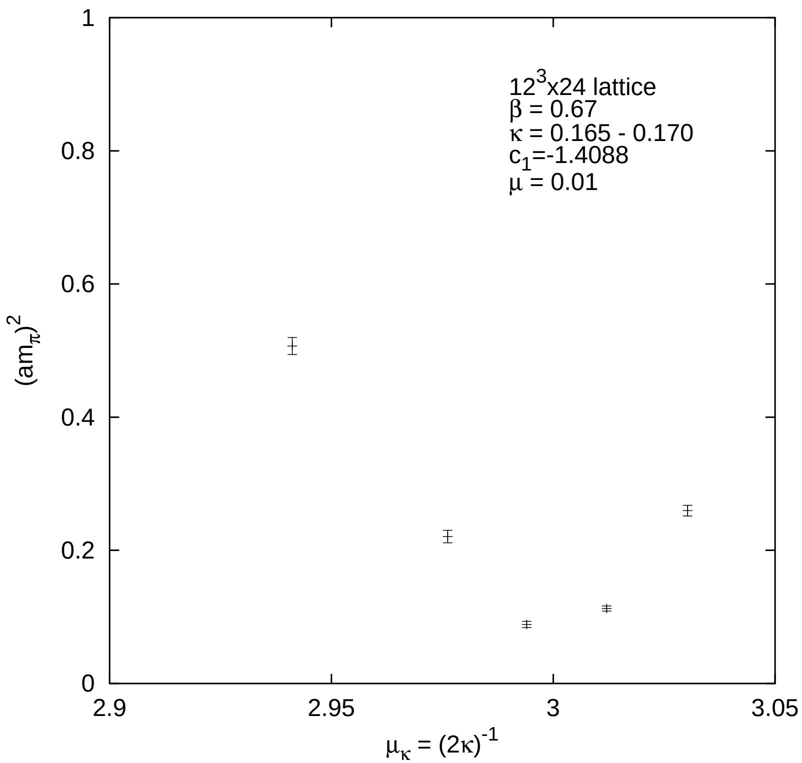
Comparing χPT with [Farchioni *et al*,hep-lat/0410031]

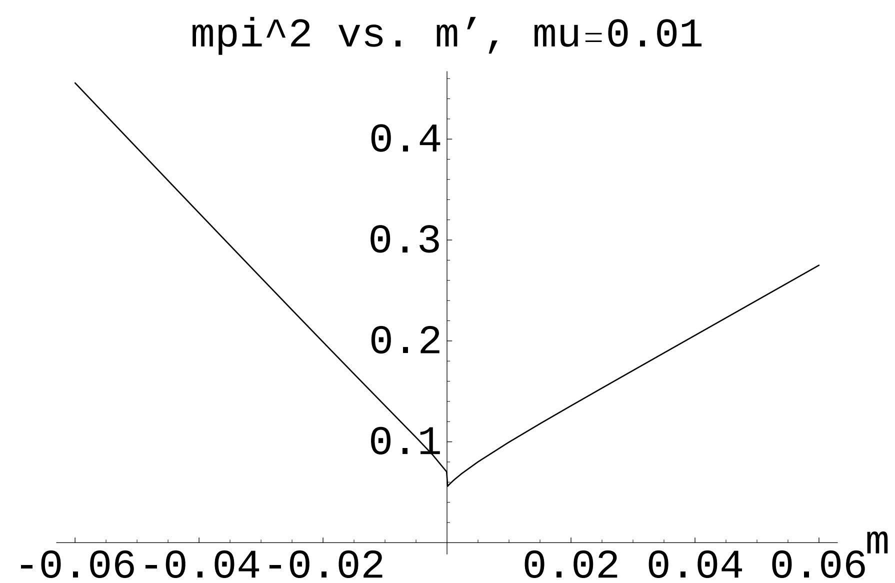




- Qualitative comparison only
- Difference in slopes for positive and negative m' caused by a 30% $\hat{a}(2W - \tilde{W})\cos(\omega_A)$ correction

Comparing χPT with [Farchioni *et al*,hep-lat/0410031]





