Progress on computing the muon anomalous magnetic moment from lattice QCD(+QED)

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Collaborators

HVP	HLbL
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Outline I

1. Introduction
   - Nature - Standard Model
2. HVP
   - Doing the integral: fits, moments, sums, ...
   - finite volume effects
   - strange
   - disconnected diagrams
3. HLbL
   - non-perturbative QED
   - Perturbative QED in configuration space
   - next steps
4. Summary/Outlook
5. References

The magnetic moment of the muon

Interaction of particle with static magnetic field

$$V(\vec{x}) = -\vec{\mu} \cdot \vec{B}_{\rm ext}$$

The magnetic moment $\vec{\mu}$ is proportional to its spin ($c = \hbar = 1$)

$$\vec{\mu} = g \left(\frac{e}{2m}\right) \vec{S}$$

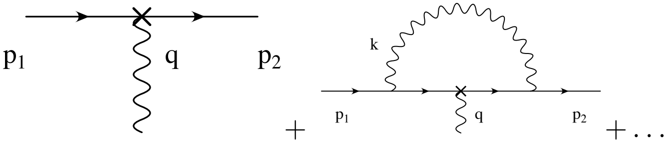
The Landé g-factor is predicted from the free Dirac eq. to be

$$g = 2$$

for elementary fermions

The magnetic moment of the muon

In interacting quantum (field) theory g gets corrections



$$\gamma^\mu \quad \rightarrow \quad \Gamma^\mu(q) = \left(\gamma^\mu F_1(q^2) + i \, \frac{[\gamma^\mu, \gamma^\nu] \, q^\nu}{2} \, \frac{F_2(q^2)}{2m} \right)$$

which results from Lorentz and gauge invariance when the muon is on-mass-shell.

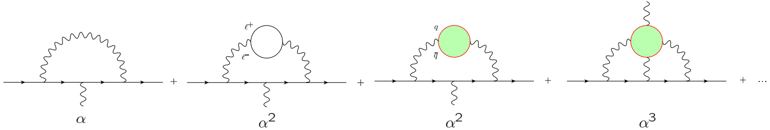
$$F_2(0) \quad = \quad \frac{g-2}{2} \equiv a_\mu \qquad\qquad (F_1(0) = 1)$$

(the anomalous magnetic moment, or anomaly)

The magnetic moment of the muon

Compute these corrections order-by-order in perturbation theory by expanding $\Gamma^\mu(q^2)$ in QED coupling constant

$$\alpha = \frac{e^2}{4\pi} = \frac{1}{137} + \ldots$$



Corrections begin at $\mathcal{O}(\alpha)$; Schwinger term $= \frac{\alpha}{2\pi} = 0.0011614\ldots$

<u>hadronic contributions</u> $\sim 6 \times 10^{-5}$ smaller, dominate theory error.

Experiment - Standard Model Theory = difference

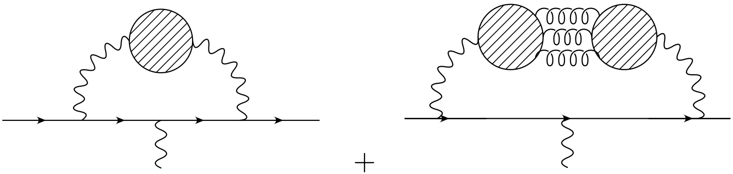
SM Contribution	Value $\pm$ Error ($\times 10^{11}$)	Ref
QED (5 loops)	116584718.951 ± 0.080	[Aoyama et al., 2012]
HVP LO	6923 ± 42	[Davier et al., 2011]
	6949 ± 43	[Hagiwara et al., 2011]
HVP NLO	-98.4 ± 0.7	[Hagiwara et al., 2011]
		[Kurz et al., 2014]
HVP NNLO	12.4 ± 0.1	[Kurz et al., 2014]
HLbL	105 ± 26	[Prades et al., 2009]
Weak (2 loops)	153.6 ± 1.0	[Gnendiger et al., 2013]
SM Tot (0.42 ppm)	116591802 ± 49	[Davier et al., 2011]
(0.43 ppm)	116591828 ± 50	[Hagiwara et al., 2011]
(0.51 ppm)	116591840 ± 59	[Aoyama et al., 2012]
Exp (0.54 ppm)	116592089 ± 63	[Bennett et al., 2006]
Diff (Exp $-$ SM)	287 ± 80	[Davier et al., 2011]
	261 ± 78	[Hagiwara et al., 2011]
	249 ± 87	[Aoyama et al., 2012]

New experiments+new theory=new physics

- Fermilab E989, begins in early 2017, aims for 0.14 ppm
- J-PARC E34, "late 2010's", aims for 0.1 ppm
- Today $a_\mu(\mathrm{Expt})$-$a_\mu(\mathrm{SM}) \approx 2.9 - 3.6\sigma$
- If both central values stay the same,
  - E989 ($\sim 4\times$ smaller error) $\rightarrow \sim 5\sigma$
  - E989+new HLBL theory (models+lattice, 10%) $\rightarrow \sim 6\sigma$
  - E989+new HLBL +new HVP (50% reduction) $\rightarrow \sim 8\sigma$
- Big discrepancy! (New Physics $\sim 2\times$ Electroweak)
- Lattice calculations important to trust theory errors

Outline I

Hadronic vacuum polarization (HVP)



Using lattice QCD and continuum, ∞-volume QED

[Blum, 2003, Lautrup et al., 1971]

$$a_\mu^{\mathrm{HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dq^2\, f(q^2)\, \hat{\Pi}(q^2)$$

$f(q^2)$ is known, $\hat{\Pi}(q^2)$ is subtracted HVP, $\hat{\Pi}(q^2) = \Pi(q^2) - \Pi(0)$

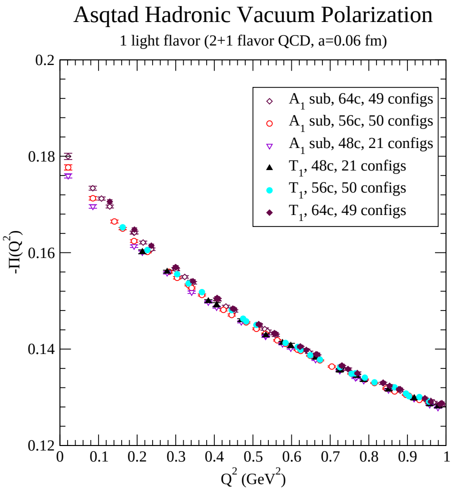
$$\Pi^{\mu\nu}(q) = \int d^4x\, e^{iqx} \langle j^\mu(x) j^\nu(0)\rangle \qquad j^\mu(x) = \sum_i Q_i \bar{\psi}(x)\gamma^\mu\psi(x)$$

$$= \Pi(q^2)(q^\mu q^\nu - q^2\delta^{\mu\nu})$$

Lattice setup (K. Wilson, 1974)

