0.

A Lattice Formulation of Super Yang - Mills Theories with Exact Supersymmetry

F. Sugino

(Okayama Institute for Quantum Physics)

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1.

§1. Introduction

Understanding nonperturbative properties of SYM theories is interesting from various points of view :

- Field Theory (beyond standard model)
- String Theory { · gauge - gravity dual
  · Matrix String Models
  ⋮

⊙ Lattice Formulation of SYM

$$(\text{SUSY}) \sim (\underset{\uparrow}{\text{infinitesimal}} \text{ translation})^{1/2}$$

SUSY parameters are Grassmann.

However, infinitesimal translation is not a symmetry of lattice.

- Accidental SUSY in continuum limit — Kaplan, Nishimura, Neuberger

4D $\mathcal{N}=1$ SU(N) (A_μ, λ)

$$(\text{YM}) + \begin{pmatrix}\text{massless}\\ \text{Majorana}\\ \text{fermion}\end{pmatrix} \xrightarrow[\text{limit}]{\text{continuum}} \text{SUSY enhanced}$$

[The discrete chiral Z_{2N} symmetry forbids the]

· Some of SUSY (not all) can be kept on lattice as a "fermionic internal symmetry".

SYM with higher SUSY $\xleftarrow{\text{dim. red.}}$ $\mathcal{N}=1$ SYM in higher dim.

SUSY corresponding to translation in the direction dimensionally reduced becomes "internal symmetry". (not generate translation)

We expect

(some SUSY manifest on lattice) $\xrightarrow{\text{continuum limit}}$ Full SUSY is restored unless relevant or marginal operators not inv. under the Full SUSY are generated by radiative corrections.

Along this scinario, (Cohen)-Kaplan-Katz-Unsal constructed various lattice models of SYM with higher SUSY using an idea of deconstruction.

(0D or 1D reduced SYM with orbifolding conditions) ⟶ (SYM on higher dim. lattice)

$\boxed{\text{expand around } (\text{Higgs VEV}) = \left(\text{lattice spacing}\right)^{-1}}$

When fixing the Higgs VEV, we must introduce SUSY breaking terms,

Here, we will present a new lattice formulation of SYM theories based on topological field theory formulation using unitary compact variables on hypercubic lattice.

Gauge Group $SU(N)$

Lattice Sites $x \in \mathbb{Z}^d$

c.f.

· Catterall

Latticization of SUSY theories without gauge sym. based on topological field theory formulation

· Cohen - Kaplan - Katz - Ünsal

Latticization of SYM theories by using deconstruction technique

· D'Adda - Kanamori - Kawamoto - Nagata

Noncommutative twisted SUSY

· Catterall

Kähler-Dirac Formulation of $\mathcal{N}=2$ 2D SYM

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Plan of Talk

§2. "Topological Field Theory Form" of $\mathcal{N}=2$ SYM

§3. Lattice Formulation for $\mathcal{N}=2$

§4. "Balanced Topological Field Theory Form" of $\mathcal{N}=4$ SYM

§5. Lattice Formulation for $\mathcal{N}=4$

§6. 3D $\mathcal{N}=4, 8$ and 2D $\mathcal{N}=8$

§7. Discussions

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§2. "Topological Field Theory Form" of $\mathcal{N}=2$ SYM

4D $\mathcal{N}=1$ (Minkowski) $(M, N = 0, 1, 2, 3)$

$$S^{(M)} = \frac{1}{g^2}\int d^4x \, \mathrm{tr}\left[-\frac{1}{4} F_{MN} F^{MN} - \frac{i}{2} \bar{\Psi} \Gamma^M D_M \Psi\right]$$

Majorana repre. $\begin{cases} \Gamma^0 & \text{imaginary anti-sym.} \\ \Gamma^i & \text{imaginary sym.} \quad (i = 1, 2, 3) \end{cases}$

$\Psi = \Psi^a T^a$

$\bar{\Psi} = \Psi^T \Gamma^0$

(T^a: generators of gauge group; Ψ^a: real Grassmann valued spinor)

Lorentz Boost along x^i : $\Psi^a \to e^{\Gamma^0 \Gamma^i \frac{\theta}{2}} \Psi^a$

$\boxed{\text{Wick Rotation } (x^0 \to -i x_4 , \quad \Gamma_4 \equiv -i \Gamma_0)}$

Euclidean Action

$$S^{(E)} = \frac{1}{g^2}\int d^4x \, \mathrm{tr}\left[\frac{1}{4} F_{MN} F_{MN} + \frac{i}{2} \bar{\Psi} \Gamma_M D_M \Psi\right]$$

Rotation in $(i, 4)$-plane : $\Psi^a \to e^{-\Gamma_i \Gamma_4 \frac{\theta}{2}} \Psi^a$

Note: The rotation does not keep Ψ^a real.

We may regard Ψ^a as $\underbrace{\text{complexified Grassmann}}_{\mathbb{C} \times (\text{real Grassmann})}$

in expressions for correlation function or partition function

with $\bar{\Psi} = \Psi^T \Gamma^0$

Euclidean 4D $\mathcal{N}=1$ 5.

[dim. red. (3,4)-directions] $\longrightarrow$ Euclidean 2D $\mathcal{N}=2$ SYM

Rotation in (3,4)-plane $\longrightarrow$ $U(1)_R$ internal symmetry (does not keep Ψ^a real)

SUSY $\left(\gamma_i \equiv -i\Gamma^0\Gamma_i, \quad \gamma_4 \equiv \mathbb{1} \right)$

$$\begin{cases} \delta A_M = \epsilon^T \gamma_M \Psi \\ \delta \Psi = \frac{1}{2}\left(-F_{ij}\gamma_{ij} - 2F_{i4}\gamma_i \right)\epsilon \end{cases}$$

We can rewrite the Action of 2D $\mathcal{N}=2$ to "Topological Field Theory (TFT) Form" : Witten

$$S^{(E)}_{2D\mathcal{N}=2} = Q\frac{1}{2g^2}\int d^2x \; \mathrm{tr}\Big[\frac{1}{4}\eta[\phi,\bar\phi] - i\chi\Phi + \chi H - i\psi_\mu D_\mu \bar\phi \Big]$$

where $\Phi \equiv 2F_{12}$ $(\mu = 1,2)$

ghost # (= $U(1)_R$ charge)	field	statistics	
2	ϕ	B	$\phi = A_3 + iA_4$
1	ψ_μ	F	
0	A_μ, H	B	H : auxiliary field
-1	χ, η	F	
-2	$\bar\phi$	B	$\bar\phi = A_3 - iA_4$

6.

