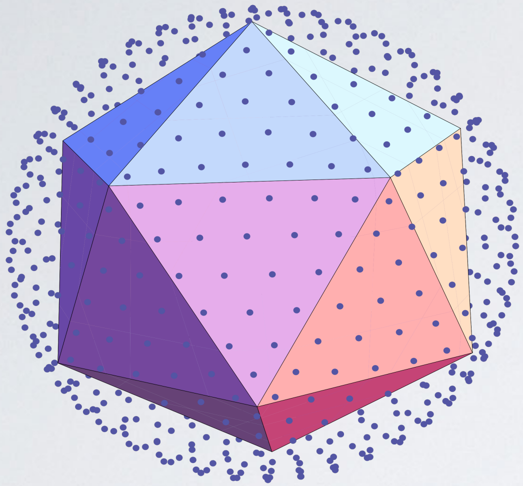
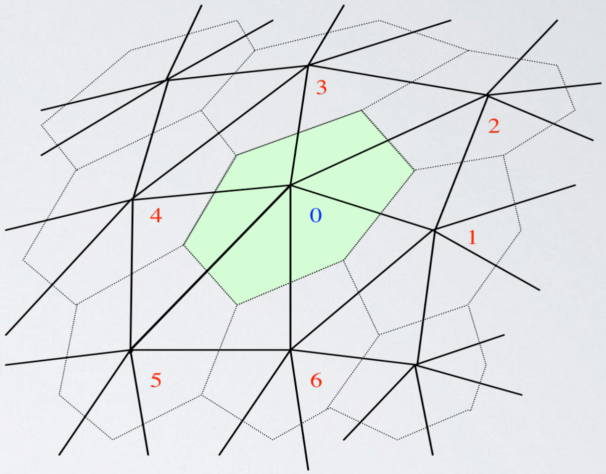
*LATTICE CFT ON THE RIEMANN SPHERE: TESTING QFE**





* QFE:Quantum Finite Element Method

Can it be made into a general approach for Lattice Field Theory on curved manifolds?

Rich Brower LATTICE 15 JULY 17, 2015
with G. Fleming, A. Gasbarro, T. Raben, C-I Tan, E. Weinberg

OUTLINE

1. WHY? Radial Quantization of CFT
2. HOW?: QFEM for Scalars + Fermions*
3. TEST? : Exact c=1/2 Ising CFT**

*see Spherical FEM in Andy's talk

** see QFEM counter terms in George's talk Text

Radial Quantization: *Early History*

- S. Fubini, A. Hanson and R. Jackiw PRD 7, 1732 (1972)

  Abstract: A field theory is quantized covariantly on Lorentz-invariant surfaces. Dilatations replace time translations as dynamical equations of motion. .... The Virasoro algebra of the dual resonance model is derived in a wide class of 2-dimensional Euclidean field theories.

- J. Cardy J. Math. Gen 18 757 (1985).

  Abstract: The relationship between the correlation length and critical exponents in finite width strips in two dimensions is generalised to cylindrical geometries of arbitrary dimensionality d. For d > 2 these correspond however, to curved spaces. The result is verified for the spherical model

Radial Quantization

Evolution: $H = P_0 \text{ in } t \implies D \text{ in } \tau = \log(r)$

$$ds^2 = dx^\mu dx_\mu = e^{2\tau}[d\tau^2 + d\Omega^2]$$

Can drop
Weyl factor!

$$\mathbb{R}^d \to \mathbb{R} \times \mathbb{S}^{d-1}$$

$$\text{"time"} \quad \tau = log(r) \; , \; \text{"mass"} \quad \Delta = d/2 - 1 + \eta$$

$$D \to x_\mu \partial_\mu = r\partial_r = \frac{\partial}{\partial \tau}$$

SCALING VS FULL CONFORMAL SYMMETRY

- General Field Theory with Scale invariance and Poincare Invariance
- O(d) ==> O(d,1) (Isometries of AdS space)

$$x_\mu \to \lambda x_\mu \quad , \quad x_\mu \to \frac{x_\mu}{x^2}$$

$$K_\mu : (inv \to trans \to inv)$$

$$[K_\mu, \mathcal{O}(x)] = i(x^2\partial_\mu - 2x_\mu x^\nu \partial_\nu + 2x_\mu \Delta)\mathcal{O}(x)$$

$$[D, \mathcal{O}(x)] = i(x^\mu \partial_\mu - \Delta)\mathcal{O}(x)$$

$$[D, P_\mu] = -iP_\mu \quad , \quad [D, K_\mu] = +iK_\mu \quad , \quad [K_\mu, P_\mu] = 2iD$$

EXACT CFT: POWER LAW

Conformal correlator: $\langle\phi(x_1)\phi(x_2)\rangle = C\frac{1}{|x_1 - x_2|^{2\Delta}}$

$$\begin{aligned} r_1^\Delta r_2^\Delta \langle\phi(\tau_1,\Omega_1)\phi(\tau_2,\Omega_2)\rangle &= C\frac{1}{[r_2/r_1 + r_1/r_2 - 2\cos(\theta_{12})]^\Delta} \\ &\simeq Ce^{-(\log(r_2) - \log(r_1)\Delta} \\ &= Ce^{-\tau\Delta} \end{aligned}$$

With $|x_1 - x_2|^2 = r_1 r_2[r_2/r_1 + r_1/r_2 - 2\cos(\theta_{12})]$

as $\tau = \log(r_2) - \log(r_1) \to \infty$

BACK TO THE BOOTSTRAP! (CFTS : NO LOCAL LAGRANGIAN)

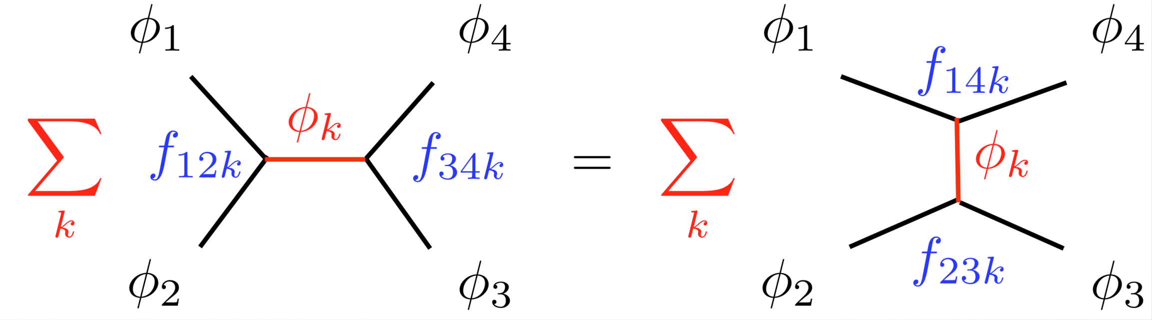
(i.e. Data: spectra + couplings to conformal blocks)

Exact 2 and 3 correlators

$$\langle \phi(x_1)\phi(x_2) \rangle = C \frac{1}{|x_1 - x_2|^{2\Delta}}$$

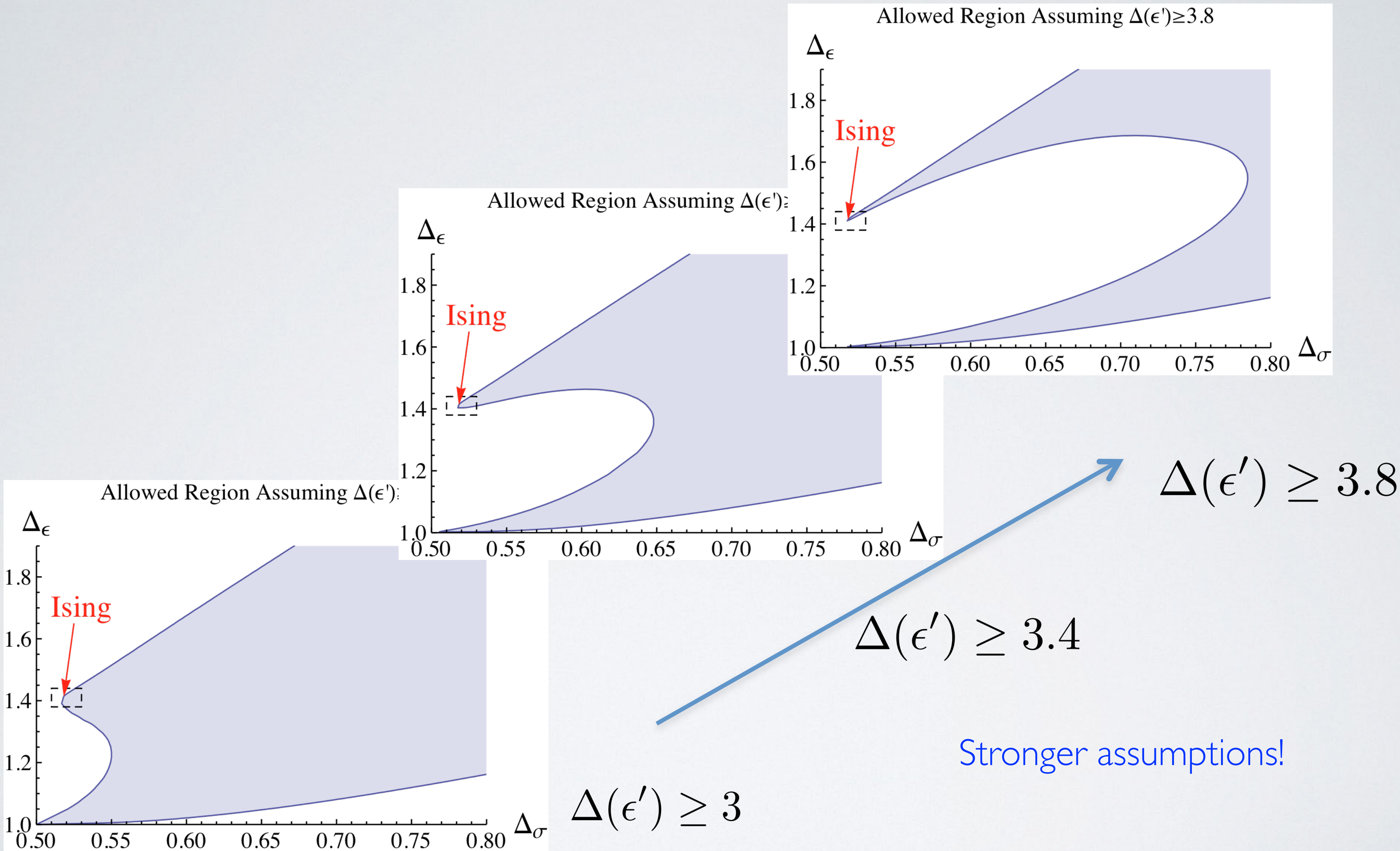
$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k \frac{f_{ijk}}{|x_1 - x_2|^{\Delta_i + \Delta_j - \Delta_k}} \mathcal{O}_k(0)$$