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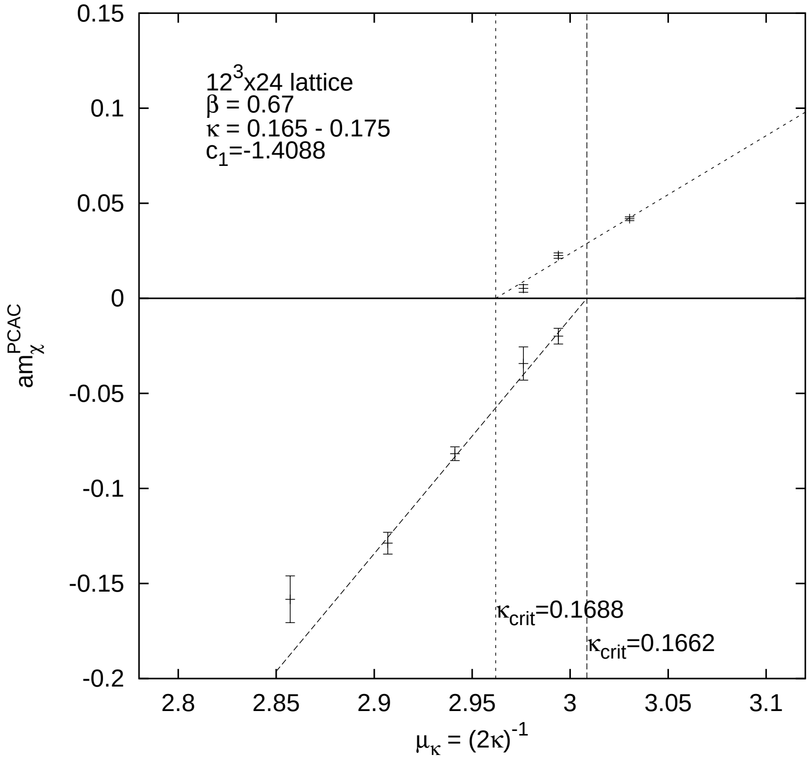
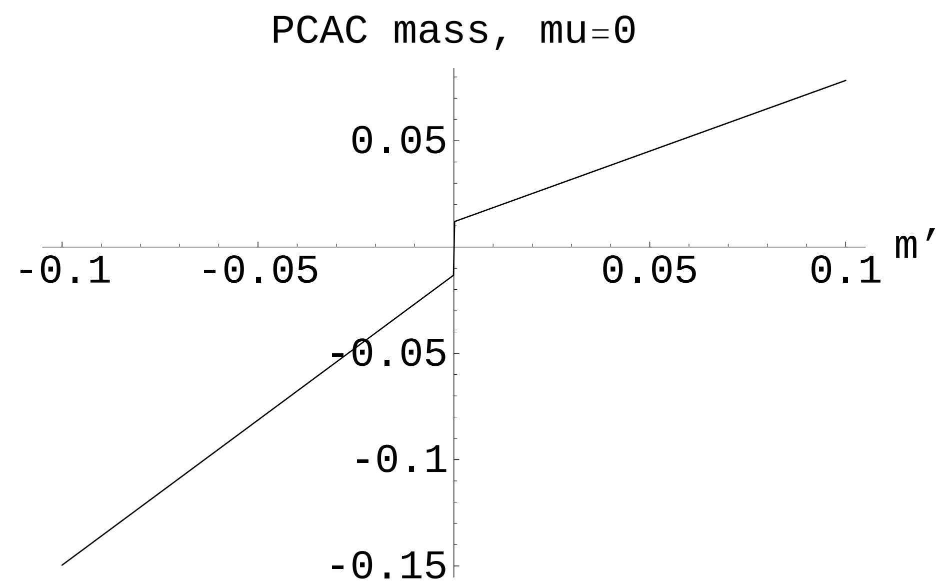
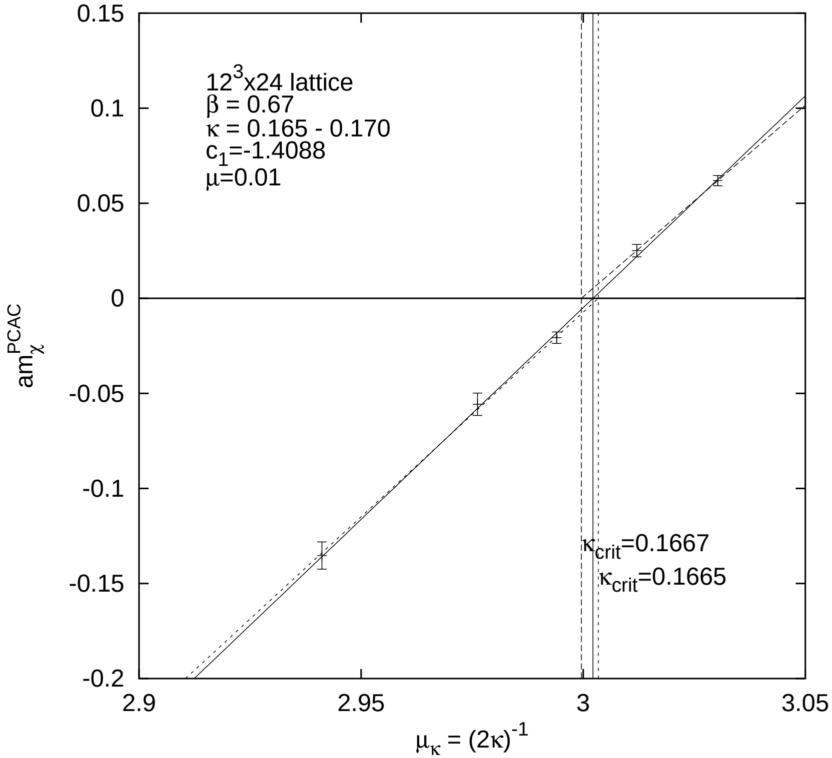
Comparing χPT with [Farchioni *et al*,hep-lat/0410031]

