Instanton Quarks II

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I. Main Goals and Results

1. We argue that the dual representation of the low-energy effective chiral Lagrangian corresponds to a statistical system of interacting pseudo-particles with fractional $1/N_c$ charges.

2. We shall identify these particles with **instanton quarks** suspected long ago consequently demonstrating a link between confinement and instantons, V.Fateev et al, Nucl. Phys. **B154** (1979) 1; B.Berg and M.Luscher, Commun.Math.Phys. **69**(1979) 57. A. Belavin et al, Phys. Lett. **83B** (1979) 317.

3. We shall demonstrate that the integration measure for the statistical ensemble exactly matches for the instanton moduli space for arbitrary gauge group G. This is because the θ dependence in physical observables will appear in the combination $\frac{\theta}{C_2(G)}$. In particular, $C_2(SU(N)) = N$, $C_2(SO(N)) = N-2$, $C_2(Sp(2N)) = N+1$. This number $4kC_2(G)$ exactly corresponds to the number of zero modes in the k-instanton background for gauge group G.

II. Main Logic of the Presentation

1. I use a TRICK which allows me to represent the low energy effective lagrangian in terms of dual variables (a statistical system of some interacting pseudo-particles).

2. I test this trick in the weak coupling regime in QCD (large chemical potential, Color Superconductor) where all calculations are under complete theoretical control.

3. I observe that the instanton-instanton(II) and instanton -anti-instanton ($\bar{I}I$) interactions at large distances are very different from the naive semiclassical calculations.

4. I apply the same trick to QCD at zero chemical potential and $T = 0$. I advocate the picture that in the strongly coupled theories the instantons and anti-instantons lose their individual properties (instantons will "melt") their sizes become very large and they overlap.

5. The description in terms of the instantons and anti-instantons is not appropriate any more, and alternative degrees of freedom should be used to describe the physics. The relevant description is that of instanton-quarks.

(This is a typical approach by CM people: they try to understand what are the GOOD quasiparticles.)

6. The key observation here is there existence of the free parameter θ (it plays the crucial role in mapping of one problem to another). The θ plays the role of the messenger between colorless (chiral effective Lagrangian fields) and colorful (instanton quarks) objects. On one side, the dependence of the physical observables as a function of θ can be studied exclusively in terms of the colorless degrees of freedom by using the effective Lagrangian approach. On the other side, the θ is defined as parameter in front of the topological charge operator $\tilde{F}F$.

III. Color Superconductivity for Pedestrians

($\mu\bar{\Psi}\gamma_0\Psi$ - term, $\mu \gg \Lambda_{QCD}$)

1. If there is a channel in which the quark-quark interaction is attractive, than the true ground state of the system will be a complicated coherent state of Cooper pairs like in **BCS** theory (ordinary superconductor).

2. Diquark condensates break color symmetry (CFL phase, $N_c = N_f = 3$):

$$\langle\psi^{ia}_{L\alpha}\psi^{jb}_{L\beta}\rangle^* \sim \epsilon_{\alpha\beta\gamma}\epsilon^{ij}\epsilon^{abc}X^\gamma_c,$$
$$\langle\psi^{ia}_{R\alpha}\psi^{jb}_{R\beta}\rangle^* \sim \epsilon_{\alpha\beta\gamma}\epsilon^{ij}\epsilon^{abc}Y^\gamma_c$$

3. $SU(3)_c \times U(1)_{EM} \times SU(3)_L \times SU(3)_R \times U(1)_B$

$$\Downarrow$$

$$SU(3)_{c+L+R} \times U(1)^*_{EM}$$

a) Color gauge group is completely broken;
b) $U(1)_B$ is spontaneously broken;
c) $U(1)_{EM}$ is not broken;
d) $U(1)_A$ is broken spontaneously and explicitly (by instantons)

4. Goldstone fields are the phases of the condensate

$$\Sigma_\gamma^\beta = \sum_c X_c^\beta Y_\gamma^{c*} \sim e^{i\lambda^a \pi^a} e^{i\varphi_A}. \qquad (1)$$

5. $U(1)_A$ is spontaneously broken. The symmetry is broken also explicitly by the instantons. Effective lagrangian is

$$L_A \sim f_A^2[(\partial_0\varphi_A)^2 - u^2(\partial_i\varphi_A)^2] + a\mu^2\Delta^2\cos(\varphi_A - \theta)$$

Coefficient a can be explicitly calculated from the t'Hooft formula *(Son, Stephanov, AZ, 2001)*.