Q : Nilpotent SUSY (up to gauge tr. generated by ϕ)

$$Q A_\mu = \psi_\mu, \qquad Q\psi_\mu = iD_\mu\phi$$

$$Q\phi = 0$$

$$Q\chi = H, \qquad QH = [\phi,\chi]$$

$$Q\bar\phi = \eta, \qquad Q\eta = [\phi,\bar\phi]$$

$$\Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \chi \\ \frac{1}{2}\eta \end{bmatrix}, \qquad \begin{aligned} \gamma_1 &= -i\sigma_1\otimes\sigma_1 \\ \gamma_2 &= i\sigma_1\otimes\sigma_3 \\ \gamma_3 &= i\sigma_3\otimes\mathbb{1}_2 \end{aligned}$$

Q corresponds to the SUSY tr. δ as

$$\delta = i\varepsilon Q, \quad \text{with} \quad \epsilon = -\varepsilon\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \quad (\varepsilon : \text{Grassmann})$$

<u>Remark</u> Here, we regard the "TFT Form" as just renaming of fields in $\mathcal{N}=2$ SYM.

(e.g.) Index μ of ψ_μ does not respect the Lorentz tr. property.

· The similar transcription is possible for 4D $\mathcal{N}=2$, 8D $\mathcal{N}=2$.

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Note

$$\epsilon_1 = -\varepsilon_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \epsilon_2 = -\varepsilon_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$(\mathrm{SUSY})^2 \sim$ (translation with the parameter $\epsilon_1^T \gamma_M \epsilon_2$)

$$\left.\begin{array}{l} \epsilon_1^T \gamma_1 \epsilon_2 = 0 \\ \epsilon_1^T \gamma_2 \epsilon_2 = 0 \end{array}\right\} \rightarrow$$ No translation along $\hat{1}$ and $\hat{2}$

$$\left.\begin{array}{l} \epsilon_1^T \gamma_3 \epsilon_2 = -i\varepsilon_1\varepsilon_2 \\ \epsilon_1^T \gamma_4 \epsilon_2 = \varepsilon_1\varepsilon_2 \end{array}\right\} \rightarrow$$ Dimensionally reduced directions

7.

§3. Lattice Formulation for $\mathcal{N}=2$

2D $\mathcal{N}=2$ case
SU(N)

$U_\mu(x) = e^{iaA_\mu(x)}$

SU(N) link variables on x —— $x+\hat{\mu}$

All the other fields are put on sites.

- It is possible to extend the SUSY Q so that $Q^2 =$ (gauge tr. generated by ϕ) holds on lattice:

$$Q\,U_\mu(x) = i\psi_\mu(x)\,U_\mu(x)$$

$$Q\,\psi_\mu(x) = \underline{i\psi_\mu(x)\psi_\mu(x)} - i\left(\phi(x) - U_\mu(x)\phi(x+\hat{\mu})U_\mu(x)^\dagger\right)$$

$$\left.\begin{array}{ll} Q\,\phi(x) = 0 & \\ Q\,\chi(x) = H(x), & Q\,H(x) = [\phi(x), \chi(x)] \\ Q\,\bar{\phi}(x) = \eta(x), & Q\,\eta(x) = [\phi(x), \bar{\phi}(x)] \end{array}\right\}$$

same form as in the continuum

$$\left.\begin{array}{l} \phi,\ \bar{\phi} = O(a) \\ (\text{fermions}) = O(a^{3/2}) \\ H = O(a^2) \end{array}\right\} \rightarrow$$ $\sim\sim$ is subleading.

$$Q = O(a^{1/2})$$

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$$Q U_\mu(x) = i\psi_\mu(x) U_\mu(x)$$

$$\downarrow$$

$$Q^2 U_\mu(x) = i(Q\psi_\mu(x))U_\mu(x) \underbrace{- i\psi_\mu(x)(QU_\mu(x))}_{\substack{\| \\ +\psi_\mu(x)\psi_\mu(x)U_\mu(x)}}$$

$$= \phi(x)U_\mu(x) - U_\mu(x)\phi(x+\hat\mu)$$

$$\therefore \quad Q\psi_\mu(x) = i\psi_\mu(x)\psi_\mu(x) - i\left(\phi(x) - U_\mu(x)\phi(x+\hat\mu)U_\mu(x)^\dagger\right)$$

$$\left(\begin{aligned} \cdot\ \psi_\mu(x) &= \sum_A \psi_\mu^A(x)\, T^A \\ \psi_\mu(x)\psi_\mu(x) &= \frac{1}{2}\sum_{A,B}\psi_\mu^A(x)\psi_\mu^B(x)\,[T^A, T^B] \\ &= \frac{i}{2} f_{ABC}\,\psi_\mu^A(x)\psi_\mu^B(x)\, T^C \neq 0 \end{aligned} \right)$$

8.

$\rightarrow$ Lattice Action with Exact SUSY Q is

$$S_{2D\,N=2} = Q\,\frac{1}{2g_0^2}\sum_x \mathrm{tr}\Big[\frac{1}{4}\eta(x)[\phi(x), \bar\phi(x)]$$

$$-i\chi(x)\left(\Phi(x) + \Delta\Phi(x)\right) + \chi(x)H(x)$$

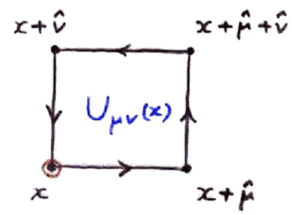
$$+i\sum_\mu \psi_\mu(x)\left(\bar\phi(x) - U_\mu(x)\bar\phi(x+\hat\mu)U_\mu(x)^\dagger\right)\Big]$$

where

$$\Phi(x) = -i\left(U_{12}(x) - U_{21}(x)\right)$$

$$\Delta\Phi(x) = -\underset{\substack{\uparrow \\ \text{constant}}}{r}\left(2 - U_{12}(x) - U_{21}(x)\right)$$

$$U_{\mu\nu}(x) = U_\mu(x)U_\nu(x+\hat\mu)U_\mu(x+\hat\nu)^\dagger U_\nu(x)^\dagger$$



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$$S_{2D\,\mathcal{N}=2} = \frac{1}{2g_0^2}\sum_x \mathrm{tr}\Big[\frac{1}{4}[\phi(x),\bar\phi(x)]^2 + \underbrace{H(x)^2 - iH(x)\big(\Phi(x)+\Delta\Phi(x)\big)}_{\sim\sim\sim}$$

$$+\sum_\mu\big(\phi(x) - U_\mu(x)\phi(x+\hat\mu)U_\mu(x)^\dagger\big)\big(\bar\phi(x) - U_\mu(x)\bar\phi(x+\hat\mu)U_\mu(x)^\dagger\big)$$

$$-\frac{1}{4}\eta(x)[\phi(x),\eta(x)] - \chi(x)[\phi(x),\chi(x)]$$

$$+i\chi(x)\big(Q\Phi(x) + Q\Delta\Phi(x)\big)$$

$$-\sum_\mu\psi_\mu(x)\psi_\mu(x)\big(\bar\phi(x) + U_\mu(x)\bar\phi(x+\hat\mu)U_\mu(x)^\dagger\big)$$

$$-i\sum_\mu\psi_\mu(x)\big(\eta(x) - U_\mu(x)\eta(x+\hat\mu)U_\mu(x)^\dagger\big)\Big]$$