Only "tree" diagrams!
"partial waves" exp: sum over conformal blocks



CFT Bootstrap: OPE & factorization completely fixed the theory

*INEQUALITIES FROM BOOTSTRAP**



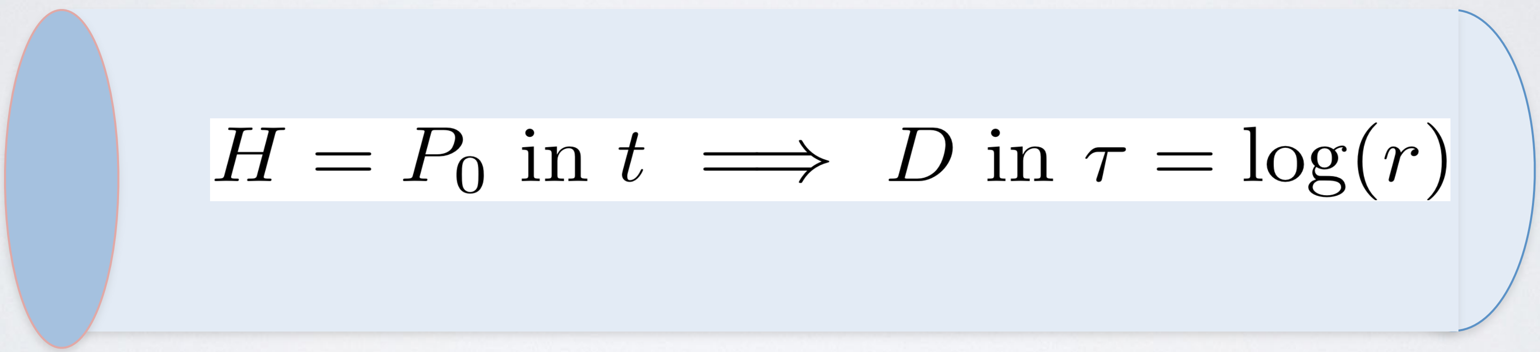
- "Solving the 3D Ising Model with the Conformal Bootstrap" (El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin and Vichi) arXiv:1203.6064v1v [hep-th] (2012)

RADIAL QUANTIZATION: NATURAL FOR CFT

Conformal (near conformal) theories are interesting for

- *BSM composite Higgs*
- *AdS/CFT weak-strong duality*
- *Model building & Critical Phenomena in general*

$$\mathbb{R} \times \mathbb{T}^3 \quad \text{vs} \quad \mathbb{R} \times \mathbb{S}^3$$



$$H = P_0 \text{ in } t \implies D \text{ in } \tau = \log(r)$$

Potential advantage: Scales increases exponentially in lattice size L!

$$1 < t < aL \implies 1 < \tau = log(r) < L$$

QFEM barrows from two traditions.

I. CLASSICAL FEM for PDEs on smooth surface.

Alexander Hrennikoff (1941) Richard Courant (1943)*

Discrete Exterior Calculus (de Rahm Complex, Whitney, etc, etc.),

Topology/Chirality 'tHooft, Leuscher et al for QCD!

II. QUANTUM FEILDS on random Lattices.

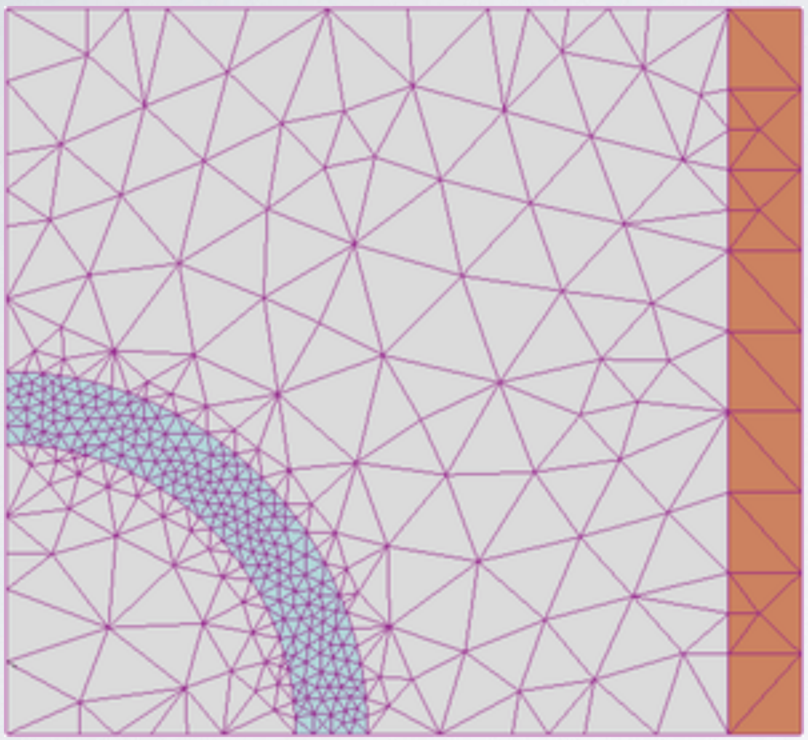
Regge Calculus T. Regge, Nuovo Cimento 19 (1961) 558. *

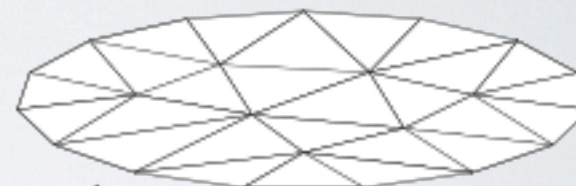
Random Lattices: N. H. Christ, R. Friedberg, and T. D. Lee, Nucl. Phys. B 202, 89 (1982).* Fermion Fields on a Random Lattice: R. Friedberg, T.D.Lee and Hai-Cang Ren Prog. of Th. Physics 86 (1986).

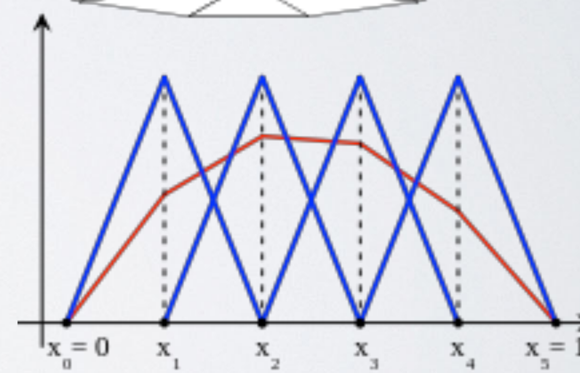
***ALL EXACTLY THE SAME FOR LINEAR ELEMENTS FOR FREE SCALAR FIELD**

Finite Element Method: What is it?

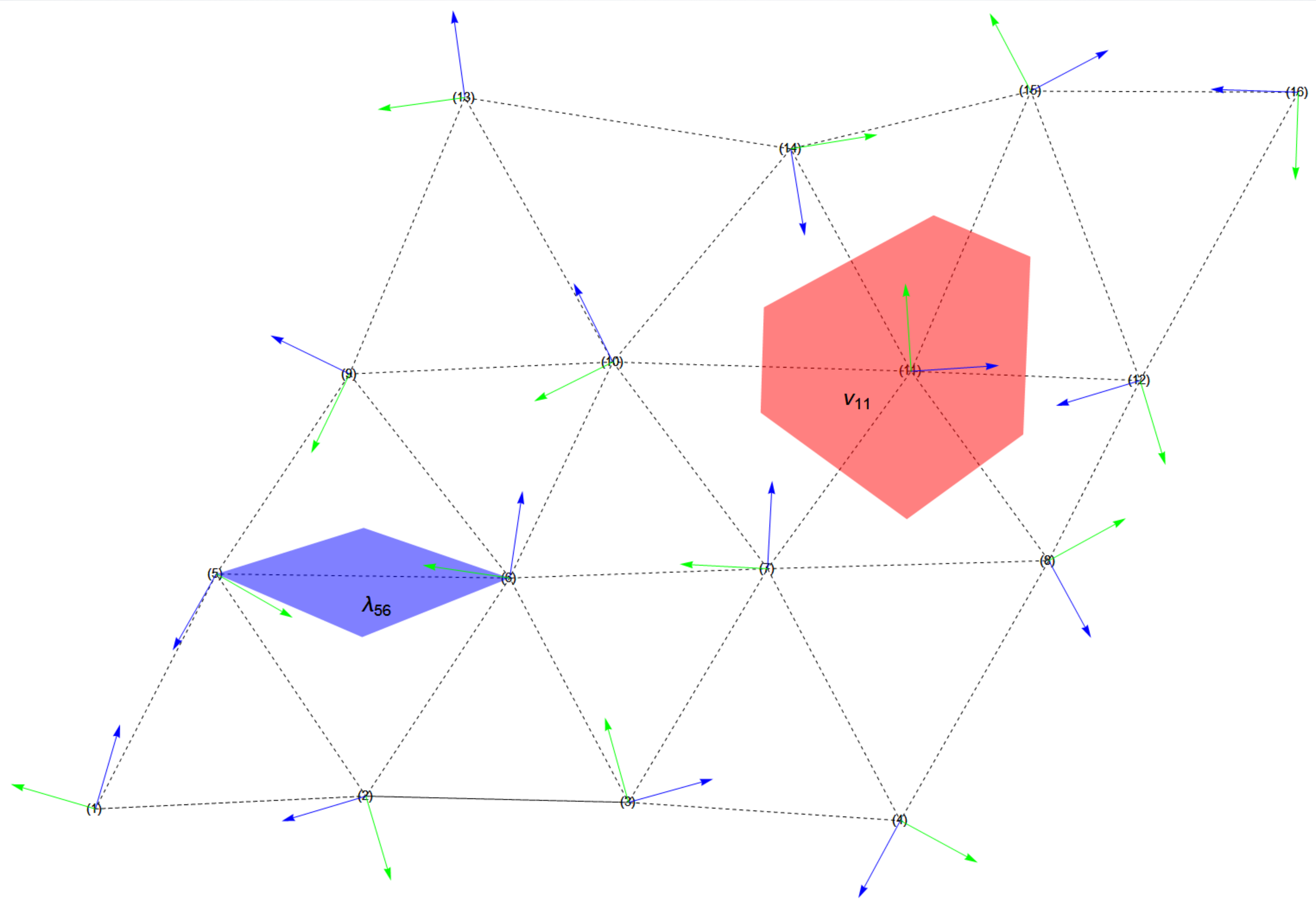
GOOGLE: 3,410,000 RESULTS







RCB, M. Cheng and G.T. Fleming,
"Improved Lattice Radial Quantization" PoS LATTICE2013 (2013) 335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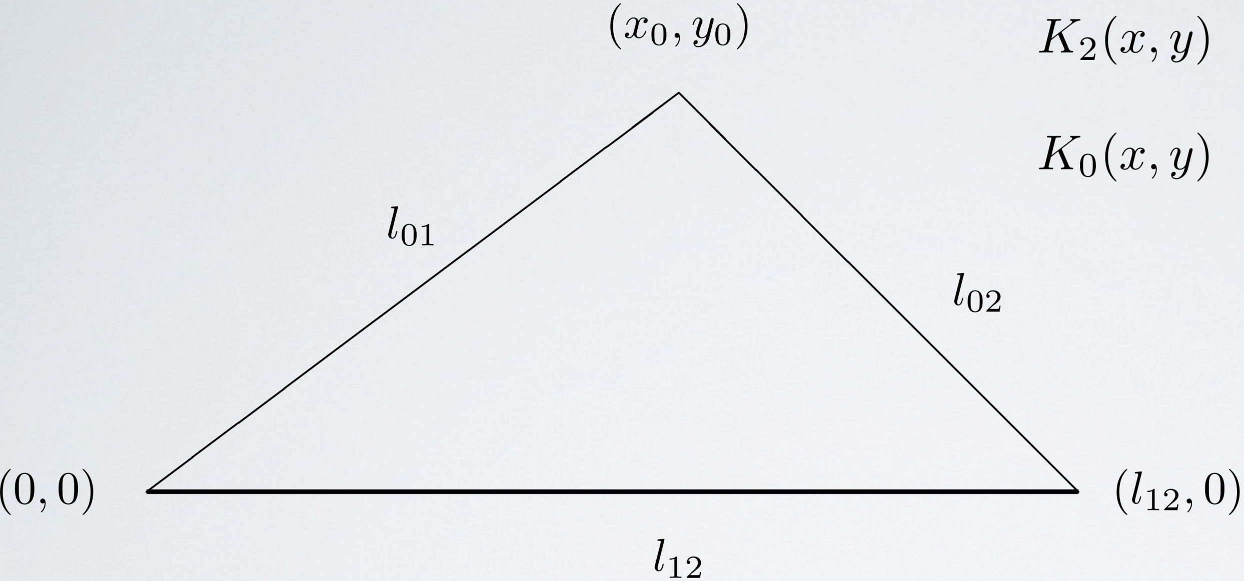
v_{11}
λ_{56}