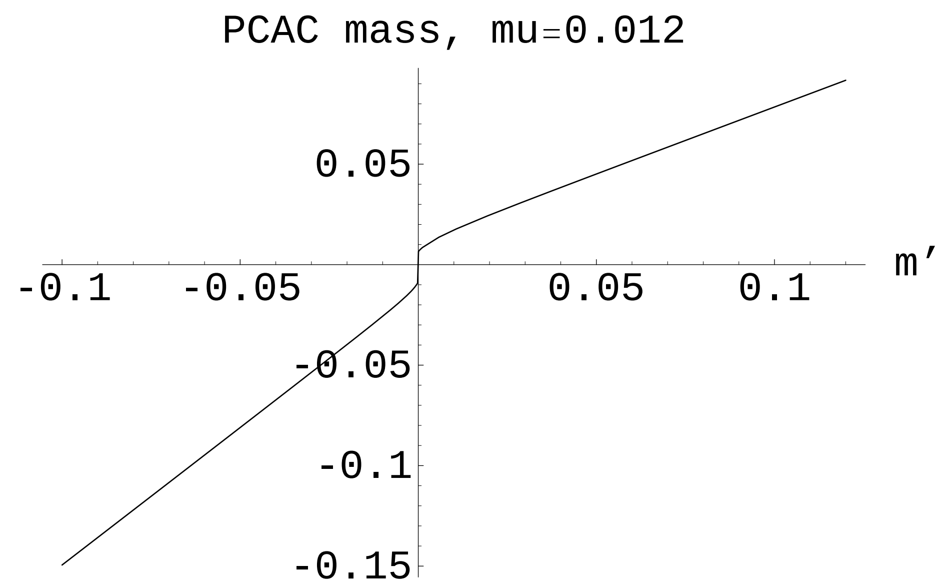




- Qualitative comparison only
- Difference in slopes for positive and negative m' caused by $\hat{a}\cos(\omega_A)$ correction