6. To compute a we start from the instanton induced effective four-fermion interaction,

$$
\begin{aligned}
L_{\rm inst} &= e^{-i\theta}\int d\rho\, n_0(\rho)\left(\frac{4}{3}\pi^2\rho^3\right)^2\Big\{(\bar u_R u_L)(\bar d_R d_L) + \\
&+ \frac{3}{32}\Big[(\bar u_R\lambda^a u_L)(\bar d_R\lambda^a d_L) \\
&- \frac{3}{4}(\bar u_R\sigma_{\mu\nu}\lambda^a u_L)(\bar d_R\sigma_{\mu\nu}\lambda^a d_L)\Big]\Big\} + {\rm H.c.} \qquad (2)
\end{aligned}
$$

where $n_0(\rho)$ in the presence of $\mu \neq 0$ is given by

$$
n_0(\rho) = C_N\left(\frac{8\pi^2}{g^2}\right)^{2N_c}\rho^{-5}\exp\left(-\frac{8\pi^2}{g^2(\rho)}\right)e^{-N_f\mu^2\rho^2} \qquad (3)
$$

with

$$
C_N = \frac{0.466e^{-1.679N_c}1.34^{N_f}}{(N_c-1)!(N_c-2)!}\,, \qquad (4)
$$

7. Averaging Eq. (2) in the superconducting background, we find

$$V_{\rm inst}(\varphi) = -\int d\rho\, n_0(\rho) \left(\frac{4}{3}\pi^2\rho^3\right)^2 12|X|^2 \cos(\varphi_A - \theta).$$

where

$$|X| = \frac{3}{2\sqrt{2}\pi}\frac{\mu^2\Delta}{g}.$$

Using formula for $n_0(\rho)$ we get the final result

$$a(\mu \gg \Lambda_{QCD}) = 5\times 10^4 \left(\ln\frac{\mu}{\Lambda_{QCD}}\right)^7 \left(\frac{\Lambda_{QCD}}{\mu}\right)^{29/3} \ll 1$$

8. The η' is light: $m^2_{\eta'} \sim \frac{\mu^2\Delta^2}{f^2} \cdot a \sim (\frac{\Lambda_{QCD}}{\mu})^b \to 0.$

9. Weak coupling regime: dilute gas approximation leads exactly to the combination $\left(e^{i(\varphi_A-\theta)} + e^{-i(\varphi_A-\theta)}\right)$ which is expected from the very beginning.

IV. Instanton interactions in dense QCD

1. Partition function for η'

$$Z = \int \mathcal{D}\varphi_A \; e^{-f^2 u \int d^4x (\partial\varphi_A)^2} \, e^{a' \int d^4x \cos(\varphi_A(x) - \theta)} ,$$

2. Different representation for η'

$$\mathrm{e}^{a' \int d^4x \cos(\varphi_A(x) - \theta)} \equiv$$

$$\sum_{M=0}^{\infty} \frac{(a'/2)^M}{M!} \left(\int d^4x \sum_{Q=\pm 1} e^{iQ(\varphi(x)-\theta)} \right)^M$$

$$= \sum_{M_\pm=0}^{\infty} \frac{(a'/2)^M}{M_+! M_-!} \int d^4x_1 \ldots \int d^4x_M \; e^{i \sum_{a=0}^{M} Q_a(\varphi_A(x_a) - \theta)} .$$

The last line is aclassical partition function of the gas of $M = M_+ + M_-$ identical particles of charges $+1$ or -1 placed in an external potential given by $i(\theta - \varphi(x))$.