Note

• $\sim\sim\sim \;\to\; \mathrm{tr}\,(\Phi(x)+\Delta\Phi(x))^2$

In the case without $\Delta\Phi(x)$,

$$\mathrm{tr}\,\Phi(x)^2 = \mathrm{tr}\big[2 - U_{12}(x)^2 - U_{21}(x)^2\big]$$

Problem of the degenerate minima

$$U_{12}(x) = \begin{pmatrix} \pm1 & & \\ & \ddots & \\ & & \pm1\end{pmatrix}$$ (up to gauge tr.)

(# of (−1)) = even

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↓

Expansion around a single minimum $U_{12}(x) = 1$ is not justified.

↓

Connection to the desired continuum theory becomes unclear.

• $\Delta\Phi(x)$ resolves the problem keeping the SUSY Q.

Consider

$$\Delta\Phi(x) = -r\big(2 - U_{12}(x) - U_{21}(x)\big)$$

$r = \cot\varphi$ with $e^{i2\ell\varphi} \neq 1$ for $\forall \ell = 1,\cdots,N$.

(One choice of φ is $0 < \varphi \le \frac{\pi}{2N}$.)

For this $\Delta\Phi(x)$,

$$\Phi(x) + \Delta\Phi(x) = 0 \;\to\; U_{12}(x) = 1.$$

$$\Phi(x) + \Delta\Phi(x) = \frac{-1}{\sin\varphi}\left[e^{-i\varphi}(1-U_{12}(x)) + e^{i\varphi}(1-U_{21}(x))\right]$$

$\Phi(x) + \Delta\Phi(x) = 0 \;\Rightarrow\; U_{12}(x) = 1 - ie^{i\varphi}\, T(x)$

$T(x)$: hermitian

$U_{12}(x)\, U_{12}(x)^{\dagger} = 1 \;\Rightarrow\;$ $t_i(x)$ (eigenvalues of $T(x)$) are

$t_i(x) = 0$ or $-2\sin\varphi$

$(i = 1, \cdots, N)$

$\det U_{12}(x) = 1 \;\Rightarrow\;$ For

$$\{t_i(x)\} = \left\{ \begin{array}{l} \left.\begin{array}{c} 0 \\ \vdots \\ 0 \end{array}\right\} N-\ell \\ \left.\begin{array}{c} -2\sin\varphi \\ \vdots \\ -2\sin\varphi \end{array}\right\} \ell \end{array}\right. ,$$

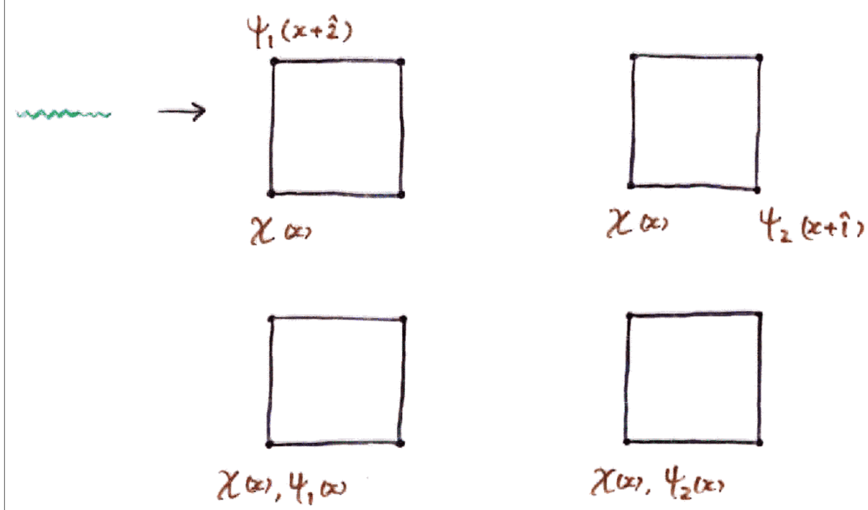
$$e^{i2\ell\varphi} = 1 .$$

$\therefore$ With the choice of φ such that

$$e^{i2\ell\varphi} \neq 1 \quad \text{for } {}^{\forall}\ell = 1, \cdots, N$$

$\Phi(x) + \Delta\Phi(x) = 0$ has the unique solution

$$t_1(x) = \cdots = t_N(x) = 0 .$$



No discrete rotational symmetry

But, in the continuum limit, rotational symmetry is considered to be restored due to the renormalization argument as we will see.

Note In the naive continuum limit, $\Delta\Phi(x)$ leads only irrelevant interaction terms.

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Symmetry of the lattice action

- $SU(N)$ gauge symmetry
- lattice translation
- supersymmetry Q
- $U(1)_R$ rotation

$$U_\mu \to U_\mu, \quad \psi_\mu \to e^{i\theta}\psi_\mu$$
$$\phi \to e^{i2\theta}\phi, \quad \bar\phi \to e^{-i2\theta}\bar\phi$$
$$H \to H$$
$$\chi \to e^{-i\theta}\chi, \quad \eta \to e^{-i\theta}\eta$$

- $x \equiv (x_1, x_2) \to \tilde{x} \equiv (x_2, x_1)$ as

$$U_1(x) \to U_2(\tilde x), \quad \psi_1(x) \to \psi_2(\tilde x)$$
$$U_2(x) \to U_1(\tilde x), \quad \psi_2(x) \to \psi_1(\tilde x)$$
$$\phi(x) \to \phi(\tilde x), \quad \bar\phi(x) \to \bar\phi(\tilde x)$$
$$H(x) \to -H(\tilde x)$$
$$\chi(x) \to -\chi(\tilde x), \quad \eta(x) \to -\eta(\tilde x)$$

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Naive Continuum Limit $\left(a \to 0 \text{ with } \frac{1}{g^2} \equiv \frac{a^2}{g_0^2} \text{ fixed}\right)$ $\longrightarrow$ Continuum Action of 2D $\mathcal{N}=2$

- Bosonic sector has no doubling.