- Compute correlation functions (*e.g.* $\langle j^\mu(x) j^\nu(y)\rangle$, $j^\mu = \bar\psi\gamma_\mu\psi$) in Feynman path integral formalism
- 4(5)D hypercubic lattice regularization, non-zero lattice spacing a and finite volume, $V = L^3 T$
- Handle fermion integrals analytically. Propagators inverse of large sparse matrix, lattice Dirac operator $\not{D} + m_q$ (domain wall, staggered, Wilson, ...)
- Treat path integrals over gauge fields stochastically, using Monte Carlo techniques: generate ensemble of gauge field configurations $\{U\}$ with weight $\det M(U) \exp -S_g$, $\langle \cdots \rangle$ simple average over ensemble
- Work entirely in Euclidean space time, analytically continue back to Minkowski at the end (usually trivial)

HVP from lattice QCD calculation



Aubin, Blum, Golterman, and Peris (MILC gauge ensembles)

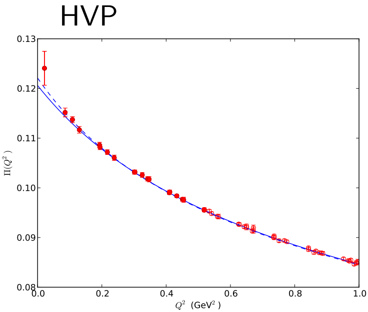
- 2+1f Imp. staggered
- MILC ensembles
- $a = 0.06$ fm, $(3.84 \text{ fm})^3$ box
- $220 \leq m_\pi \leq 315$ MeV
- $m_\pi L \sim 4.3 - 4.5$

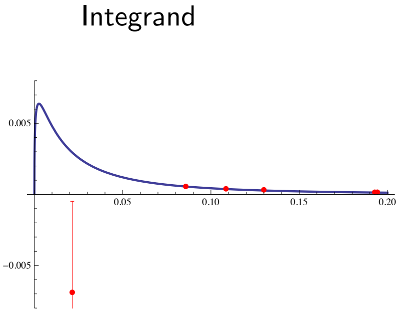
Fits

- Need smooth parametrization of lattice HVP
- Integral dominated by low momentum, $m_\mu/2 \lesssim 2\pi/L$
- Fit HVP, plug into integral. Use polynomials [Blum, 2003], VMD [Gockeler et al., 2004], chiral perturbation theory+VMD [Aubin and Blum, 2007]
- Integral ($= a_\mu$) sensitive to model dependence because of low Q uncertainties [Aubin and Blum, 2007, Aubin et al., 2012, Golterman et al., 2013]
- VMD does not work [Golterman et al., 2013]
- Use Padé approximants, model independent, based on Stieltjes functions (nice convergence properties) [Aubin et al., 2012]

$$\Pi(Q^2) = \Pi(0) - Q^2 \left(a_0 + \sum_{n=1}^{N} \frac{a_n}{b_n + Q^2} \right)$$

Fits circa 2012 [Aubin and Blum, 2007, Aubin et al., 2012]





- 2+1f Imp. staggered (MILC), 220 MeV pion, $(3.84\ \mathrm{fm})^3$
- 1,1 Padé
- Dominated by $q \sim m_\mu/2$ (large box needed for access)
- Fit uncertainty $\leftrightarrow$ large uncertainty in a_μ (10-20%)
- Need improved statistical errors and larger box for small q

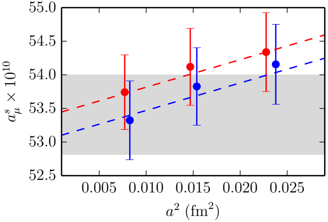
Moments method [Chakraborty et al., 2014] (HPQCD)

- Alternative to fits: compute time moments of two-point correlation function. Coefficients of Taylor exp. about $q^2 = 0$

$$\sum_t \sum_{\vec{x}} t^{2n} \langle j^i(\vec{x}, t) j^i(0) \rangle \quad = \quad (-1)^n \frac{\partial^{2n}}{\partial q^{2n}} \hat{\Pi}(q^2)\Big|_{q^2=0}$$

($\partial/\partial q \rightarrow$ finite difference in FV $\rightarrow$ FVE)

- Use moments to construct Padé approximants for $\hat{\Pi}$,
- Higher moments $\rightarrow$ more statistical noise. OK since Padé's converge rapidly, integral dominated by low Q^2



- all systematics controlled
- $a_\mu^{\text{strange}} = 53.41(59) \times 10^{-10}$
  [Chakraborty et al., 2014] (HPQCD)
- Next, apply to light quark HVP (same difficulty as $q \rightarrow 0$ or $t \rightarrow \infty$)

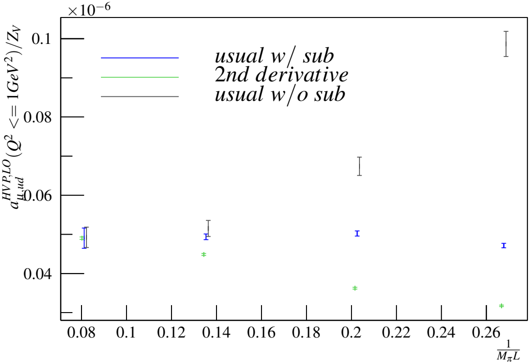
Finite volume HVP [Aubin et al., 2015] see also [Bernecker and Meyer, 2011]

- FV effects important for low q, cutoff effects small
- Finite volume $\Pi^{\mu\nu}$ transforms under 5 Irreps (1, 1, 2, 3, 3)d: A_1, A_2, E, T_1, T_2 for $L \neq T$
- $\Pi^{\mu\mu}(0) \neq 0$ in FV because Euclidean $O(4)$ symmetry is broken. Terms not constrained by WI, exponentially small
- $\Pi^{\mu\nu}(q)$ is discontinuous at $q = 0$, more singular in FV @ low q
- Suggests we should subtract $\Pi^{\mu\mu}(0)$
- $\Pi(q^2)$ depends on irrep
- full $SO(4)$ symmetry restored as $L, T \to \infty$