FREE SCALAR FIELD WITH LINEAR ELEMENTS

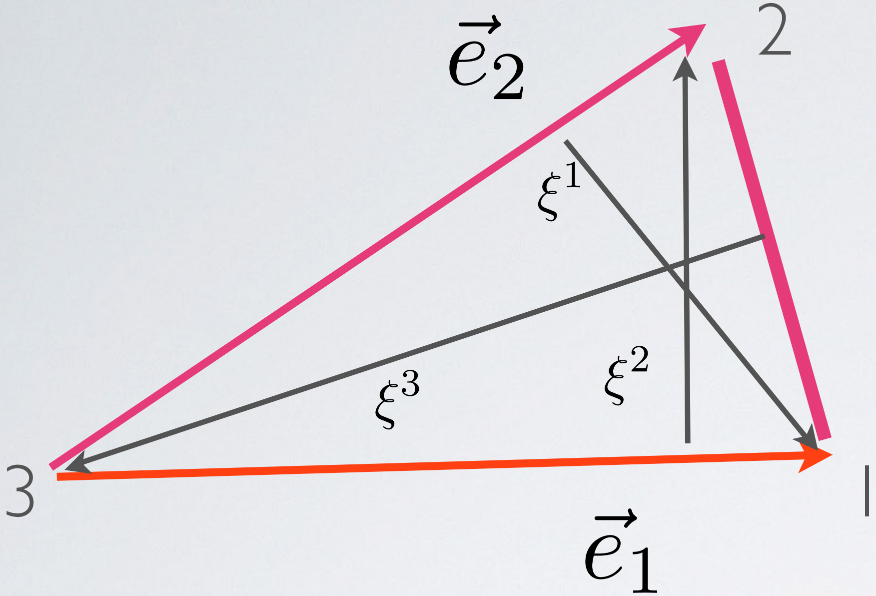


$$
\begin{aligned}
K_1(x,y) &= [l_{12} - x - \frac{(l_{12} - x_0)y}{y_0}]/l_{12} \\
K_2(x,y) &= [x - \frac{x_0 y}{y_0}]/l_{12} \\
K_0(x,y) &= \frac{y}{y_0}
\end{aligned}
$$

On each triangle expand: $\phi(x,y) = \sum_i K_i(x,y)\phi_i$ an integrate

$$
\int_{A_{012}} dx dy \partial_\mu \phi(x,y) \partial_\mu \phi(x,y) = \frac{1}{2A_{012}}[(l_{01}^2 + l_{20}^2 - l_{12}^2)(\phi_1 - \phi_2)^2 + \text{cyclic}]
$$

BARYCENTRIC CO-ORDINATES ON SIMPLEX



$$\xi^1 + \xi^2 + \xi^3 = 1 \quad , \quad 0 \le \xi^i \le 1$$

$$d\vec{x} = \vec{e}_i d\xi^i \qquad \vec{\nabla} = \vec{e}^{\,i} \partial_{\xi^i}$$

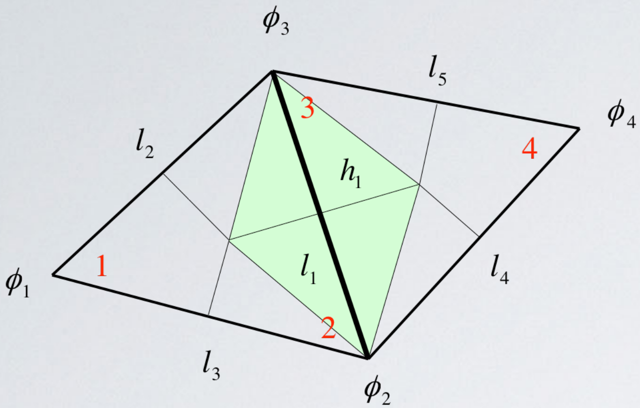
$$g_{ij} = \vec{e}_i \cdot \vec{e}_j \quad i, j = 1, 2$$

$$\vec{x} = \xi^1 \vec{r}_1 + \xi^2 \vec{r}_2 + \xi^3 \vec{r}_3$$ Planar triangle

$$\phi(\vec{x}) = \xi^1 \phi_1 + \xi^2 \phi_2 + \xi^3 \phi_3$$ Linear interpolation

$$ds^2 = d\vec{x} \cdot d\vec{x} = \frac{\partial \vec{x}}{\partial \xi^i} \cdot \frac{\partial \vec{x}}{\partial \xi^j} = g_{ij} d\xi^i d\xi^j$$

REGGE CALCULUS FORMULATION



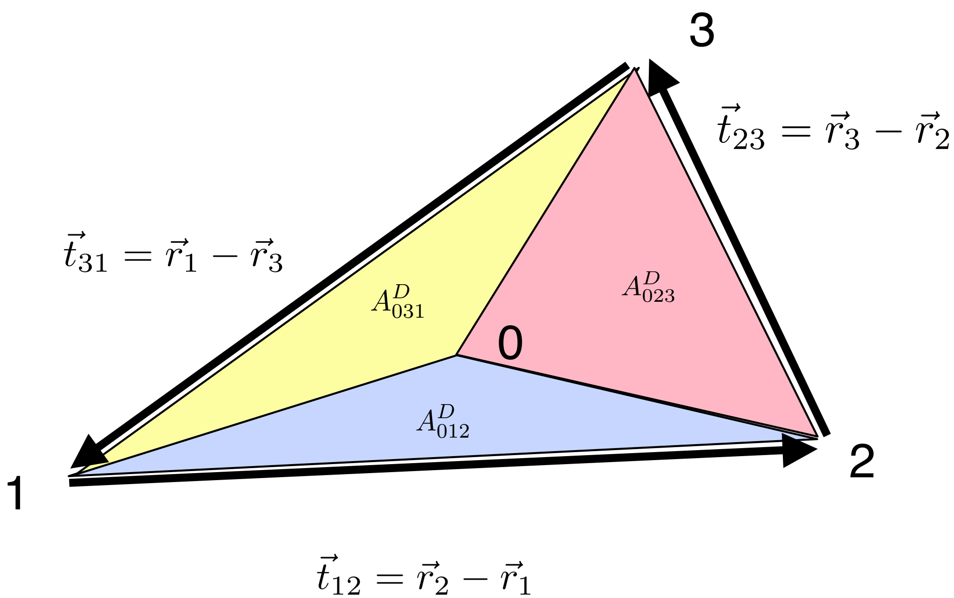
$$FEM : A_d \frac{(\phi_2 - \phi_3)^2}{l_1^2}$$

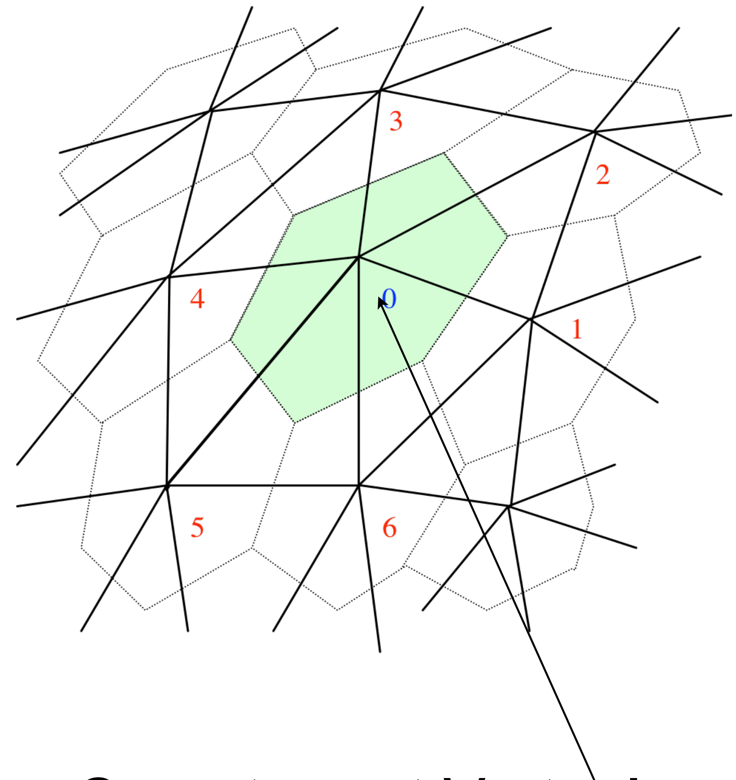
$$\text{Delaunay Link Area:} \quad A_d = h_1 l_1$$

$$\sum_{\triangle_{kij}} \sqrt{g(k)} g^{ij}(k) \frac{(\phi_k - \phi_i)(\phi_k - \phi_j)}{l_{ki} l_{kj}}$$

H. Hamber, S. Liu, **Feynman rules for simplicial gravity,** NP B475 (1996)

FEM geometry on edges.





$$\int_{A_{123}} d^2x \partial_\mu \phi \partial_\mu \phi(x) = K_{ij}(\phi_i - \phi_j)^2$$

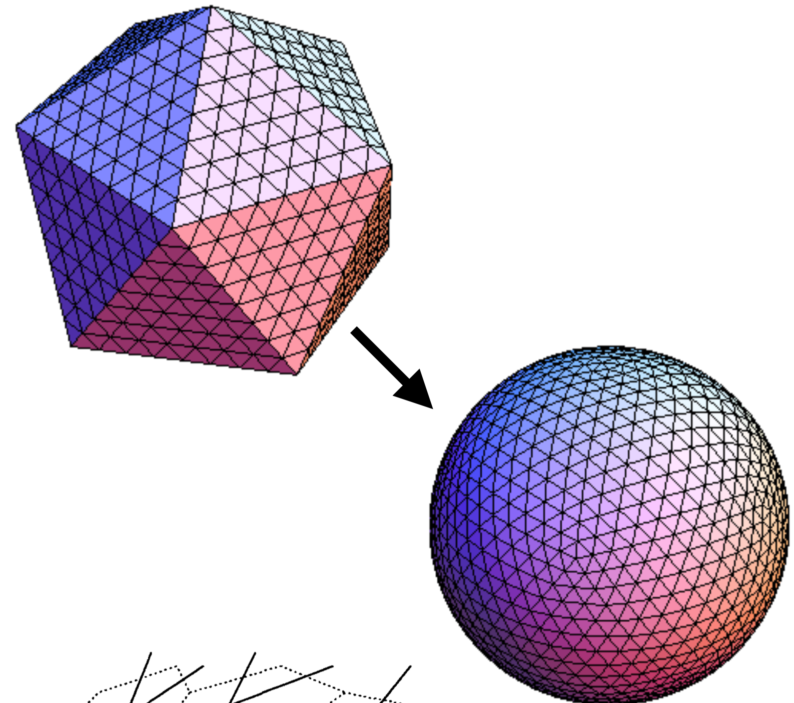
$$\sum_j K_{0j} \vec{t}_{0j} \to 0 \quad \text{in flat space}$$

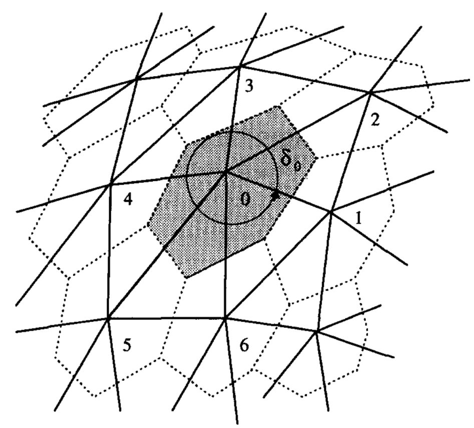
Singular Curvature at Vertex!