3. For each term the path integral is Gaussian and can be easily taken:

$$\int \mathcal{D}\varphi \, e^{-f^2 u \int d^4x (\partial\varphi)^2} \, e^{i \sum_{a=0}^{M} Q_a (\varphi(x_a) - \theta)} =$$

$$e^{-i\theta \sum_{a=0}^{M} Q_a} \, e^{-\frac{1}{2f^2 u} \sum_{a>b=0}^{M} Q_a Q_b G(x_a - x_b)} \, .$$

4. Thus we obtain the dual CG representation for the partition function

$$Z = \sum_{M_\pm = 0}^{\infty} \frac{(a/2)^M}{M_+! M_-!} \int d^4x_1 \ldots \int d^4x_M \; e^{-i\theta \sum_{a=0}^{M} Q_a} \; \cdot$$

$$e^{-\frac{1}{2f^2 u} \sum_{a>b=0}^{M} Q_a Q_b G(x_a - x_b)} \, , \quad G(x_a - x_b) = \frac{1}{4\pi^2 (x_a - x_b)^2} \, .$$

The two representations of the partition function are equivalent.

5. Physical Interpretation.

a) Since $Q_{\rm net} \equiv \sum_a Q_a$ is the total charge and it appears in the action multiplied be the parameter θ, one concludes that $Q_{\rm net}$ *is* the total topological charge of a given configuration.

b) Each charge Q_a in a given configuration should be identified with an integer topological charge well localized at the point x_a. This, by definition, corresponds to a small instanton positioned at x_a.

c) Further support for the identification: every particle with charge Q_a brings along a factor of fugacity $\sim a'$ which contains the classical one-instanton suppression factor $\exp(-8\pi^2/g^2(\rho))$ in the density of instantons.

6. The following hierarchy of scales exists: The typical size of the instantons $\bar{\rho} \sim \mu^{-1}$ is much smaller than the short-distance cutoff of our effective low-energy theory, Δ^{-1},

$$
\begin{array}{ccccccc}
\text{(size)} & \ll & \text{(cutoff)} & \ll & (II\ \text{distance}) & \ll & \text{(Debye)} \\
\mu^{-1} & \ll & \Delta^{-1} & \ll & (\sqrt{a}\mu\Delta)^{-1/2} & \ll & (\sqrt{a}\Delta)^{-1}
\end{array}
$$

Due to this hierarchy, ensured by large $\mu/\Lambda_{\rm QCD}$, we acquire analytical control.

7. The starting low-energy effective Lagrangian contains only a colorless field φ_A, we have ended up with a representation of the partition function in which objects carrying color (the instantons) can be studied.

8. In particular, II and $I\bar{I}$ interactions (at very large distances) are exactly the same up to a sign, order g^0, and are Coulomb-like. This is in **contrast with semiclassical expressions** when II interaction is zero and $I\bar{I}$ interaction is order $1/g^2$.

9. Very complicated picture of the **bare** II and $I\bar{I}$ interactions becomes very simple for **dressed** instantons/anti-instantons when all integrations over all possible sizes, color orientations and interactions with background fields are properly accounted for!

10. As expected, the ensemble of small $\rho \sim 1/\mu$ instantons can not produce confinement. This is in accordance with the fact that CS phase is not in confining phase.

V.Chiral Lagrangian ($\mu = 0 \quad \theta \neq 0$)

1. We keep only the diagonal elements of the chiral matrix $U = \exp\{i \mathrm{diag}(\phi_1, \ldots, \phi_{N_f})\}$ which are relevant in the description of the ground state. Singlet combination is defined as $\phi = \mathrm{Tr}\, U$.

2. Effective lagrangian for the ϕ is

$$L_{\eta'} = f^2 (\partial_\mu \phi)^2 + E \cos\left(\frac{\phi - \theta}{N_c}\right) + \sum_{a=1}^{N_f} m_a \cos \phi_a \quad (6)$$

3. A Sine-Gordon structure for the singlet combination corresponds to the following behavior of the $(2k)^{\rm th}$ derivative of the vacuum energy in pure gluodynamics (Veneziano, 1979)

$$\left.\frac{\partial^{2k} E_{vac}(\theta)}{\partial\, \theta^{2k}}\right|_{\theta=0} \sim \int \prod_{i=1}^{2k} dx_i \langle Q(x_1)...Q(x_{2k})\rangle \sim (\frac{i}{N_c})^{2k},$$

where $Q = \frac{g^2}{32\pi^2} G_{\mu\nu} \widetilde{G}_{\mu\nu}$ is topological density. *Veneziano originally thought that this relation implies the periodicity to be $2\pi N_c$ rather than 2π.*