Finite volume effects

- Zero mom subtraction $\Pi_{\nu\nu}(0)$ seen to reduce FV effect

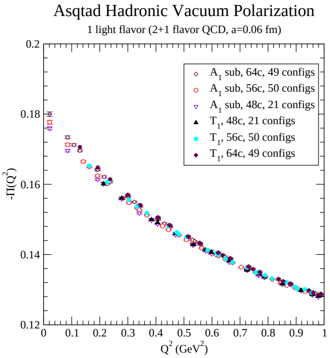
[Malak et al., 2015] (BMWc)

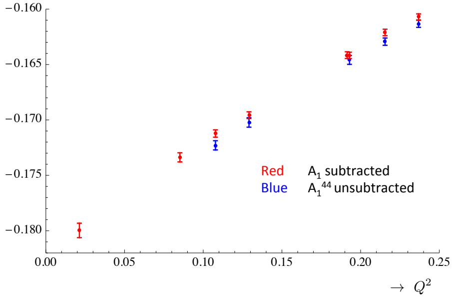


- $2.5 \leq L \leq 8.3$ fm, $5 \leq T \leq 10$ fm, $a = 0.104$ fm, $m_\pi = 292$ MeV, $3.7 \leq m_\pi L \leq 12.3$
- 100% error for "small" box, 40% even for $m_\pi L = 4.9$

FV effects: M. Golterman's talk at Lattice 2015 [Aubin et al., 2015]

- FVE small, but visible, so fit HVP for each irrep separately



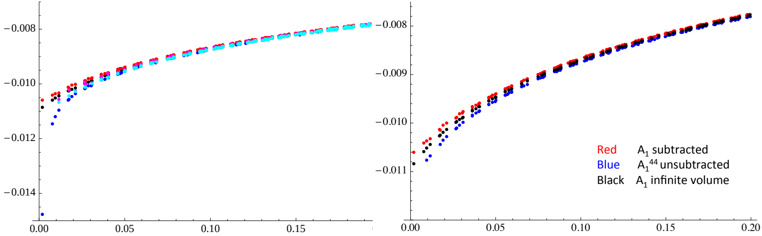


Statistical errors < 0.4% ! All mode averaging [Izubuchi et al., 2013]

Finite volume effects [Aubin et al., 2015]

- Use FV SU2 chiral perturbation theory to compute differences between irreps, and same irreps with and without subtraction

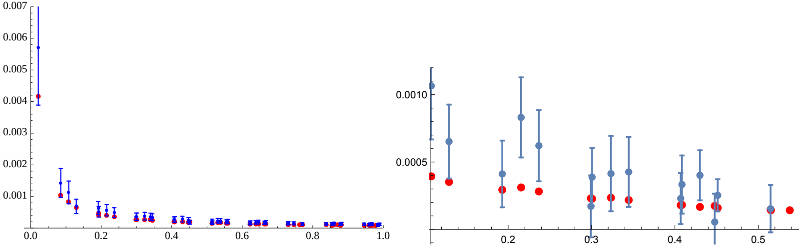
$$\Pi^{\mu\nu} = \frac{4}{L^3 T}\sum_p \frac{\sin(p+q/2)_\mu \sin(p+q/2)_\nu}{(2\sum_k(1-\cos p)_k + m_\pi^2)(2\sum_k(1-\cos p+q/2)_k + m_\pi^2)} - \delta_{\mu\nu}\frac{2}{L^3 T}\sum_p \frac{\cos p_\mu}{2\sum_k(1-\cos p)_k + m_\pi^2}$$



- A_1 irrep has lowest Q^2, largest FVE, $O(\sim 40\,\%)$!, $m_\pi L = 4.2$
- FVE $O(\text{few}\,\%)$ of full HVP, $m_\pi L = 4.2$

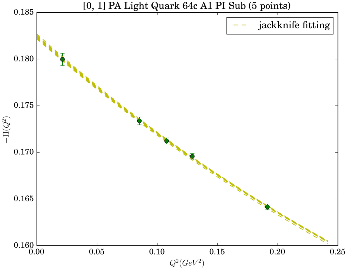
Finite volume effects [Aubin et al., 2015]

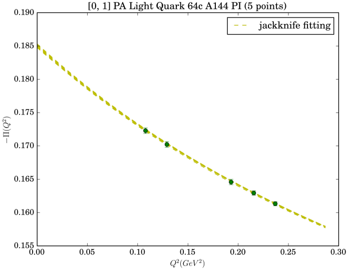
Compare lattice and NLO χPT (both in FV)



- Difference of A_1 (subtracted) and A_1^{44} irreps
- Differences are $\lesssim 0.5\%$ of total HVP @ $m_\pi L = 4.2$ after zero mom subtraction
- χPT does a reasonable job describing lattice calculation
- Reasonable assumption: FV effects dominated my pions

Finite volume errors [Aubin et al., 2015]

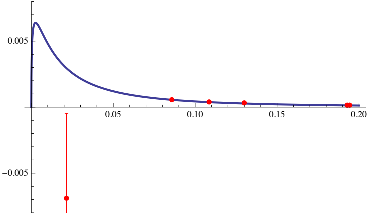




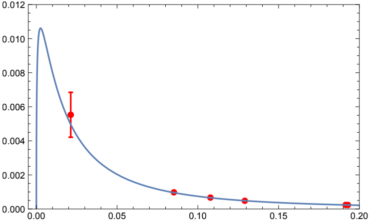
- 2+1f Imp. staggered (MILC), 220 MeV pion, (3.84 fm)^3
- A_1 sub, $a_\mu = 8.4 \pm 0.4 \times 10^{-8}$ $\quad 8.4 \pm 0.5 \times 10^{-8}$ (conf. pol.)
- A_1^{44}, $\quad a_\mu = 9.2 \pm 0.3 \times 10^{-8}$ $\quad 9.6 \pm 0.4 \times 10^{-8}$ (conf. pol.)
- Difference $9 - 13\%$ due to FVE
- χPT: irreps straddle ∞ volume result, so FV error $\lesssim 5 - 7\%$ in this case?