The l's fix metric and the local co-ordinates (diffeomorphism) and the angles the intrinsic curvature.

Discretization of $\mathbb{S}^2$

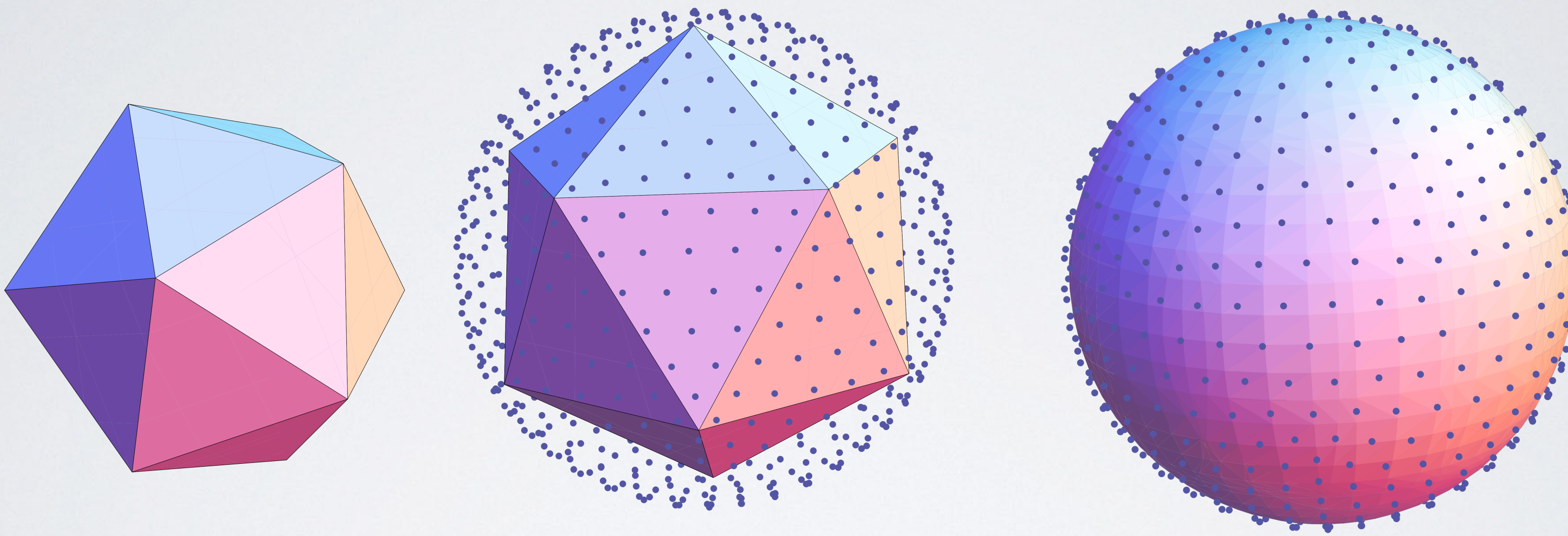
- Start with icosahedron, isometries form maximal discrete subgroup I_h of $SO(3)$.
- Tesselate triangular faces and project to surface of inscribing sphere.
- Delaunay triangulation gives nearest neighbors links of length ℓ_{ij}.
- Voronoï cells give areas of each site ω_i
- WARNING: Procedure does not generalize easily to $\mathbb{S}^d$.





PROBLEM: CURVED MANIFOLDS CAN NOT HAVE UNFORM LATTICES: (E.G. 2-SPHERE)

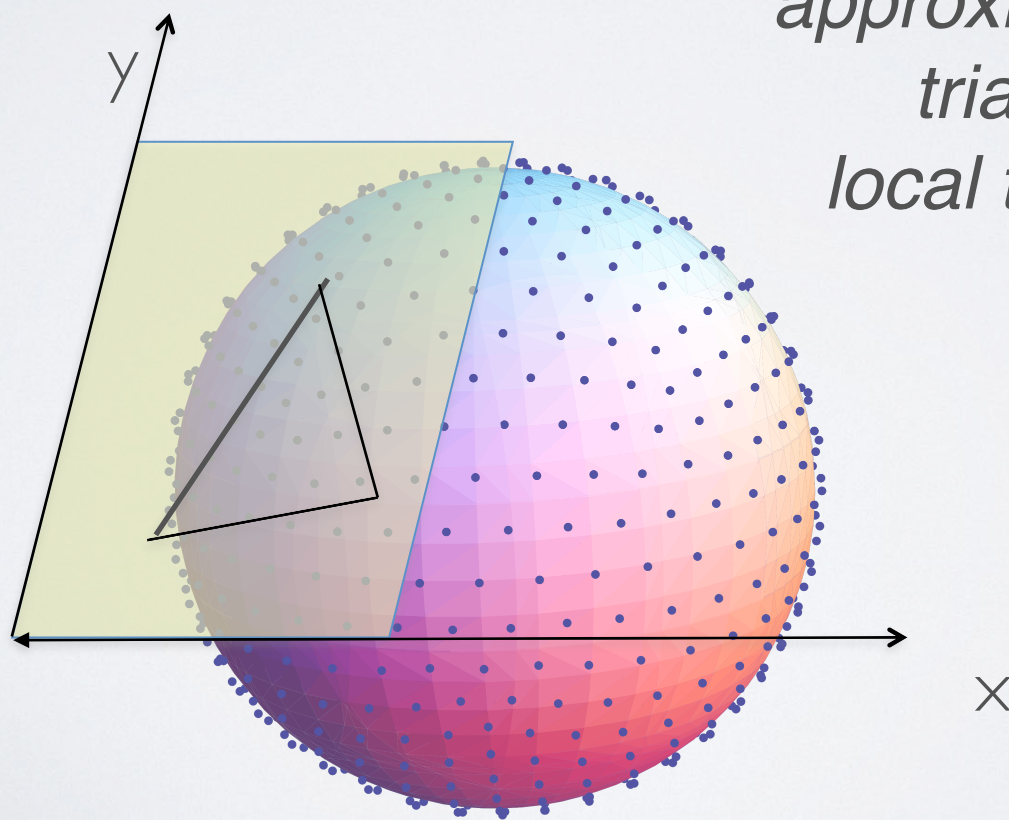
$s = 1$ $s = 8$ Projection Dilates Triangles



l = 0 (A),1 (T1) , 2 (H) are irreducible 120 Icosahedral subgroup of O(3)

TEST CFT: PHI 4TH WILSON-FISHER FIXED POINT IN 3D.

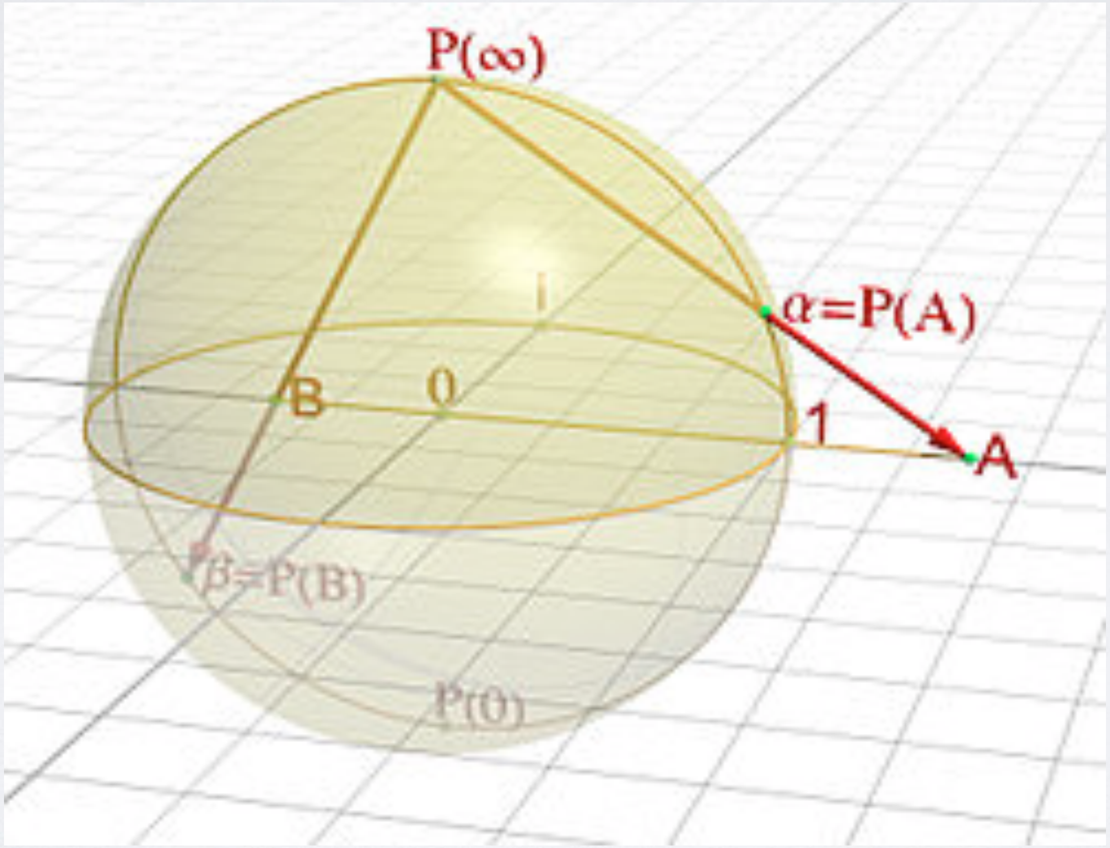
$$L = \int d^3x [\sqrt{g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \lambda \sqrt{g} (\phi^2 - \mu^2/2\lambda)^2]$$



approximate spherical triangles onto local tangent plane

EVEN "EASIER" 2D ISING/PHI 4TH ON THE RIEMANN SPHERE!

projection $$\xi = \tan(\theta/2)e^{-i\phi} = \frac{x+iy}{1+z}$$



Conformal Projection + Weyl Rescaling to the Sphere

EXACT SOLUTION TO CFT

Exact Two point function

$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{1}{|x_1 - x_2|^{2\Delta}} \to \frac{1}{|1 - cos\theta_{12}|^{\Delta}}$$

$$\Delta = \eta/2 = 1/8 \qquad\qquad x^2 + y^2 + z^2 = 1$$

4 pt function $\quad (x_1, x_2, x_3, x_4) = (0, \xi, 1, \infty)$

$$g(0,\xi,1,\infty) = \frac{1}{2|\xi|^{1/4}|1-\xi|^{1/4}}[|1+\sqrt{1-\xi}| + |1-\sqrt{1-\xi}|]$$

Critical Binder Commulant $\quad U_B^* = 1 - \frac{\langle M^4\rangle}{3\langle M^2\rangle\langle M^2\rangle} = 0.567336$