$\downarrow$ Exact SUSY Q of $S_{2D\,\mathcal{N}=2}$

No fermion doubling

In fact, after the rescaling $\Psi \to a^{3/2}\Psi$, the fermion kinetic term becomes

$$S_f^{(2)} = \frac{a^4}{2g_0^2}\sum_{x,\mu} \mathrm{tr}\Big[-\frac{1}{2}\Psi(x)^T(P_\mu + \gamma_\mu)\Delta_\mu\Psi(x) + \frac{1}{2}\Psi(x)^T(P_\mu - \gamma_\mu)\Delta_\mu^*\Psi(x)\Big]$$

$$\begin{cases} \Delta_\mu f(x) \equiv \frac{1}{a}\left(f(x+\hat\mu) - f(x)\right) \\ \Delta_\mu^* f(x) \equiv \frac{1}{a}\left(f(x) - f(x-\hat\mu)\right) \end{cases}$$

$$P_1 \equiv \sigma_1 \otimes \sigma_2, \quad P_2 \equiv \sigma_2 \otimes \mathbb{1}_2$$
$$\{P_\mu, P_\nu\} = 2\delta_{\mu\nu}, \quad \{\gamma_\mu, P_\nu\} = 0$$

$$\rightarrow S_f^{(2)} = \frac{a^4}{2g_0^2}\sum_{x,\mu}\mathrm{tr}\Big[-\frac{1}{2}\Psi(x)^T\gamma_\mu(\Delta_\mu + \Delta_\mu^*)\Psi(x) \underbrace{- \frac{a}{2}\Psi(x)^T P_\mu \Delta_\mu\Delta_\mu^*\Psi(x)}\Big]$$

Similar structure to the Wilson term

This term removes doublers keeping $U(1)_R$ symmetry

12'

In the momentum space,

$$S_f^{(2)} = \frac{a^2}{2g_0^2}\int_{-\pi/a}^{\pi/a} \frac{d^2q}{(2\pi)^2}\, \mathrm{tr}\left[\tilde{\Psi}(-q)^T\, \tilde{D}(q)\, \tilde{\Psi}(q)\right]$$

$$\tilde{D}(q) = \frac{1}{a}\sum_\mu \left[-i\gamma_\mu \sin(q_\mu a) + 2P_\mu \sin^2\left(\frac{q_\mu a}{2}\right)\right]$$

$$\tilde{D}(q)^2 = \frac{1}{a^2}\sum_\mu \left[\sin^2(q_\mu a) + 4\sin^4\left(\frac{q_\mu a}{2}\right)\right]$$

$$\therefore\ \tilde{D}(q) = 0 \ \rightarrow\ (q_1, q_2) = (0, 0)$$

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But, $U(1)_R$ is chiral.

$\downarrow$

This model must break some assumptions in Nielsen-Ninomiya's no go Theorem:

1. Lattice translation symmetry
2. Lattice Dirac operator $\not{D}$ is local.
3. $\not{D}$ is hermitian
4. Some conserved charge concerning with fermion # Q_F exists.
5. Q_F is expressed as a sum of local charge densities
6. Q_F is quantized.

④ is broken.

Recombine as

$$\zeta = \frac{1}{\sqrt{2}}\begin{pmatrix}\psi_1 + i\psi_2 \\ \chi + i\frac{1}{2}\eta\end{pmatrix}, \quad \bar{\zeta} = \frac{1}{\sqrt{2}}\left(\psi_1 - i\psi_2,\ \chi - i\frac{1}{2}\eta\right)$$

$$U(1)_J : \begin{cases}\zeta \rightarrow e^{i\theta}\zeta \\ \bar{\zeta} \rightarrow e^{-i\theta}\bar{\zeta}\end{cases} \quad \text{corresponds to } Q_F.$$

$$\sim\sim = -\frac{a}{2}\left\{-\varepsilon_{\alpha\beta}\,\bar{\zeta}_\alpha\left(\Delta_1\Delta_1^* - i\Delta_2\Delta_2^*\right)\zeta_\beta\right.$$

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Renormalization

- Naive continuum limit $\rightarrow$ Full SUSY, Rotational Sym. } are restored.

Check whether relevant or marginal operators, which violate the symmetry restoration, are generated due to radiative corrections.

Consider around $g=0$. ($D \leq 4$ SYM : asymptotic free)

Mass dimension $[g^2] = 2$

Operator of the type $\varphi^a \partial^b \psi^{2c}$

$$\left(\begin{array}{l} \varphi : \text{ generic boson field} \\ \psi : \text{ generic fermion field} \\ \partial : \ x_\mu \text{ derivative} \end{array} \right) \qquad \begin{array}{l} [\varphi^a \partial^b \psi^{2c}] \\ = a + b + 3c \equiv p \end{array}$$

Possible Radiative Corrections $\qquad g^2 \leftrightarrow \hbar$

$$\left(\underset{\text{tree}}{\frac{a^{p-4}}{g^2}} + \underset{\text{1-loop}}{(\text{const})\, a^{p-2}} + \underset{\text{2-loop}}{(\text{const})' a^p g^2} + \cdots \right) \int d^D x\, \varphi^a \partial^b \psi^{2c}$$

(up to possible log-corrections.)

tree : $p \leq 4$ operators are relevant or marginal

1-loop : $p \leq 2$

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Operators with $p \leq 4$

p	$\varphi^a \partial^b \psi^{2c}$
0	1
1	φ
2	φ^2
3	φ^3, $\psi\psi$, $\varphi\partial\varphi$
4	φ^4, $\varphi^2\partial\varphi$, $(\partial\varphi)^2$, $\psi\partial\psi$, $\psi\psi\varphi$

$\rightarrow$ We need to check the operators of the type φ, φ^2 only.

Gauge Symmetry, $U(1)_R$ symmetry } $\rightarrow$ $\cancel{\mathrm{tr}\, \phi\bar{\phi}}$

$\uparrow$ Q supersymmetry

$\Downarrow$

No relevant or marginal operators except the id. are radiatively generated.

$\therefore$ In the continuum limit, full SUSY and rotational symmetry are considered to be restored without fine tuning.

16.

§4. "Balanced Topological Field Theory Form" of $\mathcal{N}=4$ SYM

$$2D\ \mathcal{N}=4 \xleftarrow{\text{dim. red.}} 6D\ \mathcal{N}=1$$

"Balanced Topological Field Theory (BTFT) Form" of 2D $\mathcal{N}=4$

Dijkgraaf - Moore

$$S^{(E)}_{2D\,\mathcal{N}=4} = Q_+ Q_- \frac{1}{2g^2}\int d^2x\, \mathrm{tr}\Big[-iB\Phi - \psi_{+\mu}\psi_{-\mu} - \chi_+\chi_- - \frac{1}{4}\eta_+\eta_- \Big]$$

where $\Phi \equiv 2F_{12}$ $(\mu = 1, 2)$

ghost # (= $U(1)_R$ charge)	field	statistics	
2	ϕ	B	$\phi = A_5 + iA_6$
1	$\psi_{+\mu}, \chi_+, \eta_+$	F	
0	$A_\mu, B, C, \tilde{H}_\mu, H$	B	$B = A_3,\ C = 2A_4$
-1	$\psi_{-\mu}, \chi_-, \eta_-$	F	
-2	$\bar{\phi}$	B	$\bar{\phi} = A_5 - iA_6$

$\tilde{H}_\mu$, H : auxiliary fields

11.