Fits and the a_μ integrand [Aubin et al., 2012, Aubin et al., 2015]

Why is a_μ so sensitive to FVE?



old statistics (2012)



new statistics (AMA)

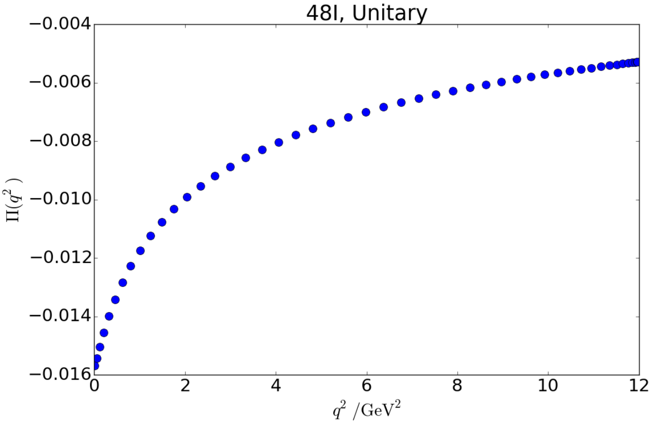
- dominated by $q \sim m_\mu/2$ (large box needed to access)
- better, but still larger box needed
- moment method has similar problem, *e.g.*, t^2 moment:

$$\Pi(0) \quad = \sum_{n=-T/2, n\neq 0}^{T/2-1} 4(-1)^n \Pi(n\Delta) \qquad (\Delta \equiv \frac{2\pi}{L}, \quad \mathrm{pbc})$$

- Solid understanding of low Q^2 region emerging

Strange: Matt Spraggs's talk at Lattice 2015 [RBC/UKQCD 2015]

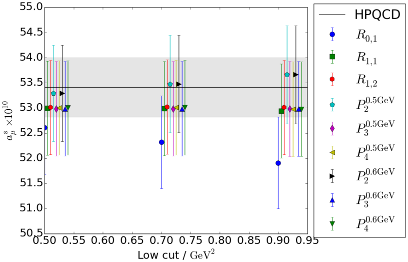
- Strange contribution to HVP, 2+1 flavor Möbius DWF, physical quark mass ensemble



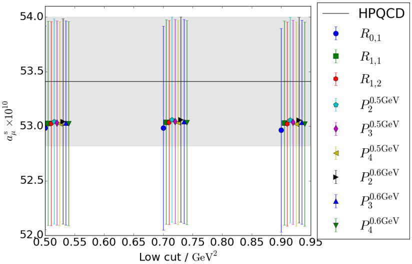
Strange: Matt Spraggs's talk at Lattice 2015 [RBC/UKQCD 2015]

Analysis strategy to parametrize $\Pi(q^2)$

- Use fits, moments, continuous FT (sin cardinal constr)
- Padé approximants, conformal polynomials
- various q^2 cuts (high and low)



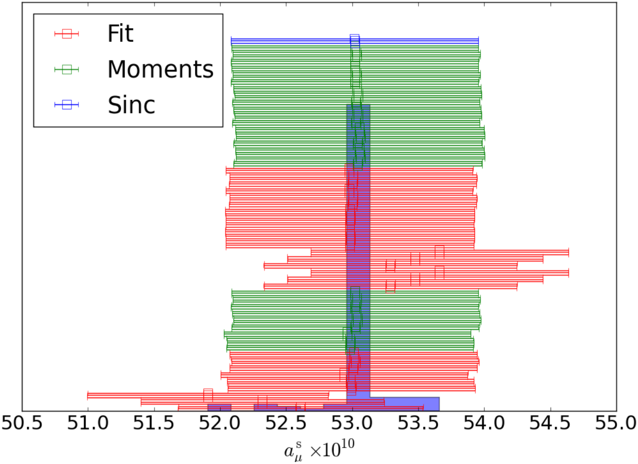
Fits



Moments

Strange: Matt Spraggs's talk at Lattice 2015 [RBC/UKQCD 2015]

Histogram of results from various strategies. Results insensitive



Strange: Matt Spraggs's talk at Lattice 2015 [RBC/UKQCD 2015]

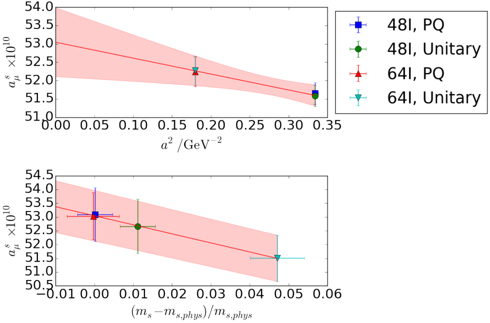
Extrapolations

- Strange quark mistuning: $\sim 1\%$ 48I, $\sim 5\%$ 64I
- Partially quenched strange mass extrapolation
- continuum limit

$$a_\mu^s = a_{\mu,0}^s + \alpha a^2 + \beta \frac{m_s - m_s^{phys}}{m_s^{phys} + m_{res}}$$

Strange: Matt Spraggs's talk at Lattice 2015 (Kobe) [RBC/UKQCD 2015]

- Strange contribution, 2+1 f Möbius DWF, continuum limit



- Physical masses
- $a = 0.114$ and 0.09 fm
- $(5.5\ \text{fm})^3$ boxes

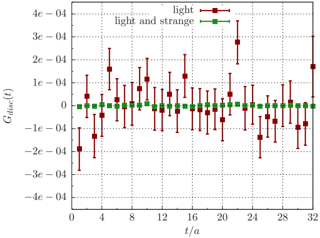
RBC/UKQCD

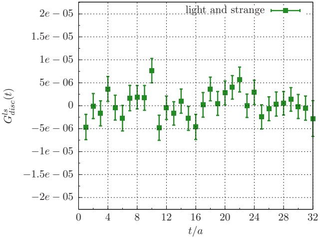
- results independent of analysis method (fits or moments)
- remarkable agreement with HPQCD 2+1+1 staggered fermion result 53.41 (59) (1% level) [Chakraborty et al., 2014]

Disconnected diagrams

- Zero contribution in the SU3 flavor limit
- 10% of connected in χPT [Della Morte and Juttner, 2010]
- Computed by several groups so far