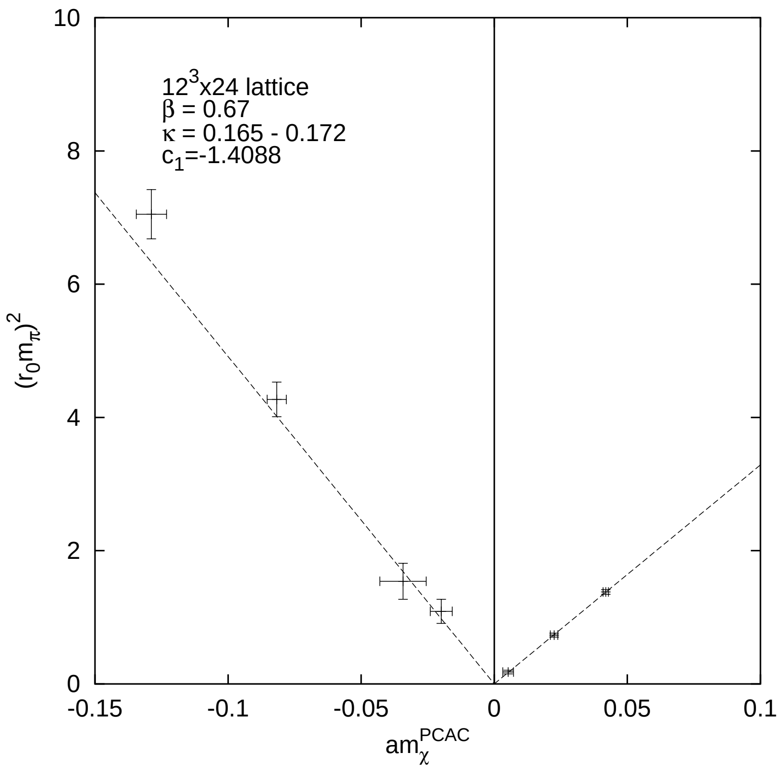
Comparing χPT with [Farchioni *et al*,hep-lat/0410031]