Nilpotent SUSY $Q_\pm$

$$\begin{cases} Q_+^2 = (\text{gauge tr. generated } \phi) \\ Q_-^2 = (\text{gauge tr. generated } -\bar{\phi}) \\ \{Q_+, Q_-\} = (\text{gauge tr. generated } C) \end{cases}$$

A_μ

$$Q_+ A_\mu = \psi_{+\mu}, \quad Q_+ \psi_{+\mu} = iD_\mu \phi$$

$$Q_- A_\mu = \psi_{-\mu}, \quad Q_- \psi_{-\mu} = -iD_\mu \bar{\phi}$$

$$Q_- \psi_{+\mu} = \frac{i}{2} D_\mu C - \tilde{H}_\mu$$

$$Q_+ \psi_{-\mu} = \frac{i}{2} D_\mu C + \tilde{H}_\mu$$

$$Q_+ \tilde{H}_\mu = [\phi, \psi_{-\mu}] - \frac{1}{2}[C, \psi_{+\mu}] - \frac{i}{2} D_\mu \eta_+$$

$$Q_- \tilde{H}_\mu = [\bar{\phi}, \psi_{+\mu}] + \frac{1}{2}[C, \psi_{-\mu}] + \frac{i}{2} D_\mu \eta_-$$

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(B)

$$Q_+ B = \chi_+ \;, \qquad Q_+ \chi_+ = [\phi, B]$$

$$Q_- B = \chi_- \;, \qquad Q_- \chi_- = -[\bar\phi, B]$$

$$Q_- \chi_+ = \frac{1}{2}[C, B] - H$$

$$Q_+ \chi_- = \frac{1}{2}[C, B] + H$$

$$Q_+ H = [\phi, \chi_-] + \frac{1}{2}[B, \eta_+] - \frac{1}{2}[C, \chi_+]$$

$$Q_- H = [\bar\phi, \chi_+] - \frac{1}{2}[B, \eta_-] + \frac{1}{2}[C, \chi_-]$$

(C)

$$Q_+ C = \eta_+ \;, \qquad Q_+ \eta_+ = [\phi, C]$$

$$Q_- C = \eta_- \;, \qquad Q_- \eta_- = -[\bar\phi, C]$$

$$Q_- \eta_+ = -[\phi, \bar\phi]$$

$$Q_+ \eta_- = [\phi, \bar\phi]$$

(ϕ)

$$Q_+ \phi = 0$$

$$Q_- \phi = -\eta_+$$

($\bar\phi$)

$$Q_+ \bar\phi = \eta_-$$

$$Q_- \bar\phi = 0$$

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· The similar transcription is possible for $4D\ \mathcal{N}=4$.

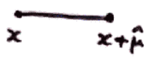
20.

§4. Lattice Formulation for $\mathcal{N}=4$

2D $\mathcal{N}=4$ case
SU(N)

$$U_\mu(x) = e^{iaA_\mu(x)}$$

on the link



All the other fields are on sites.

· Exact $Q_\pm$ symmetry on lattice

U_μ

$$Q_+ U_\mu(x) = i\psi_{+\mu}(x)\, U_\mu(x)$$

$$Q_- U_\mu(x) = i\psi_{-\mu}(x)\, U_\mu(x)$$

$$Q_+ \psi_{+\mu}(x) = i\psi_{+\mu}(x)\psi_{+\mu}(x) - i\left(\phi(x) - U_\mu(x)\,\phi(x+\hat{\mu})\,U_\mu(x)^\dagger\right)$$

$$Q_- \psi_{-\mu}(x) = i\psi_{-\mu}(x)\psi_{-\mu}(x) + i\left(\bar{\phi}(x) - U_\mu(x)\,\bar{\phi}(x+\hat{\mu})\,U_\mu(x)^\dagger\right)$$

$$Q_- \psi_{+\mu}(x) = \frac{i}{2}\left\{\psi_{+\mu}(x), \psi_{-\mu}(x)\right\} - \frac{i}{2}\left(C(x) - U_\mu(x)\,C(x+\hat{\mu})\,U_\mu(x)^\dagger\right) - \tilde{H}_\mu(x)$$

$$Q_+ \psi_{-\mu}(x) = \frac{i}{2}\left\{\psi_{+\mu}(x), \psi_{-\mu}(x)\right\} - \frac{i}{2}\left(C(x) - U_\mu(x)\,C(x+\hat{\mu})\,U_\mu(x)^\dagger\right) + \tilde{H}_\mu(x)$$

21.

$$\begin{aligned} Q_+ \tilde{H}_\mu(x) = &-\frac{1}{2}\left[\psi_{-\mu}(x),\ \phi(x) + U_\mu(x)\,\phi(x+\hat{\mu})\,U_\mu(x)^\dagger\right] \\ &+\frac{1}{4}\left[\psi_{+\mu}(x),\ C(x) + U_\mu(x)\,C(x+\hat{\mu})\,U_\mu(x)^\dagger\right] \\ &+\frac{i}{2}\left(\eta_+(x) - U_\mu(x)\,\eta_+(x+\hat{\mu})\,U_\mu(x)^\dagger\right) \\ &+\frac{i}{2}\left[\psi_{+\mu}(x),\ \tilde{H}_\mu(x)\right] \\ &+\frac{1}{4}\left[\psi_{+\mu}(x)\psi_{+\mu}(x),\ \psi_{-\mu}(x)\right] \end{aligned}$$

$$\begin{aligned} Q_- \tilde{H}_\mu(x) = &-\frac{1}{2}\left[\psi_{+\mu}(x),\ \bar{\phi}(x) + U_\mu(x)\,\bar{\phi}(x+\hat{\mu})\,U_\mu(x)^\dagger\right] \\ &-\frac{1}{4}\left[\psi_{-\mu}(x),\ C(x) + U_\mu(x)\,C(x+\hat{\mu})\,U_\mu(x)^\dagger\right] \\ &-\frac{i}{2}\left(\eta_-(x) - U_\mu(x)\,\eta_-(x+\hat{\mu})\,U_\mu(x)^\dagger\right) \\ &+\frac{i}{2}\left[\psi_{-\mu}(x),\ \tilde{H}_\mu(x)\right] \\ &-\frac{1}{4}\left[\psi_{-\mu}(x)\psi_{-\mu}(x),\ \psi_{+\mu}(x)\right] \end{aligned}$$

are subleadings.