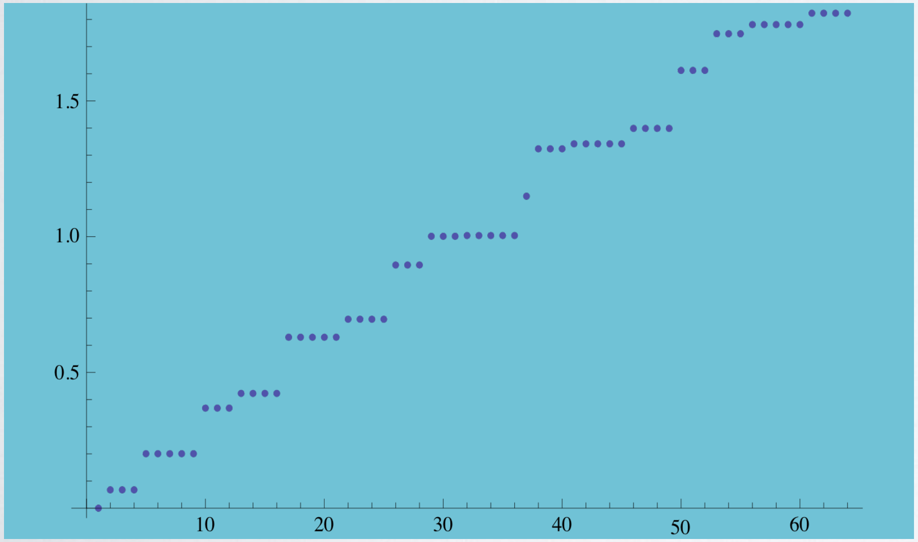
Dual to Free Fermion: 2D string theory on any manifold etc

FEM FIXES THE HUGE SPECTRAL DEFECTS OF THE LAPLACIAN ON THE SPHERE

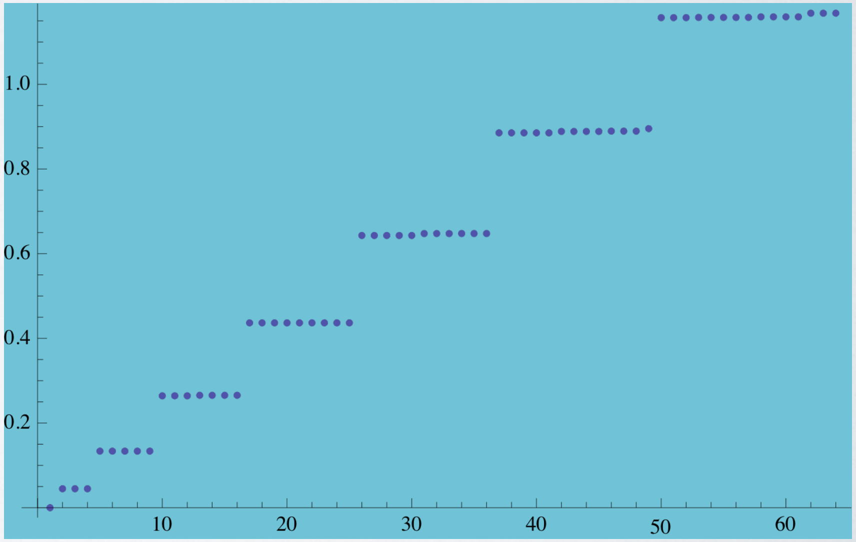
For s = 8 first (l+1)*(l +1) = 64 eigenvalues

BEFORE (K = 1)

AFTER (FEM K's)

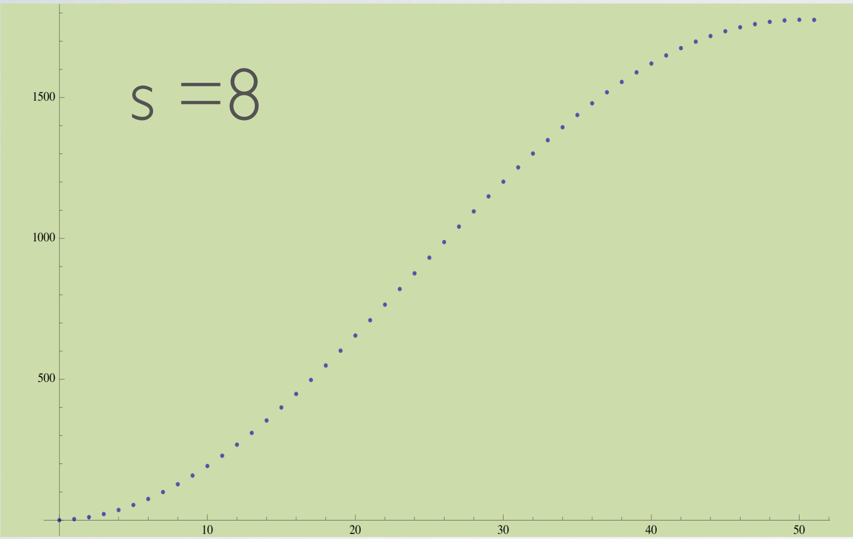


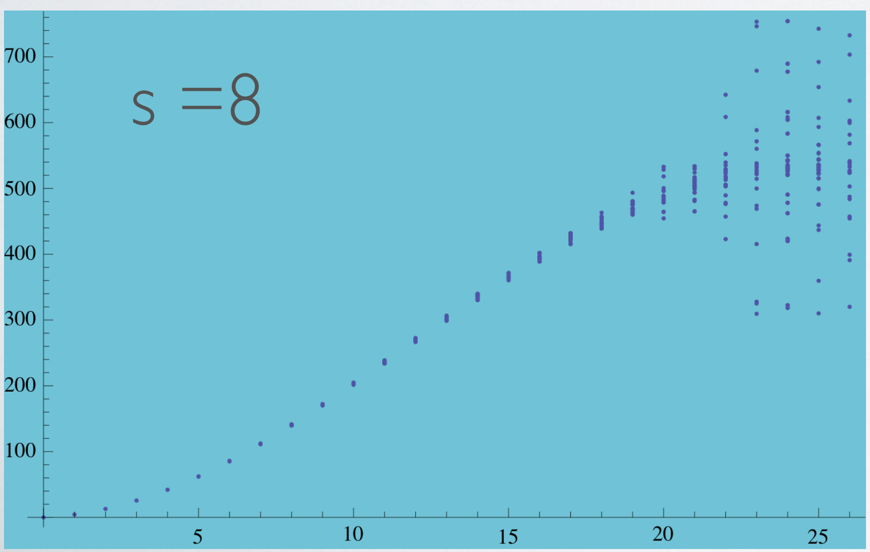
l, m

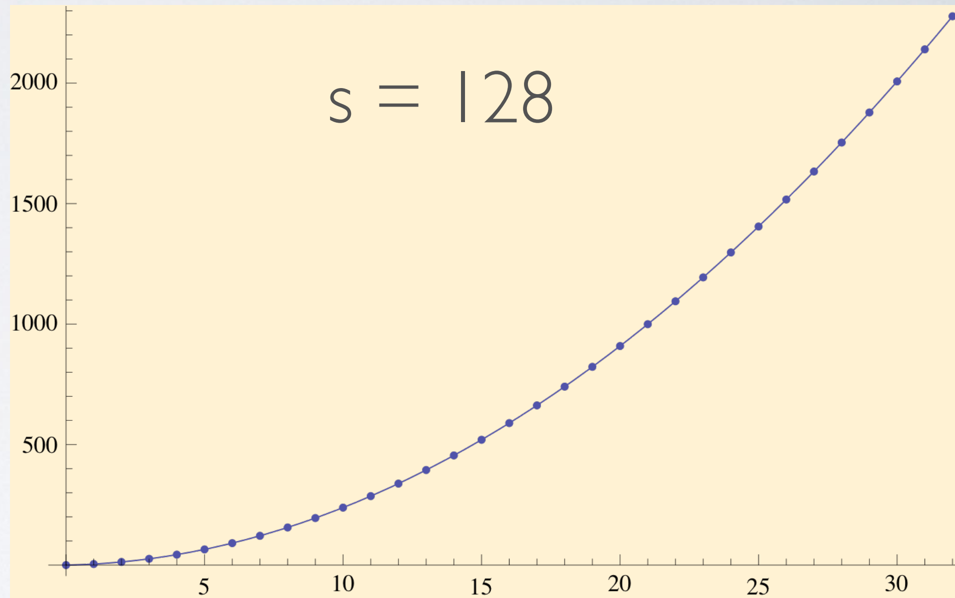


l, m

SPECTRUM OF FE LAPLACIAN ON A SPHERE





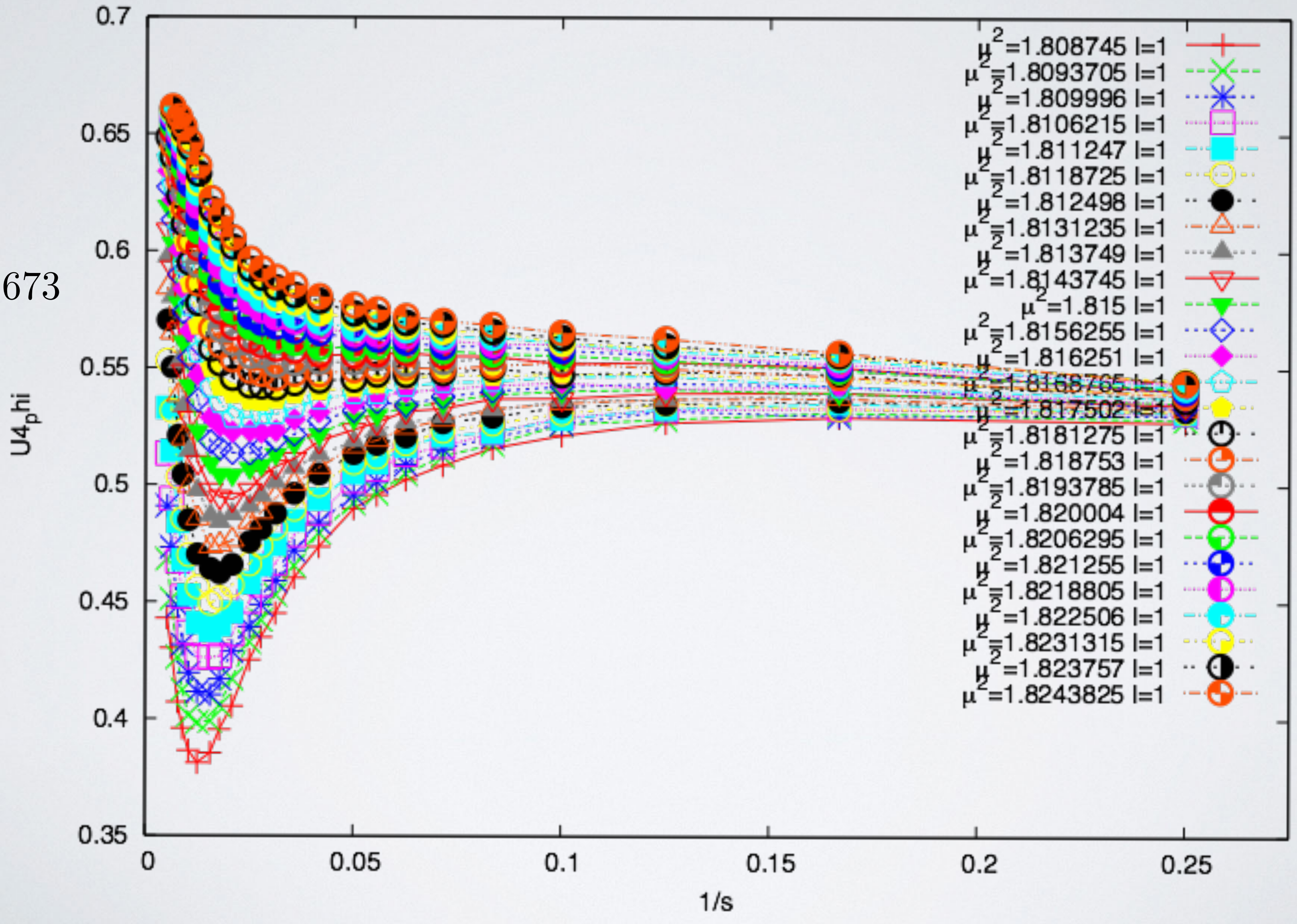


Fit

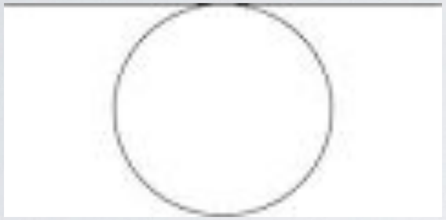
$$l + 1.00012\, l^2 - 13.428110^{-6}\, l^3 - 5.5724410^{-6}\, l^4$$