  [Feng et al., 2011, Gulpers et al., 2014, Burger et al., 2015]
- Compute light-strange to cancel noise (Mainz Group)

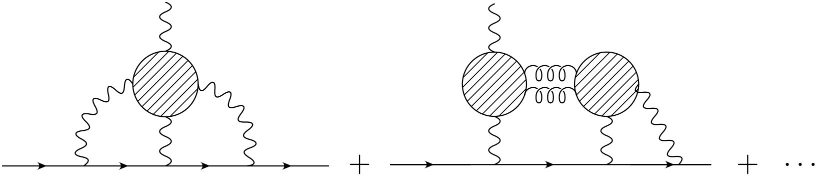




Zero within $\sim$ 3% statistical errors for heavier quarks

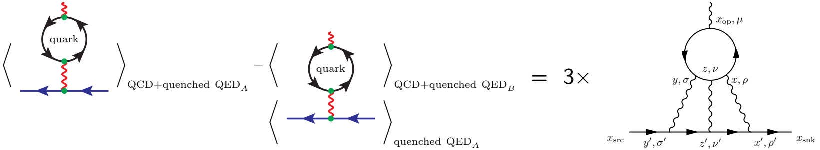
Outline I

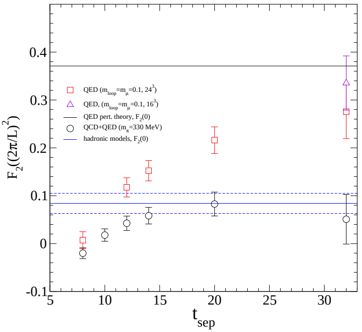
Hadronic light-by-light (HLbL) scattering



- Models: $(105 \pm 26) \times 10^{-11}$ [Prades et al., 2009, Benayoun et al., 2014]
  $(116 \pm 40) \times 10^{-11}$ [Jegerlehner and Nyffeler, 2009]
  systematic errors difficult to quantify
- Dispersive approach difficult, but progress is being made
  [Colangelo et al., 2014b, Colangelo et al., 2014a, Pauk and Vanderhaeghen, 2014b, Pauk and Vanderhaeghen, 2014a, Colangelo et al., 2015]
- First non-PT QED+QCD calculation [Blum et al., 2015]
- Very rapid progress with pQED+QCD [Jin et al., 2015]
- New HLbL scattering calculation by Mainz group [Green et al., 2015]

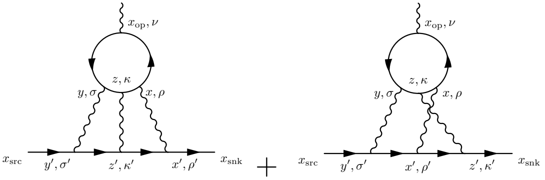
Non-perturbative QED method [Blum et al., 2015]





- quark-connected part of HLbL
- $a^{-1} = 1.7848$ GeV, $(2.7 \text{ fm})^3$
- $m_\pi = 330$ MeV, $m_\mu = 190$ MeV
- Consistent with model expectations (J. Bijnens)
- Agreement with models accidental
- $O(\alpha^2)$ noise, $O(\alpha^4)$ corrections

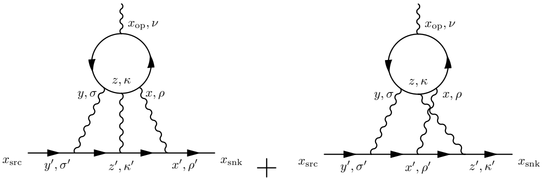
HLbL: pQED, L. Jin's talk, Lattice 2015 [Jin et al., 2015]



+ 4 more

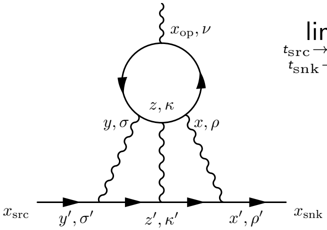
- Compute quark loop non-perturbatively
- Photons, muon on lattice, but use (exact) tree-level propagators
- Work in configuration space
- Do QED (two) loop integrals stochastically
- Key insight: quark loop exponentially suppressed with x and y separation. Concentrate on "short distance" (π Compton λ)
- Chiral (DW) fermions at finite lattice spacing: UV properties like in continuum, modified by $O(a^2)$

HLbL: pQED [Jin et al., 2015]



$$
\begin{aligned}
\mathcal{F}_\nu(x,y,z,x_{\rm op},x_{\rm snk},x_{\rm src}) &= \\
-(-ie)^3 \sum_{q=u,d,s} (ie_q)^4 &\left\langle {\rm tr}\left[\gamma_\nu S_q\left(x_{\rm op},x\right)\gamma_\rho S_q(x,z)\gamma_\kappa S_q(z,y)\gamma_\sigma S_q\left(y,x_{\rm op}\right)\right]\right\rangle_{\rm QCD} \\
&\cdot \sum_{x',y',z'} G_{\rho\rho'}(x,x')G_{\sigma\sigma'}(y,y')G_{\kappa\kappa'}(z,z') \\
&\cdot\Big[S_\mu\left(x_{\rm snk},x'\right)\gamma_{\rho'}S_\mu(x',z')\gamma_{\kappa'}S_\mu(z',y')\gamma_{\sigma'}S_\mu\left(y',x_{\rm src}\right) \\
&\quad +S_\mu\left(x_{\rm snk},z'\right)\gamma_{\kappa'}S_\mu(z',x')\gamma_{\rho'}S_\mu(x',y')\gamma_{\sigma'}S_\mu\left(y',x_{\rm src}\right) \\
&\quad +4 \text{ other permutations}\Big].
\end{aligned}
$$

HLbL: pQED, point source method [Jin et al., 2015]



FT muon src, snk $\quad \mathcal{F}_\nu(\vec{q}, x, y, z, x_{\rm op}) =$

$$\lim_{\substack{t_{\rm src}\to-\infty \\ t_{\rm snk}\to\infty}} e^{E_{q/2}(t_{\rm snk}-t_{\rm src})} \sum_{\vec{x}_{\rm snk},\vec{x}_{\rm src}} e^{-i\frac{\vec{q}}{2}\cdot(\vec{x}_{\rm snk}+\vec{x}_{\rm src})} e^{i\vec{q}\cdot\vec{x}_{\rm op}} \mathcal{F}_\nu(x, y, z, x_{\rm op}, x_{\rm snk}, x_{\rm src})$$

with mom. transfer $\vec{q} = 2\pi\vec{z}/L$, and use translational invariance to shift origin:

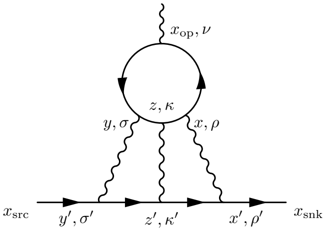
$$\begin{aligned}
\mathcal{M}_\nu(\vec{q}) &= \sum_{x,y,z} \mathcal{F}_\nu(\vec{q}, \frac{x-y}{2}, -\frac{x-y}{2}, z-w, x_{\rm op}-w) \\
&= \sum_r \left\{ \sum_{z', x'_{\rm op}} \mathcal{F}_\nu(\vec{q}, r, -r, z', x'_{\rm op}) \right\} \\
&= \left(\frac{\not{q}^+ + m_\mu}{2E_{q/2}}\right) \left(F_1(q^2)\gamma_\nu + \frac{F_2(q^2)}{2m}\frac{i}{2}[\gamma_\nu, \gamma_\beta](q_\beta) \right) \left(\frac{\not{q}^- + m_\mu}{2E_{q/2}}\right)
\end{aligned}$$

$$w = \frac{x+y}{2}, \quad r = \frac{x-y}{2}, \quad z' = z - w \quad \text{and} \quad x'_{\rm op} = x_{\rm op} - w$$

Sum over r and w stochastically, do $x'_{\rm op}$ and z' sums exactly

HLbL: pQED, point source method [Jin et al., 2015]

$$G(x,x')_{\rho\rho'} = \sum_k \frac{1}{(2\sin k/2)^2} e^{ik(x-x')}$$



- QED$_L$ [Hayakawa and Uno, 2008]
- Muon propagators FV (analytic), tree-level DWF with $L_s = \infty$
- Randomly choose w
- Compute 2 point source props in QCD at x, y, connect sink points at $x'_{\rm op}$ and z', do the latter sums exactly
- $t_{\rm src}$, $t_{\rm snk} = w^0 \pm T/2$ for each w
- Do sums over r, w (x, y) stochastically, average over QCD configurations then yields $\mathcal{M}_\nu(\vec{q})$

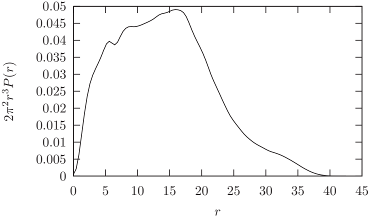
HLbL: pQED, point source method [Jin et al., 2015]

- Use importance sampling to do sum over r efficiently (sample $|r| \lesssim 1$ fm most frequently). Empirical choice:

$$p(|x_i - w|) \propto \begin{cases} 1 & (|x_i - w| < R) \\ 1/|x_i - w|^{3.5} & (|x_i - w| \geqslant R) \end{cases},$$

The distribution of the relative distance $|r|$ between any two points drawn from this set is:

$$P(r) = \sum_x p(|x-r|)p(|x|)$$



- 2+1f DWF+I-DSDR ensemble RBC/UKQCD
- 171 MeV pion mass
- $R = 4$, so do *all* points with $r = 3$ or less in this case

HLbL: point source method results [Jin et al., 2015]

Label	size	$m_\pi L$	m_π/GeV	#qcdtraj	t_{sep}	$\frac{F_2 \pm \mathrm{Err}}{(\alpha/\pi)^3}$	$\frac{\mathrm{Cost}}{\mathrm{BG/Q\ rack\ days}}$
16I	$16^3 \times 32$	3.87	0.423	16	16	0.1235 ± 0.0026	0.63
24I	$24^3 \times 64$	5.81	0.423	17	32	0.2186 ± 0.0083	3.0
24IL	$24^3 \times 64$	4.57	0.333	18	32	0.1570 ± 0.0069	3.2
32ID	$32^3 \times 64$	4.00	0.171	47	32	0.0693 ± 0.0218	10

Table 2. Central values and errors. $a^{-1} = 1.747\mathrm{GeV}$ except for 32ID where $a^{-1} = 1.371\mathrm{GeV}$. Muon mass and pion mass ratio is fixed at physical value. For comparison, at physical point, model estimation is 0.08 ± 0.02.

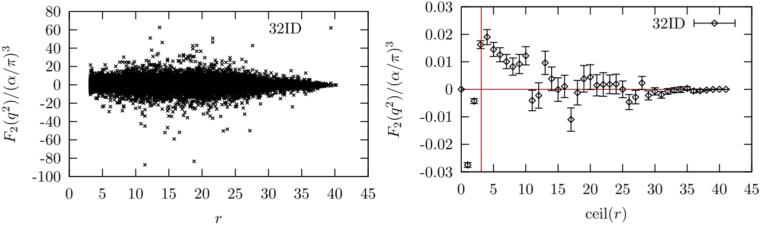


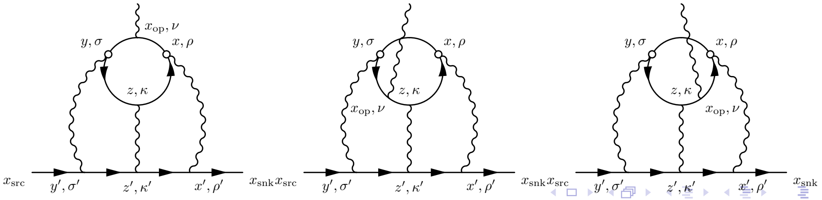
Figure 13. $32^3 \times 64$ lattice, with $a^{-1} = 1.371\mathrm{GeV}$, $m_\pi = 171\mathrm{MeV}$, $m_\mu = 134\mathrm{MeV}$.