Transformation rule for other fields is same as in the continuum case.

22

↑ Lattice Action with Exact SUSY Q_+, Q_- is

$$S_{2D,N=4} = Q_+Q_-\frac{1}{2g_0^2}\sum_x \mathrm{tr}\Big[-iB(x)\left(\Phi(x)+\Delta\Phi(x)\right) - \sum_\mu \psi_{+\mu}(x)\psi_{-\mu}(x) - \chi_+(x)\chi_-(x) - \frac{1}{4}\eta_+(x)\eta_-(x)\Big]$$

with

$$\Phi(x) = -i\left(U_{12}(x) - U_{21}(x)\right)$$

$$\Delta\Phi(x) = -r\left(2 - U_{12}(x) - U_{21}(x)\right)$$

22'

$$\begin{aligned}
S_{2D,N=4} = \frac{1}{2g_0^2}\sum_x \mathrm{tr}\Big[&-i\left(\frac{1}{2}[C(x),B(x)] + H(x)\right)\left(\Phi(x)+\Delta\Phi(x)\right)\\
&+i\chi_-(x)Q_+\left(\Phi(x)+\Delta\Phi(x)\right) - i\chi_-(x)Q_-\left(\Phi(x)+\Delta\Phi(x)\right)\\
&-iB(x)Q_+Q_-\left(\Phi(x)+\Delta\Phi(x)\right)\Big]\\
+\frac{1}{2g_0^2}\sum_{x,\mu}\mathrm{tr}\Big[&\tilde{H}_\mu(x)^2 - \frac{1}{2}\psi_{+\mu}(x)\psi_{+\mu}(x)\psi_{-\mu}(x)\psi_{-\mu}(x)\\
&+\left(\phi(x) - U_\mu(x)\phi(x+\hat\mu)U_\mu(x)^\dagger\right)\left(\bar\phi(x) - U_\mu(x)\bar\phi(x+\hat\mu)U_\mu(x)^\dagger\right)\\
&+\frac{1}{4}\left(C(x) - U_\mu(x)C(x+\hat\mu)U_\mu(x)^\dagger\right)^2\\
&-\psi_{+\mu}(x)\psi_{+\mu}(x)\left(\bar\phi(x) + U_\mu(x)\bar\phi(x+\hat\mu)U_\mu(x)^\dagger\right)\\
&+\psi_{-\mu}(x)\psi_{-\mu}(x)\left(\phi(x) + U_\mu(x)\phi(x+\hat\mu)U_\mu(x)^\dagger\right)\\
&-i\psi_{+\mu}(x)\left(\eta_-(x) - U_\mu(x)\eta_-(x+\hat\mu)U_\mu(x)^\dagger\right)\\
&-i\psi_{-\mu}(x)\left(\eta_+(x) - U_\mu(x)\eta_+(x+\hat\mu)U_\mu(x)^\dagger\right)\\
&-\frac{1}{2}\{\psi_{+\mu}(x),\psi_{-\mu}(x)\}\left(C(x) + U_\mu(x)C(x+\hat\mu)U_\mu(x)^\dagger\right)\Big]\\
+\frac{1}{2g_0^2}\sum_x\mathrm{tr}\Big[&-[\phi,B][\bar\phi,B] + H^2 - \frac{1}{4}[C,B]^2\\
&+\chi_+[\bar\phi,\chi_+] - \chi_-[\phi,\chi_-] + \chi_-[C,\chi_+]\\
&-\chi_-[B,\eta_+] - \chi_+[B,\eta_-]\\
&+\frac{1}{4}[\phi,\bar\phi]^2 - \frac{1}{4}[\phi,C][\bar\phi,C]\\
&-\frac{1}{4}\eta_-[\phi,\eta_-] + \frac{1}{4}\eta_+[\bar\phi,\eta_+] - \frac{1}{4}\eta_+[C,\eta_-]\Big]
\end{aligned}$$

23.

Symmetry of the lattice action

- $SU(N)$ gauge symmetry
- lattice translation
- supersymmetry Q_+, Q_-
- $U(1)_R$ rotation

$$
\begin{aligned}
U_\mu &\to U_\mu, & \psi_{+\mu} &\to e^{i\theta}\psi_{+\mu}\\
\tilde{H}_\mu &\to \tilde{H}_\mu, & \psi_{-\mu} &\to e^{-i\theta}\psi_{-\mu}\\
B &\to B, & \chi_+ &\to e^{i\theta}\chi_+\\
H &\to H, & \chi_- &\to e^{-i\theta}\chi_-\\
\phi &\to e^{i2\theta}\phi, & \eta_+ &\to e^{i\theta}\eta_+\\
\bar{\phi} &\to e^{-i2\theta}\bar{\phi}, & \eta_- &\to e^{-i\theta}\eta_-\\
C &\to C
\end{aligned}
$$

- $+ \leftrightarrow -$ exchanging symmetry $(Q_\pm \to Q_\mp)$

$$
\begin{aligned}
\phi &\to -\bar{\phi}, & \psi_{\pm\mu} &\to \psi_{\mp\mu}\\
\bar{\phi} &\to -\phi, & \eta_\pm &\to \eta_\mp\\
B &\to -B, & \chi_\pm &\to -\chi_\mp\\
\tilde{H}_\mu &\to -\tilde{H}_\mu,
\end{aligned}
$$

Other fields do not change.

24.

- $x \equiv (x_1, x_2) \to \tilde{x} \equiv (x_2, x_1)$ as

$$
\begin{aligned}
U_1(x) &\to U_2(\tilde{x}), & \psi_{\pm 1}(x) &\to \psi_{\pm 2}(\tilde{x})\\
U_2(x) &\to U_1(\tilde{x}), & \psi_{\pm 2}(x) &\to \psi_{\pm 1}(\tilde{x})\\
\tilde{H}_1(x) &\to \tilde{H}_2(\tilde{x}), & \eta_\pm(x) &\to \eta_\pm(\tilde{x})\\
\tilde{H}_2(x) &\to \tilde{H}_1(\tilde{x}), & \chi_\pm(x) &\to -\chi_\pm(\tilde{x})\\
H(x) &\to -H(\tilde{x})\\
B(x) &\to -B(\tilde{x})\\
\phi(x) &\to \phi(\tilde{x}), & \bar{\phi}(x) &\to \bar{\phi}(\tilde{x})\\
C(x) &\to C(\tilde{x})
\end{aligned}
$$

- $SU(2)_R$ $(\supset U(1)_R)$

$\begin{pmatrix}\psi_{+\mu}\\ \psi_{-\mu}\end{pmatrix}$, $\begin{pmatrix}\chi_+\\ \chi_-\end{pmatrix}$, $\begin{pmatrix}\eta_+\\ -\eta_-\end{pmatrix}$, $\begin{pmatrix}Q_+\\ Q_-\end{pmatrix}$: doublets

$\begin{pmatrix}\phi\\ C\\ -\bar{\phi}\end{pmatrix}$: triplet

25

- Naive Continuum Limit $a \to 0$ with $\frac{1}{g^2} \equiv \frac{a^2}{g_0^2}$ fixed

leads the continuum 2D $\mathcal{N}=4$ action.