VI. Dual Representation for the Chiral Lagrangian ($\mu = 0 \quad \theta \neq 0$)

1. One can represent the Sine Gordon effective field theory in terms of the classical statistical model (Coulomb Gas representation)

$$
\begin{aligned}
Z &= \sum_{Q_a^{(0)}=\pm\frac{1}{N_c}} \frac{(\frac{E}{2})^{M_0}}{M_0!} \int (dx_1^{(0)} \dots dx_{M_0}^{(0)})\ e^{-S_{CG}} \\
S_{CG} &= i\theta Q_T^{(0)} + \frac{1}{2f^2}\left\{ \sum_{b,c=1}^{M_0} Q_b^{(0)}\ G(x_b^{(0)} - x_c^{(0)})\ Q_c^{(0)} \right\}. \\
Q_T^{(0)} &= \sum_{b=1}^{M_0} Q_b^{(0)} - \text{total charge for the configuration}
\end{aligned}
$$

2. One can identify $Q_T^{(0)}$ as the total topological charge of the given configuration. Indeed, the θ parameter appears in the original Lagrangian only in the combination $i\theta\frac{G_{\mu\nu}\widetilde{G}_{\mu\nu}}{32\pi^2}d^4x$.

3. The fundamental difference in comparison with the previous case: while the total charge in integer, the individual charges are fractional $\pm 1/N_c$. The fact that species $Q_i^{(0)}$ has charges $\sim 1/N_c$ is a direct consequence of the θ/N_c dependence in the underlying QCD with frozen (non-dynamical) quarks.

4. Due to the 2π periodicity of the theory, only configurations which contain an integer topological number contribute to the partition function. Therefore, the number of particles for each given configuration $Q_i^{(0)}$ with charges $\sim 1/N_c$ must be proportional to N_c.

5. The number of integrations over $d^4x_i^{(0)}$ exactly equals $4N_ck$, where k is integer. This number, $4N_ck$, exactly corresponds to the number of zero modes in the k-instanton background, and we *conjecture* that at low energies (large distances) the fractionally charged species-$Q_i^{(0)}$ pseudo-particles are the **instanton-quarks** suspected long ago.

VII. Interpretation. Speculations.

1. There is an interesting connection between the CG statistical ensemble and the $2d$ CP^{N_c} models. An exact accounting and resummation of the n-instanton solutions maps the original problem to a $2d$-CG with fractional charges (dubbed in 1979 as the instanton-quarks). These pseudo-particles do not exist separately as individual objects; rather, they appear in the system all together as a set of $\sim N_c$ instanton-quarks so that the total topological charge of each configuration is always integer.

2. A charge for an individual instanton quark cannot be created and measured; instead, only the total topological charge for the whole configuration is forced to be integer and has a physical meaning. However a physical understanding and interpretation of the system in terms of these (fictitious) fractionally charged objects leads to an elegant explanation of confinement (the gap) and other important properties of the CP^{N_c} model.

3. For the gauge group, G the number of integrations would be equal to $4kC_2(G)$ where $C_2(G)$ is the quadratic Casimir of the gauge group (θ dependence in physical observables comes in the combination $\frac{\theta}{C_2(G)}$). This number $4kC_2(G)$ exactly corresponds to the number of zero modes in the k-instanton background for gauge group G.

4. One immediate objection: it has long been known that instantons can explain most low energy QCD phenomenology (chiral symmetry breaking, resolution of the $U(1)$ problem, spectrum, etc) with the exception confinement; and we claim that confinement arises in this picture: how can this be consistent?

• We note that it in the dilute gas approximation, when the instantons and anti-instantons are well separated and maintain their individual properties (sizes, positions, orientations), quark confinement can not be described. However, in strongly coupled theories the instantons and anti-instantons lose their individual properties (instantons will "melt") their sizes become very large and they overlap. The relevant description is that of instanton-quarks which can be far away from each other, but still, be strongly correlated.