HLbL: Current conservation [Jin et al., 2015]

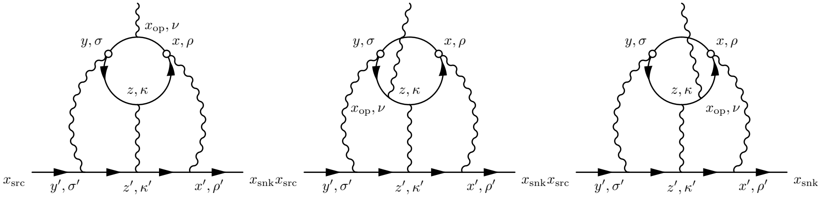
- At least one (lattice) conserved current to have convergent amplitude in continuum limit. Have chosen $J_\mu(x_{\rm op})$
- $\mathcal{M}_\mu \sim F_1(q)\gamma^\mu + i\gamma^\mu\gamma^\nu q^\nu F_2(q)/2m$ relies on WI
- To maintain constant signal-to-noise as $q \to 0$, WI (conserved current) must be exact for each config and choice of x, y, z

$$\begin{aligned}\partial_\mu\langle j^\mu(x_{\rm op})\bar\psi(x)\gamma^\rho\psi(x)\cdots\rangle &= i\delta(x_{\rm op}-x)\langle\bar\psi(x)\gamma_\nu\psi(x)\cdots\rangle \\ &\quad -i\delta(x_{\rm op}-x)\langle\bar\psi(x)\gamma_\nu\psi(x)\cdots\rangle+\cdots\end{aligned}$$

$$\langle j^\mu(x_{\rm op})\bar\psi(x)\gamma^\rho\psi(x)\bar\psi(z)\gamma^\nu\psi(z)\bar\psi(y)\gamma^\sigma\psi(y)\rangle \;=$$



HLbL: Current conservation [Jin et al., 2015]



- Compute all 3 diagrams so WI exact on each configuration
- signal *and* error vanish as $q \to 0$. Error on $F_2(q^2) \sim$ constant
- new diagrams require (6) sequential source props
- One more trick: restrict sum over z,

$$\sum_{x,y,z} \mathcal{F}_\mu(\mathbf{q}; x, y; z, x_{\rm op}) = \sum_{\substack{x,y,z \\ |x-y| < \min(|x-z|,|y-z|)}} 3\mathcal{F}_\mu(\mathbf{q}; x, y; z, x_{\rm op})$$

- Skews distribution towards small r where noise is smaller, signal larger

HLbL: Moment method for $F_2(0)$ in FV [Jin et al., 2015]

- Can do calculation directly at zero momentum for large L

$$
\begin{aligned}
\mathcal{M}_\mu(q) &= \sum_r \sum_{z,x_{\rm op}} \mathcal{F}_\mu(\vec{q}; -\frac{r}{2}, +\frac{r}{2}; z, x_{\rm op}) \\
&= \sum_{x_{\rm op}} \exp\left(iq \cdot x_{\rm op}\right) \mathcal{F}'_\mu(q, x_{\rm op}) \\
&\approx \sum_{x_{\rm op}} \left(1 + iq \cdot x_{\rm op}\right) \mathcal{F}'_\mu(q, x_{\rm op}) \\
&\approx \sum_{x_{\rm op}} iq \cdot x_{\rm op} \mathcal{F}'_\mu(q, x_{\rm op})
\end{aligned}
$$

- The “1” term vanishes in ∞ volume, exponentially small in FV

HLbL: Moment method for $F_2(0)$ in FV [Jin et al., 2015]

- Sandwich amplitude between positive energy Dirac eigenstates $\bar{u}(q_+)_{s'}, u(q_-)_s$, take $q \to 0$ limit,

$$\begin{aligned}\bar{u}(0)_{s'}\mathcal{M}_\mu(q)u(0)_s &= \bar{u}(0)_{s'}\sum_{x_{\rm op}} iq\cdot x_{\rm op}\mathcal{F}'_\mu(q=0,x_{\rm op})u(0)_s \\ &= \bar{u}(0)_{s'}\left[i\frac{F_2(q^2)}{4m}[\gamma_\mu,\gamma_\nu]q_\nu\right]u(0)_s\end{aligned}$$

where $q_\pm = (E_{\vec{q}/2}, \pm\vec{q}/2)$.

- get $F_2(0)$ directly from $\sum_{x_{\rm op}} x_{\rm op}\mathcal{F}'_\mu(0,x_{\rm op})$
- For $\mu = i, \cdots,$

$$\frac{F_2(0)}{2m}\overline{u}_{s'}\vec{\Sigma}\, u_s = \frac{1}{2}\sum_{r,z,x_{\rm op}}\vec{x}_{\rm op}\times i\overline{u}_{s'}\vec{\mathcal{F}}'\left(\frac{r}{2},-\frac{r}{2},z,x_{\rm op}\right)u_s$$

- Can use local (not conserved) current for all four currents since $x_{\rm op}=0$ kills contact terms

A word on excited state contamination [Jin et al., 2015]

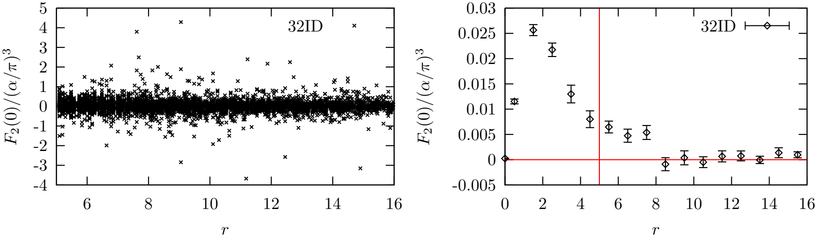
Usual method:

- (hadronic) external states "interpolated" far from operator insertion point $x_{\rm op}$
- excited states exp. suppressed relative to ground state

Our method:

- Sum over $x_{\rm op}$
- Includes points where $t_{\rm op} = t_{src}$ or $t_{\rm snk}$ or is nearby
- Origin of quark loop $x + y$ in middle of t_{src} and $t_{\rm snk}$, so these are exponentially suppressed.
- usual choice: $t_{snk} - t_{\rm src} = T/2$, but check for contamination with shorter separations

Exact photon method with all improvements [Jin et al., 2015]



- 171 MeV Pion, $m_\pi L \gtrsim 4$
- AMA used for quark propagators
  (1000 low modes, sloppy CG: 100 iters)

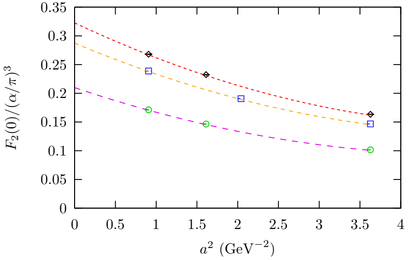
Exact photon method with all improvements [Jin et al., 2015]