Similarly to the $\mathcal{N}=2$ case, fermion doublers are removed.

Renormalization

By the same argument as in the $\mathcal{N}=2$ case, check the operators of the type φ, φ^2 only.

$$\left.\begin{array}{l}\text{Gauge Symmetry}\\ SU(2)_R\ \text{Symmetry}\end{array}\right\} \to \cancel{\mathrm{tr}(4\phi\bar{\phi}+C^2)},\ \cancel{\mathrm{tr}\, B^2}$$

$Q_\pm$ supersymmetry

$\Downarrow$

No relevant or marginal operators except the id. are radiatively generated.

$\therefore$ In the continuum limit, full SUSY and rotational symmetry are considered to be restored without any fine tuning.

26.

§6. 3D $\mathcal{N}=4, 8$ and 2D $\mathcal{N}=8$

"Topological Field Theory Form" of 4D $\mathcal{N}=2$

$$S^{(E)}_{4D\,\mathcal{N}=2} = Q\frac{1}{2g^2}\int d^4x\, \mathrm{tr}\Big[\frac{1}{4}\eta[\phi,\bar{\phi}] - i\vec{\chi}\cdot\vec{\Phi} + \vec{\chi}\cdot\vec{H} - i\psi_\mu D_\mu\bar{\phi}\Big]$$

$\vec{H}, \vec{\chi}, \vec{\Phi}$: 3 component vector

$$\Phi_A = 2\Big(F_{A4} + \frac{1}{2}\varepsilon_{ABC}F_{BC}\Big) \qquad (A=1,2,3)$$

$$\mathrm{tr}\,\vec{\Phi}^2 = \frac{1}{2}\mathrm{tr}\big(F_{\mu\nu}^2 + F_{\mu\nu}\tilde{F}_{\mu\nu}\big)$$

Lattice Action

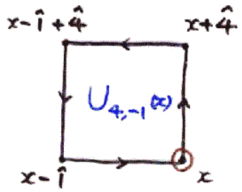
$$S_{4D\,\mathcal{N}=2} = Q\frac{1}{2g_0^2}\sum_x \mathrm{tr}\Big[\frac{1}{4}\eta(x)[\phi(x),\bar{\phi}(x)] - i\vec{\chi}(x)\cdot\big(\vec{\Phi}(x)+\Delta\vec{\Phi}(x)\big) + \vec{\chi}(x)\cdot\vec{H}(x) - i\sum_\mu \psi_\mu(x)\big(\bar{\phi}(x) - U_\mu(x)\bar{\phi}(x+\hat{\mu})U_\mu(x)^\dagger\big)\Big]$$

27.

with

$$\begin{cases} \Phi_1(x) = -i\left[U_{4,-1}(x) - U_{-1,4}(x) + U_{23}(x) - U_{32}(x) \right] \\ \Phi_2(x) = -i\left[U_{4,-2}(x) - U_{-2,4}(x) + U_{31}(x) - U_{13}(x) \right] \\ \Phi_3(x) = -i\left[U_{4,-3}(x) - U_{-3,4}(x) + U_{12}(x) - U_{21}(x) \right] \end{cases}$$

$$\begin{cases} \Delta\Phi_1(x) = -r\left[2 - U_{4,-1}(x) - U_{-1,4}(x) + 2 - U_{23}(x) - U_{32}(x) \right] \\ \Delta\Phi_2(x) = -r\left[2 - U_{4,-2}(x) - U_{-2,4}(x) + 2 - U_{31}(x) - U_{13}(x) \right] \\ \Delta\Phi_3(x) = -r\left[2 - U_{4,-3}(x) - U_{-3,4}(x) + 2 - U_{12}(x) - U_{21}(x) \right] \end{cases}$$



With the choice $r = \cot\varphi$ $\left(0 < \varphi \leqq \frac{\pi}{2N}\right)$

vacuum degeneracy is removed.

↓

Expansion around $U_\mu(x) = 1$ is justified.

$U_\mu(x) = 1 + iaA_\mu(x) + \cdots$

But,

Quadratic term of A_μ in $\mathrm{tr}\left(\vec{\Phi}(x) + \Delta\vec{\Phi}(x)\right)^2$

has surplus zero-modes having $q_4 \neq 0$.

(other than gauge d.o.f.)

Fermion kinetic term also has zero-modes at the same momenta.

(consistent to the Exact SUSY Q)

27'

In momentum space

$$\tilde{\Phi}_1(q) + \Delta\tilde{\Phi}_1(q) = \left(1 - e^{-iaq_1}\right)\tilde{A}_4(q) - e^{-iaq_1}\left(e^{iaq_4} - 1\right)\tilde{A}_1(q) + \left(e^{iaq_2} - 1\right)\tilde{A}_3(q) - \left(e^{iaq_3} - 1\right)\tilde{A}_2(q) + O(\tilde{A}^2)$$

$$\tilde{\Phi}_2(q) + \Delta\tilde{\Phi}_2(q) = \left(1 - e^{-iaq_2}\right)\tilde{A}_4(q) - e^{-iaq_2}\left(e^{iaq_4} - 1\right)\tilde{A}_2(q) + \left(e^{iaq_3} - 1\right)\tilde{A}_1(q) - \left(e^{iaq_1} - 1\right)\tilde{A}_3(q) + O(\tilde{A}^2)$$

$$\tilde{\Phi}_3(q) + \Delta\tilde{\Phi}_3(q) = \left(1 - e^{-iaq_3}\right)\tilde{A}_4(q) - e^{-iaq_3}\left(e^{iaq_4} - 1\right)\tilde{A}_3(q) + \left(e^{iaq_1} - 1\right)\tilde{A}_2(q) - \left(e^{iaq_2} - 1\right)\tilde{A}_1(q) + O(\tilde{A}^2)$$

Surplus zero-modes (other than gauge d.o.f.)

$$\vec{q} = \left(\pm\frac{\pi}{2a}, 0, 0, \mp\frac{\pi}{2a}\right), \left(0, \pm\frac{\pi}{2a}, 0, \mp\frac{\pi}{2a}\right), \left(0, 0, \pm\frac{\pi}{2a}, \mp\frac{\pi}{2a}\right), \cdots$$

For every mode, $q_4 \neq 0$.