BINDER CUMMULANT NEVER

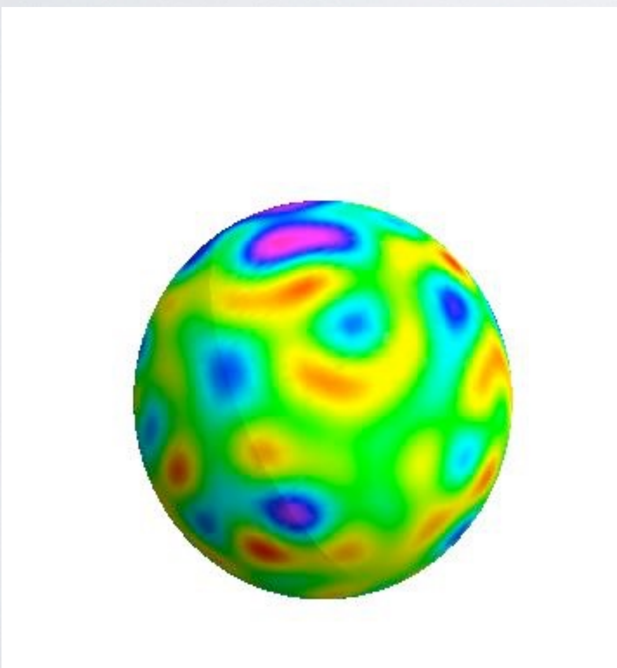
0.5673



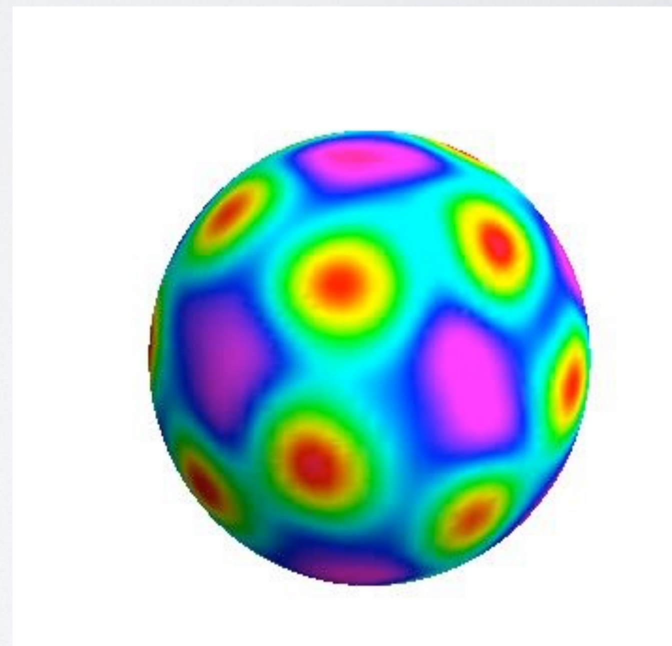
UV DIVERGENCE BREAKS ROTATIONS



$$\delta m^2 = \lambda \langle \phi(x)\phi(x) \rangle \to \frac{1}{K_{xx}}$$

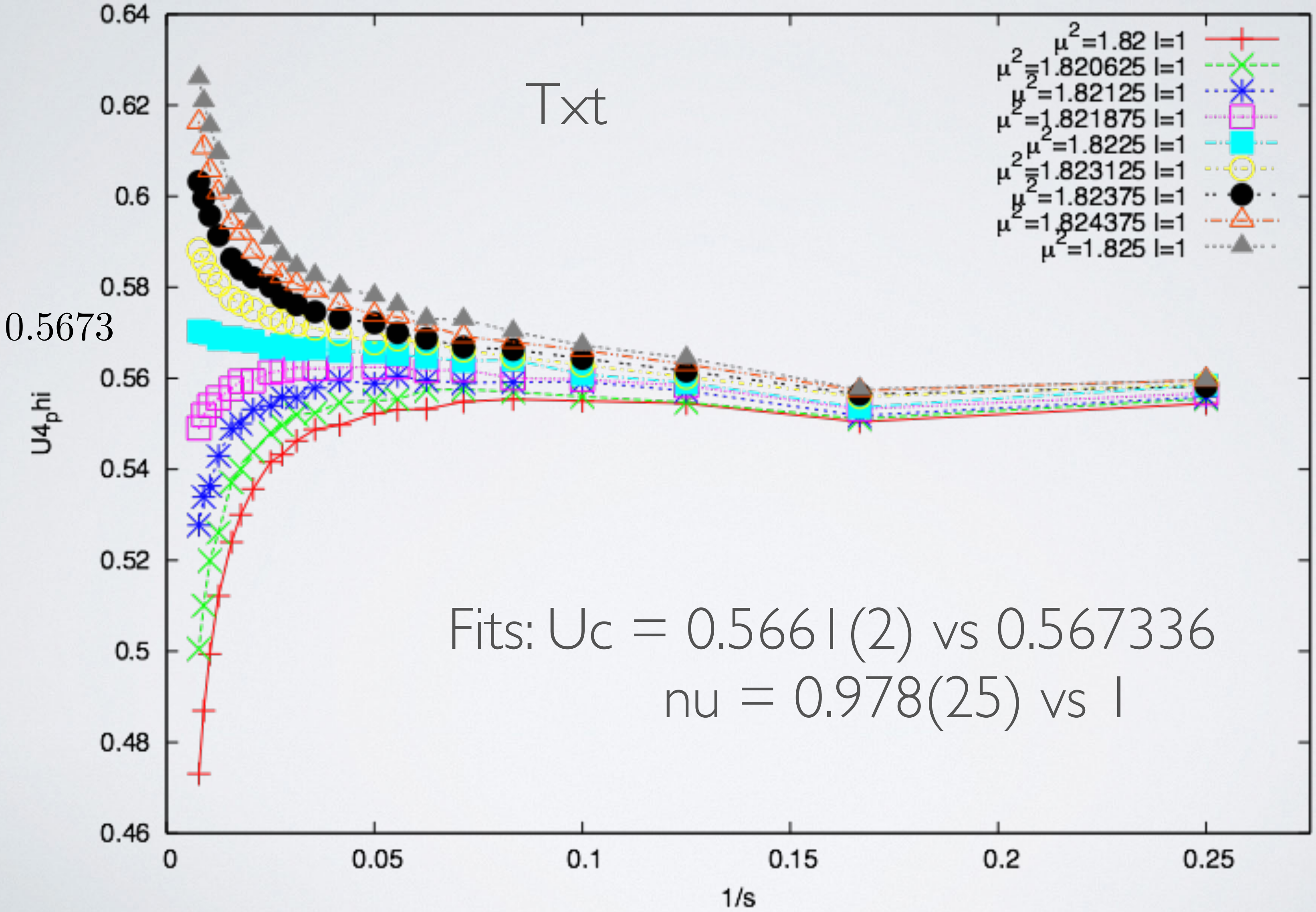


one configuration



average of config.

BINDER CUMMULANT



NEED TO IMPROVE QUANTUM LAGRANGIAN
3 POSSIBLE SOLUTIONS?

(i) Pauli-Villars* 1949

$$\frac{1}{p^2} - \frac{1}{p^2 + M_{PV}^2} \equiv \frac{1}{p^2 + p^4/M_{PV}^2}$$

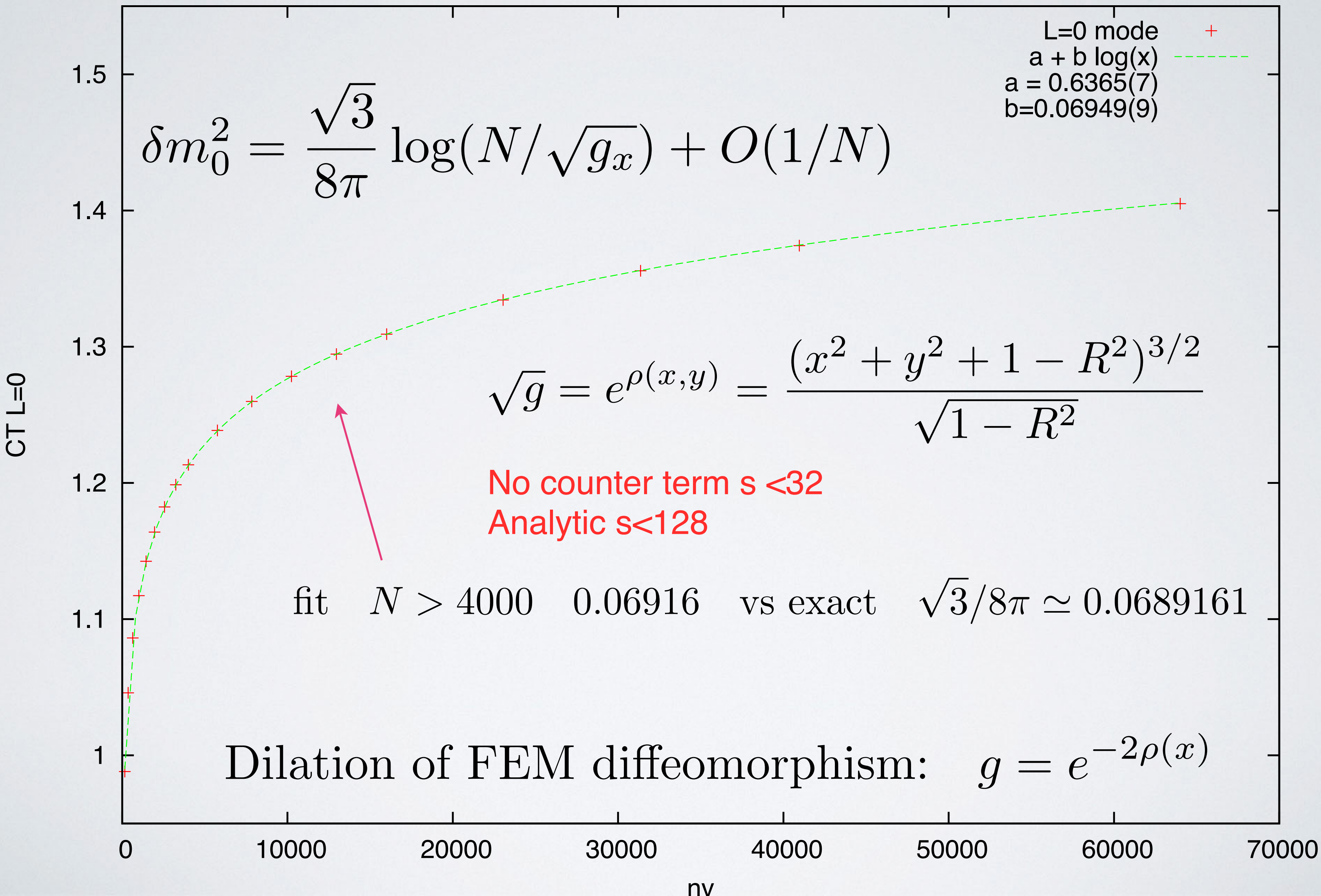
(ii) Subtract x-dependent mass Counter term

$$1/\xi \ll M_{PV} \ll \pi/a$$

(iii) Better simplex distribution (Exact density)