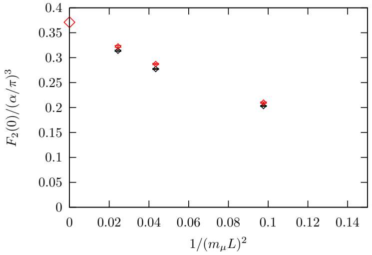
Method	$(F_2(q^2) \pm \mathrm{Err})/(\alpha/\pi)^3$	#confs	#prop-per-conf	$\sqrt{\mathrm{Var}}/(\alpha/\pi)^3$
Exact-Photon	0.0693 ± 0.0218	47	$58 + 8 \times 16$	2.04
Conserved	0.1022 ± 0.0137	13	$(58 + 8 \times 16) \times 7$	1.78
Moment*	0.0994 ± 0.0029	23	$(217 + 512) \times 2 \times 4$	1.08
Moment-AMA-corrrection	0.0060 ± 0.0043	23	$(10 + 48) \times 2 \times 4$	0.44
Moment	0.1054 ± 0.0054	23		

Method	$r_{\max}$	short-distance	long-distance	long-dis-pair-error
Exact-Photon	3	-0.0152 ± 0.0017	0.0845 ± 0.0218	0.0186
Conserved	3	0.0637 ± 0.0034	0.0385 ± 0.0114	0.0093
Moment*	5	0.0791 ± 0.0018	0.0203 ± 0.0026	0.0028
Moment-AMA-corrrection	2	0.0024 ± 0.0006	0.0036 ± 0.0044	0.0045

Table VIII. We use the 32ID lattice and set $m_\mu = 134\mathrm{MeV}$, the seperation between the muon source and sink $t_{\mathrm{sep}} = 32$. $\sqrt{\mathrm{Var}} = \mathrm{Err} \times \sqrt{\#\mathrm{confs} \times \#\mathrm{prop\text{-}per\text{-}conf}}$. We compute the short distance

Continuum and ∞ volume limits in QED [Jin et al., 2015]

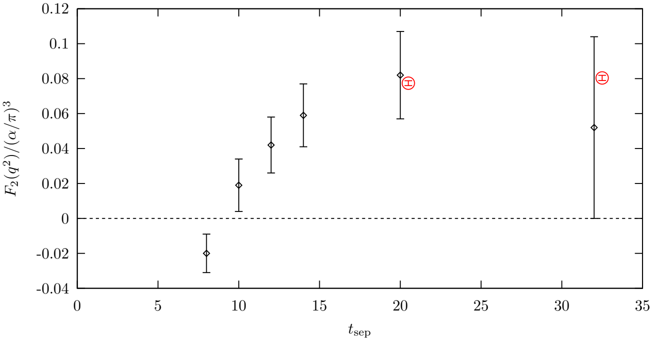




- Using all improvements
- QED systematics large but under good control
- Limits quite consistent with PT result

Dramatic improvement [Jin et al., 2015]

- Including all improvements, statistical errors reduced by 10×



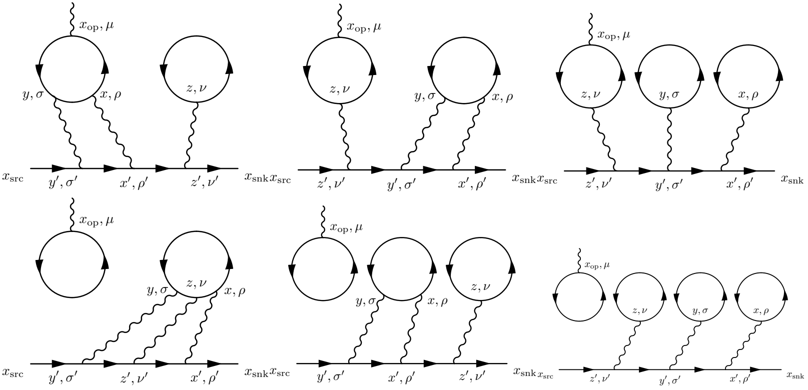
- quark-connected part of HLbL, $q = 2\pi/L$
- $a^{-1} = 1.7848$ GeV, $(2.7\text{ fm})^3$
- $m_\pi = 330$ MeV, $m_\mu = 190$ MeV
- Strong check on method(s)

Next calculation

ALCC award on MIRA at ANL ALCF,

- Applying improved point source method to physical light quark mass 2+1f Möbius DWF ensemble (RBC/UKQCD)
- $(5.5\text{ fm})^3$ QCD box, $a = 0.114$ fm ($a^{-1} = 1.7848$ GeV)
- Use AMA with 1000-2000 low-modes, $\sim$ 4500 sloppy props per configuration
- Same size QED and QCD boxes to start, but different size boxes under investigation
- Parasitic studies: HVP, mass splittings, ...

M. Hayakawa's talk at Lattice 2015 [Jin et al., 2015]



- SU(3) Flavor (only 1 survives), Zweig suppressed
- Requires explicit HVP subtraction when any quark loop with two photons is not connected to others by gluons
- Use dynamical QED+QCD or only valence quarks

Solving QED FV effects

- Integrand exponentially suppressed with distance between any pair of points on the quark loop. FV effect is small.
- Amplitude *not* suppressed with distance between points on muon line and loop. FV effect is large.
- Put QED in larger, perhaps ∞, box, QCD unchanged
- use ∞ volume photon on finite box
- Can compute average QCD loop and do muon line once, offline, so free to experiment with size of QED box

Outline I

Summary/Outlook

- HVP
  - Very high statistical precision required
  - Progress in understanding systematics, FV, fits, moments, ...
  - Strange contribution done very well ✓
  - physical quark mass, large volume calculations in progress
  - Disconnected challenging, maybe small
- HLBL
  - First calculations for connected part very promising– calculation with controlled errors clearly within reach of lattice methods.
  - 5% stat. errors already for near physical pions
  - FV effects large but controllable. ∞ volume limit consistent with PT. Put QCD and QED in different boxes
  - Applying improved point source method to physical quark mass 2+1f Möbius DWF ensemble RBC/UKQCD
  - Disconnected part challenging, new ideas under investigation
  - Lattice important to compare (SM) with experiment

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  - Ds cluster at FNAL (USQCD)
  - USQCD BQ/Q at BNL

Outline I

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