27'

In the case

$$\begin{cases} \Phi_1(x) = -i\left[U_{14}(x) - U_{41}(x) + U_{23}(x) - U_{32}(x) \right] \\ \Phi_2(x) = -i\left[U_{24}(x) - U_{42}(x) + U_{31}(x) - U_{13}(x) \right] \\ \Phi_3(x) = -i\left[U_{34}(x) - U_{43}(x) + U_{12}(x) - U_{21}(x) \right] \end{cases}$$

$$\begin{cases} \Delta\Phi_1(x) = -r\left[2 - U_{14}(x) - U_{41}(x) + 2 - U_{23}(x) - U_{32}(x) \right] \\ \Delta\Phi_2(x) = -r\left[2 - U_{24}(x) - U_{42}(x) + 2 - U_{31}(x) - U_{13}(x) \right] \\ \Delta\Phi_3(x) = -r\left[2 - U_{34}(x) - U_{43}(x) + 2 - U_{12}(x) - U_{21}(x) \right] \end{cases}$$

Surplus zero-modes (other than gauge d.o.f.)

$$\begin{aligned} \vec{q} = &\left(\pm\tfrac{\pi}{2a}, \mp\tfrac{\pi}{2a}, 0, 0\right), \left(\pm\tfrac{\pi}{2a}, 0, \mp\tfrac{\pi}{2a}, 0\right), \\ &\left(\pm\tfrac{\pi}{2a}, 0, 0, \mp\tfrac{\pi}{2a}\right), \left(0, \pm\tfrac{\pi}{2a}, \mp\tfrac{\pi}{2a}, 0\right), \\ &\left(0, \pm\tfrac{\pi}{2a}, 0, \mp\tfrac{\pi}{2a}\right), \left(0, 0, \pm\tfrac{\pi}{2a}, \mp\tfrac{\pi}{2a}\right), \\ &\left(\pm\tfrac{\pi}{3a}, \pm\tfrac{\pi}{3a}, \pm\tfrac{\pi}{3a}, \mp\tfrac{2\pi}{3a}\right), \left(\pm\tfrac{\pi}{3a}, \pm\tfrac{\pi}{3a}, \mp\tfrac{2\pi}{3a}, \pm\tfrac{\pi}{3a}\right), \\ &\left(\pm\tfrac{\pi}{3a}, \mp\tfrac{2\pi}{3a}, \pm\tfrac{\pi}{3a}, \pm\tfrac{\pi}{3a}\right), \left(\mp\tfrac{2\pi}{3a}, \pm\tfrac{\pi}{3a}, \pm\tfrac{\pi}{3a}, \pm\tfrac{\pi}{3a}\right) \end{aligned}$$

28.

- Consider the dimensional reduction w.r.t. $\hat{4}$
  (Forget the dependence on $\hat{4}$)

  $$4D\ \mathcal{N}=2 \rightarrow 3D\ \mathcal{N}=4$$

  $\rightarrow$ The surplus zero-modes are killed.

Then, the dimensional reduced lattice model reproduces 3D $\mathcal{N}=4$ theory in the naive continuum limit

$$a \to 0, \quad \frac{1}{g^2} = \frac{a}{g_0^2}$$

- The renormalization argument tells that to reach the desired continuum theory fine tuning of 3 parameters for the counter terms

  $$Q\sum_{\mu=1}^{3} \mathrm{tr}(\psi_\mu \bar{\phi}), \quad Q\,\mathrm{tr}(\psi_4 \bar{\phi}), \quad Q\sum_{A=1}^{3} \mathrm{tr}(\chi_A A_4)$$

  is necessary.

Lattice Action for "Balanced Topological Field Theory Form" 29.
of 4D $\mathcal{N}=4$

$$S_{4D\,\mathcal{N}=4} = Q_+Q_- \frac{1}{2g_0^2}\sum_x \mathrm{tr}\Big[-i\vec{B}(x)\cdot(\vec{\Phi}(x)+\Delta\vec{\Phi}(x)) - \frac{1}{3}\sum_{A,B,C=1}^{3}\varepsilon_{ABC}B_A(x)[B_B(x),B_C(x)] - \sum_{\mu=1}^{4}\psi_{+\mu}(x)\psi_{-\mu}(x) - \vec{\chi}_+(x)\cdot\vec{\chi}_-(x) - \frac{1}{4}\eta_+(x)\eta_-(x)\Big]$$

Same problem of the surplus zero-modes occurs.

→ Dimensionally reduced lattice theories $\left\{\begin{array}{l}3D\ \mathcal{N}=8\\ 2D\ \mathcal{N}=8\end{array}\right\}$ reproduce the desired theories in the naive continuum limit.

Renormalization argument (only the result)

3D $\mathcal{N}=8$: One parameter fine tuning for

$$Q_+Q_-\,\mathrm{tr}\left[(B_1+B_2+B_3)A_4\right]$$

is needed.

2D $\mathcal{N}=8$: Any fine tuning is not necessary.

§6. Discussions 30.

- We constructed lattice models for SYM theories of
  2D $\mathcal{N}=2,4,8$
  3D $\mathcal{N}=4,8$
  based on (balanced) topological field theory formulation.
  - keeping a fraction of supersymmetry exactly
  - hypercubic lattice
  - unitary link variables
  - desired continuum theories are obtained by fine-tuning $\left\{\begin{array}{c}3\\1\\0\end{array}\right\}$ parameters for $\left\{\begin{array}{l}3D\ \mathcal{N}=4\\ 3D\ \mathcal{N}=8\\ 2D\ \mathcal{N}=2,4,8\end{array}\right.$

- It is possible to impose the admissibility conditions
  $\|1-U_{12}(x)\|<\epsilon$
  removing the vacuum degeneracy instead of the modification $\Phi(x)\to\Phi(x)+\Delta\Phi(x)$ for the cases 2D $\mathcal{N}=2,4$.

  (The SUSY Q ($Q_\pm$) is also preserved.)

31.

- Problem of the surplus zero-modes in 4D theories

  (Is it related to construction of the topological term $F\tilde{F}$ on the lattice ?)

- 2D $\mathcal{N}=8$, 3D $\mathcal{N}=8$ cases could be used for nonperturbative investigation of matrix string theories.

- Coupling to matter fields

  (Topological QCD Formulation)

- Simulative Study (Fermion determinant)