*(*Richard Feynman, Ernst Stueckelberg)*

ANALYTIC FORM OF COUNTER TERM!



$$\delta m_0^2 = \frac{\sqrt{3}}{8\pi} \log(N/\sqrt{g_x}) + O(1/N)$$

$$\sqrt{g} = e^{\rho(x,y)} = \frac{(x^2+y^2+1-R^2)^{3/2}}{\sqrt{1-R^2}}$$

$$\text{fit} \quad N > 4000 \quad 0.06916 \quad \text{vs exact} \quad \sqrt{3}/8\pi \simeq 0.0689161$$

$$\text{Dilation of FEM diffeomorphism:} \quad g = e^{-2\rho(x)}$$

Using Binder Cumulants

$$U_4 = \frac{3}{2}\left(1 - \frac{m_4}{3\ m_2^2}\right) \qquad m_n = \langle \phi^n \rangle$$

$$U_6 = \frac{15}{8}\left(1 + \frac{m_6}{30\ m_2^3} - \frac{m_4}{2\ m_2^2}\right)$$

$$U_8 = \frac{315}{136}\left(1 - \frac{m_8}{630\ m_2^4} + \frac{2\ m_6}{45\ m_2^3} + \frac{m_4^2}{18\ m_2^4} - \frac{2\ m_4}{3\ m_2^2}\right)$$

$$U_{10} = \frac{2835}{992}\left(1 + \frac{m_{10}}{22680\ m_2^5} - \frac{m_8}{504\ m_2^4} - \frac{m_6\ m_4}{108 m_2^5} + \frac{m_6}{18\ m_2^3} + \frac{5\ m_4^2}{36\ m_2^4} - \frac{5\ m_4}{6\ m_2^2}\right)$$

$$U_{12} = \frac{155925}{44224}\left(1 - \frac{m_{12}}{1247400\ m_2^6} + \frac{m_{10}}{18900\ m_2^5} + \frac{m_8\ m_4}{2520\ m_2^6} - \frac{m_8}{420\ m_2^4} + \frac{m_6^2}{2700 m_2^6} - \frac{m_6 m_4}{45\ m_2^5} + \frac{m_6}{15\ m_2^3} - \frac{m_4^3}{108\ m_2^6} + \frac{m_4^2}{4\ m_2^4} - \frac{m_4}{m_2^2}\right)$$

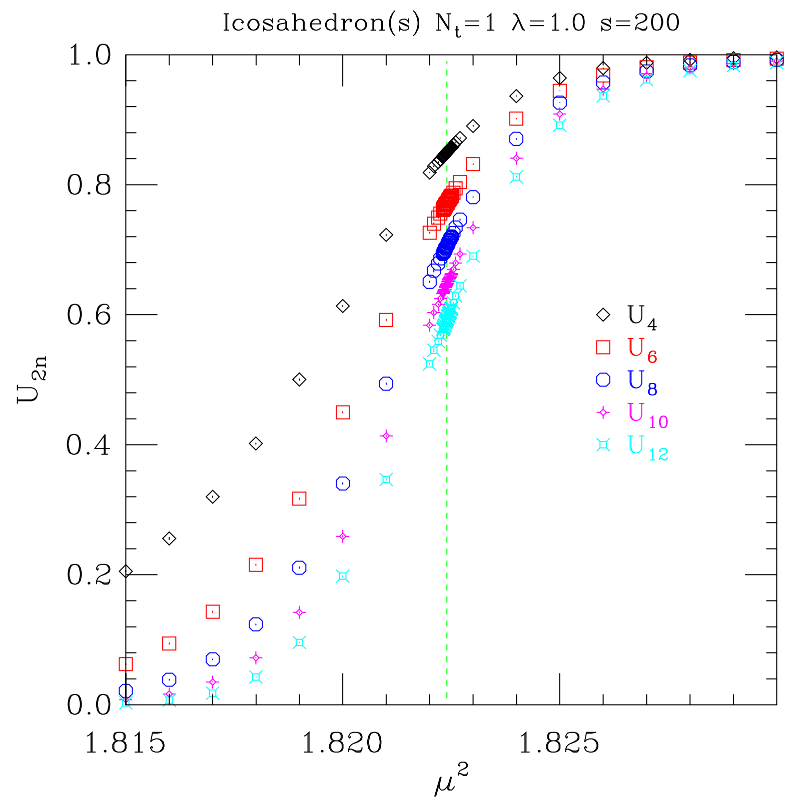
In infinite volume

$U_{2n}=0$ in disordered phase

$U_{2n}=1$ in ordered phase

$0<U_{2n}<1$ on critical surface

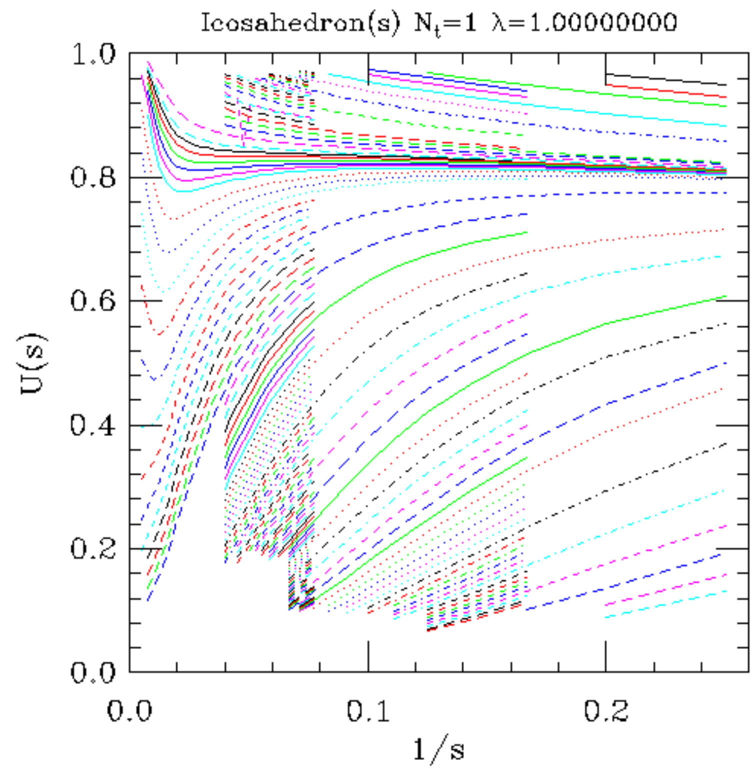
- $U_{2n,cr}$ are universal quantities.
- Deng and Blöte (2003): $U_{4,cr}=0.851001$
- Higher critical cumulants computable using conformal 2n-point functions: Luther and Peschel (1975) Dotsenko and Fateev (1984)



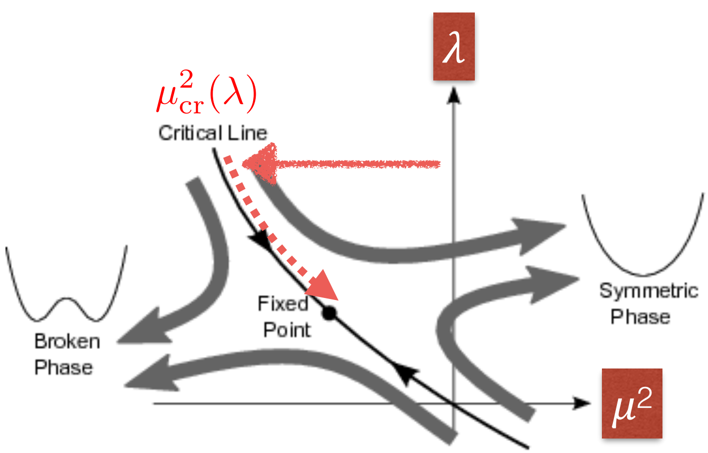
Approaching Critical Surface (1)

$$U_{2n}(\mu^2, \lambda, s) \quad = \quad U_{2n,\mathrm{cr}} + a_{2n}(\lambda)\ [\mu^2 - \mu^2_{\mathrm{cr}}(\lambda)]\ (1/s)^{-1/\nu} + b_{2n}(\lambda)\ (1/s)^{\omega}$$

- We fix λ=1 and tune μ^2 to look for critical surface.
- If we can find critical surface, expand using known critical exponents: ν=1, ω=2.
- No guarantee that critical surface exists: **frustration**.



U$_{4,cr}$ = 0.851001



Approaching Critical Surface (2)

$$U_{2n}(\mu^2, \lambda, s) \quad = \quad U_{2n,\mathrm{cr}} + a_{2n}(\lambda)\ [\mu^2 - \mu^2_{\mathrm{cr}}(\lambda)]\ (1/s)^{-1/\nu} + b_{2n}(\lambda)\ (1/s)^{\omega}$$

- Numerical calculation of CT is expensive but only needs to be done once for each s and is trivially parallel.

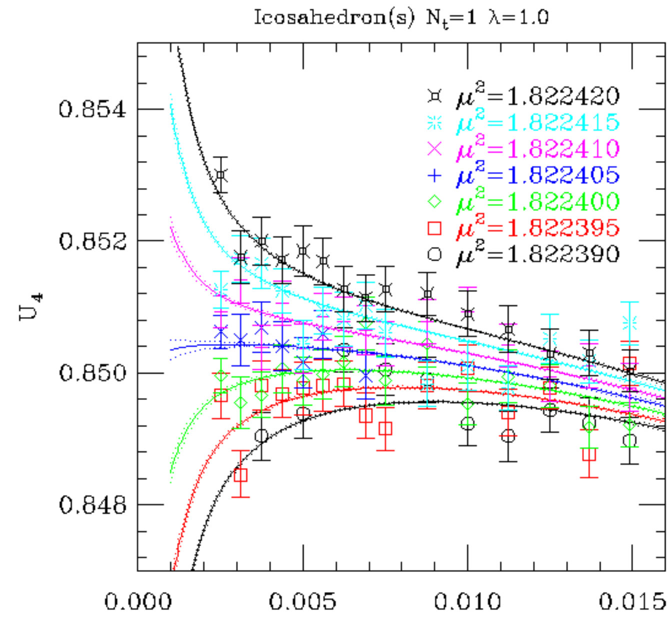
- Counterterm cures frustration.

- All cumulants critical at same μ^2_{cr}.