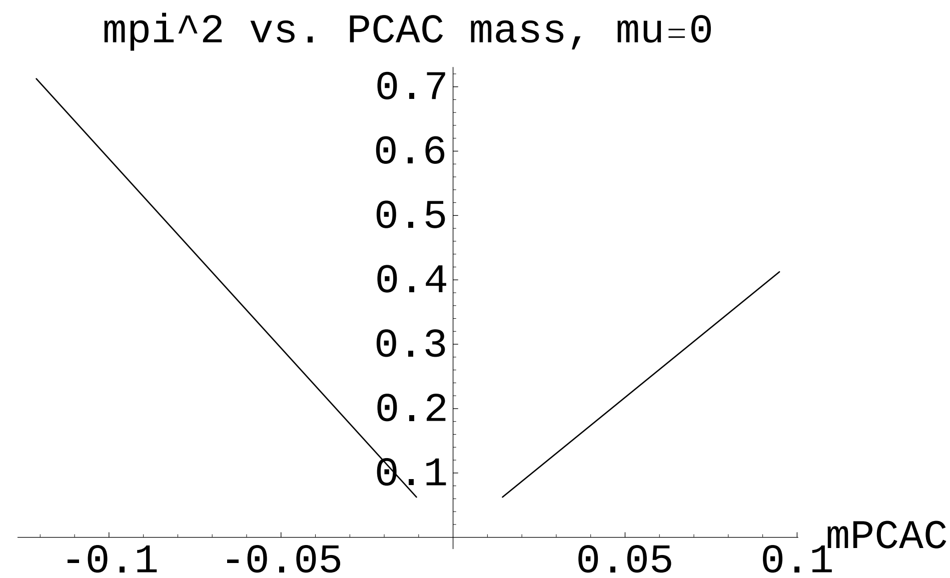




- χPT does *not* explain the change in relative slopes as μ increases

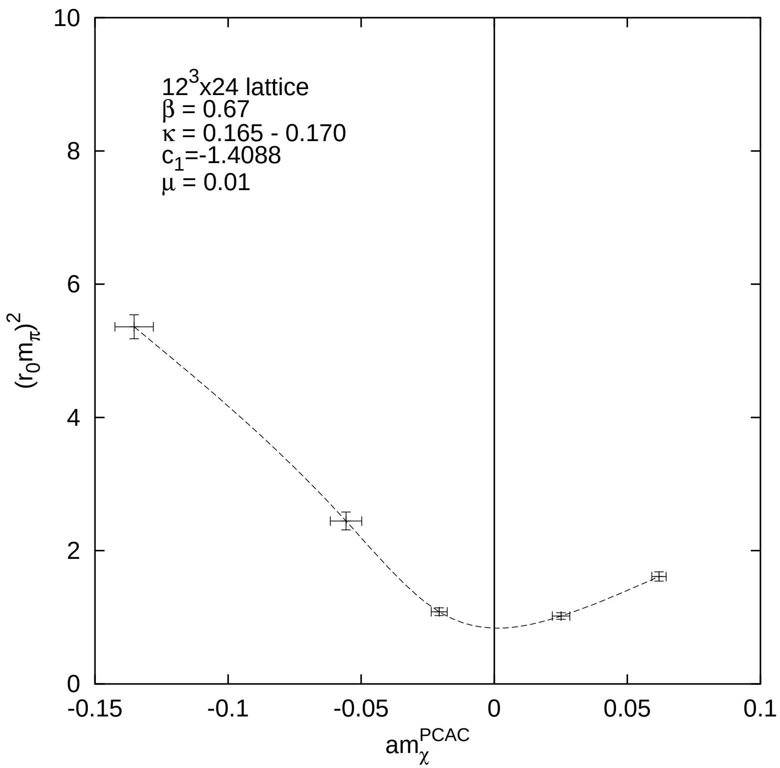
Comparing χPT with [Farchioni *et al*,hep-lat/0410031]

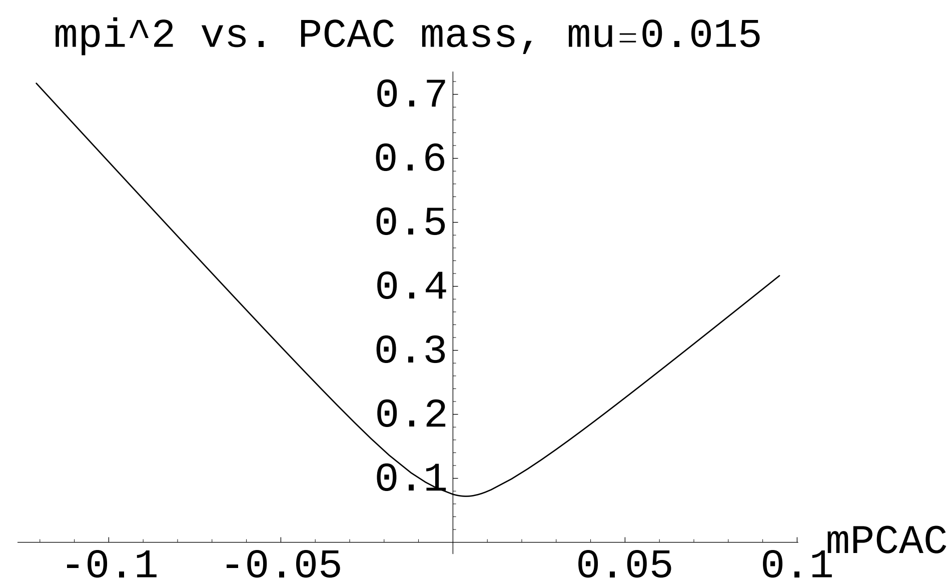




- Data goes to lighter pion mass because of metastability

Comparing χPT with [Farchioni *et al*,hep-lat/0410031]





- χPT curve is above phase boundary so no minimum m_{PCAC}

Summary and outlook

- Automatic $O(a)$ improvement works for $m_q \geq c' a^2 \Lambda^3_{QCD}$
  - But essential to use an appropriate definition of twisting angle
- Parity-flavor violating quantities provide interesting window on theory, and will allow test of our understanding
- Flavor violation in most parity conserving quantities requires NNLO calculation (underway)
- Many interesting quantities have disconnected contractions so hard to calculate
  - Use partially quenched tm χPT to separate connected and disconnected contractions?
- tmLQCD is a potential competitor to improved staggered fermions matching its advantages but without its major drawback and merits intensive study