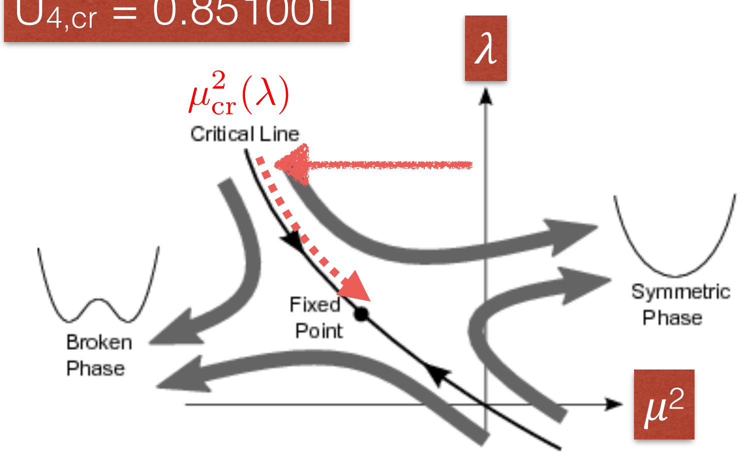
- Simultaneous fit for certain data selection:
  μ^2_{cr}=1.8224055(5), $U_{4,\mathrm{cr}}$=0.8505(1)
  χ^2/dof=1.026, dof=1701

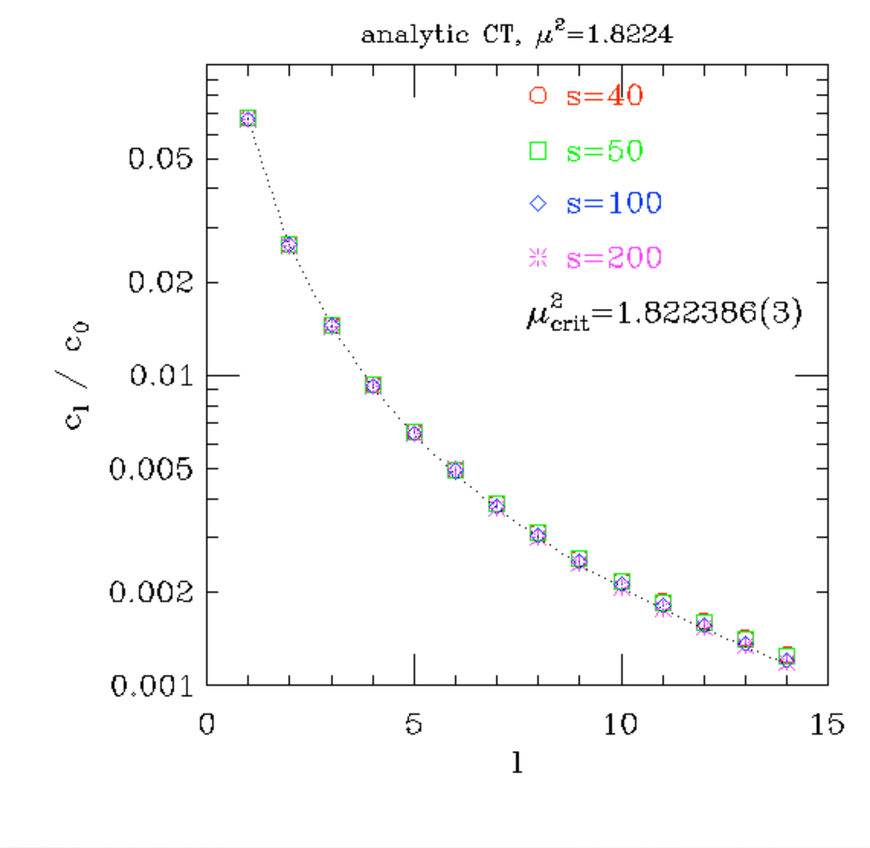
Now: 0.8508(2)
for s < 800

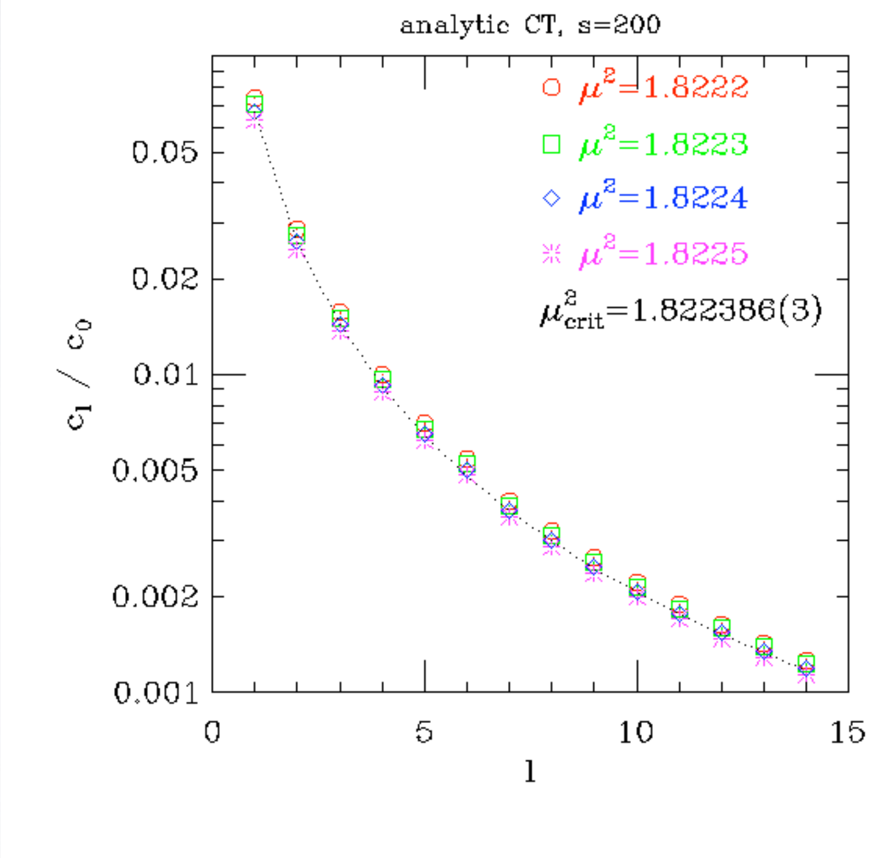
- Universal predictions:
  $U_{6,\mathrm{cr}}$=0.7724(4), $U_{8,\mathrm{cr}}$=0.7072(6)
  $U_{10,\mathrm{cr}}$=0.6483(8), $U_{12,\mathrm{cr}}$=0.5944(8)



$U_{4,\mathrm{cr}}$ = 0.851001





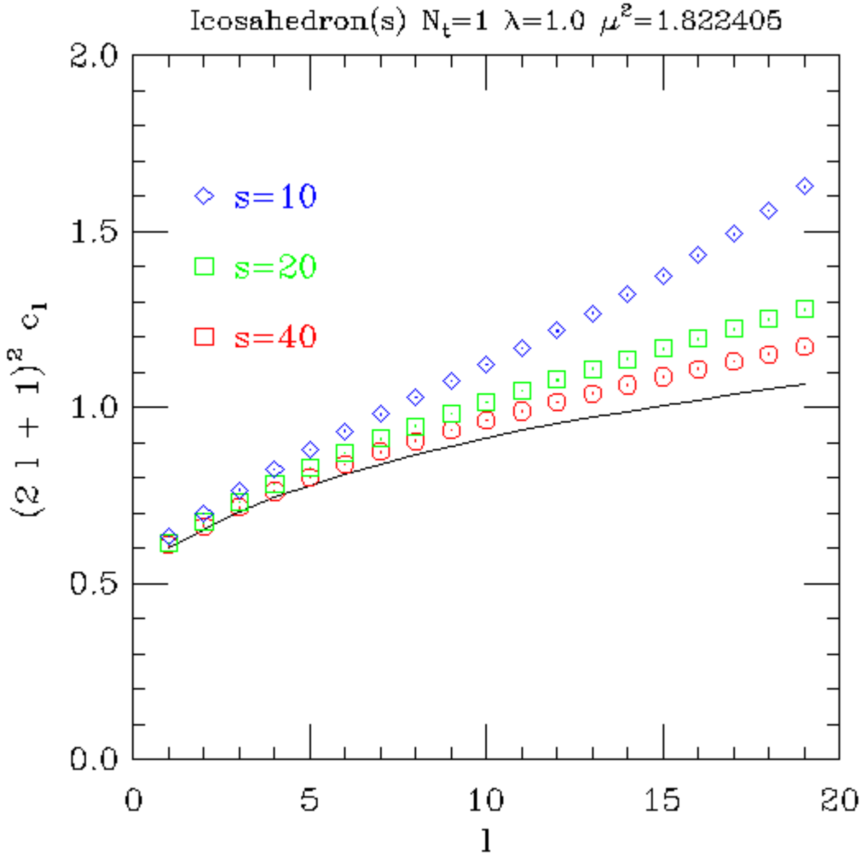


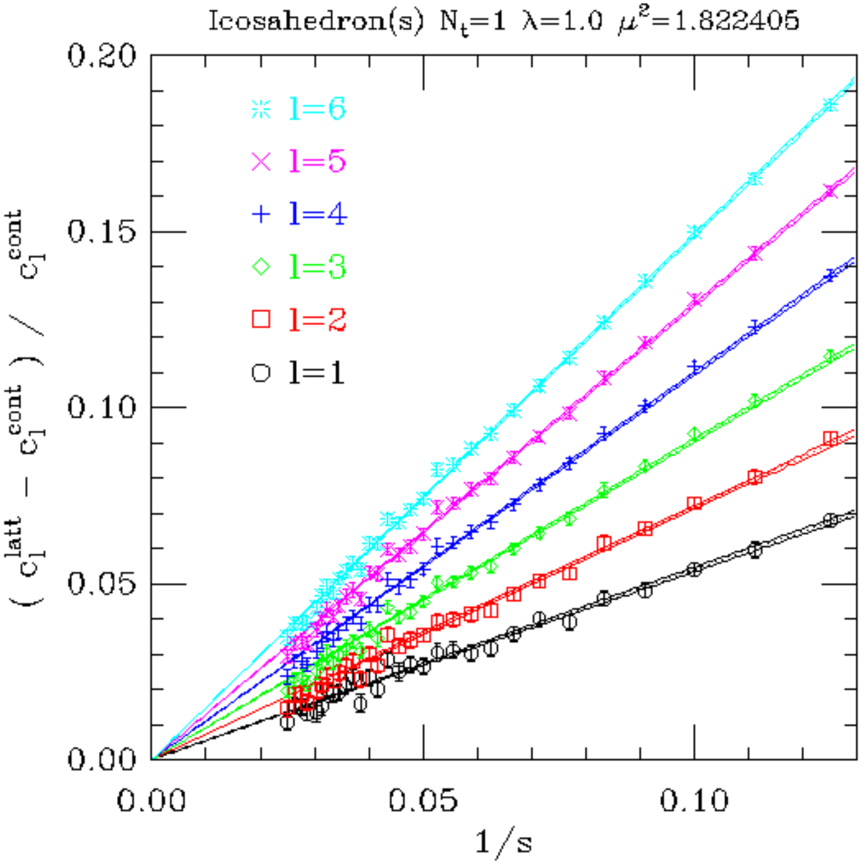
$$\int_{-1}^{1} dz \left(\frac{2}{1-z}\right)^{1/8} P_l(z) \qquad \Delta_\sigma = \eta/2 = 1/8 \simeq 0.128$$

$$\Longrightarrow \frac{16}{7}, \frac{16}{35}, \frac{48}{161}, \frac{816}{3565}, \frac{12240}{64883}, \frac{493680}{3049501}, \frac{33456}{234577}, \frac{55760}{435643}, \frac{3602096}{30930653}, \frac{20129360}{187963199}, \frac{541373840}{5450932771} \cdots$$

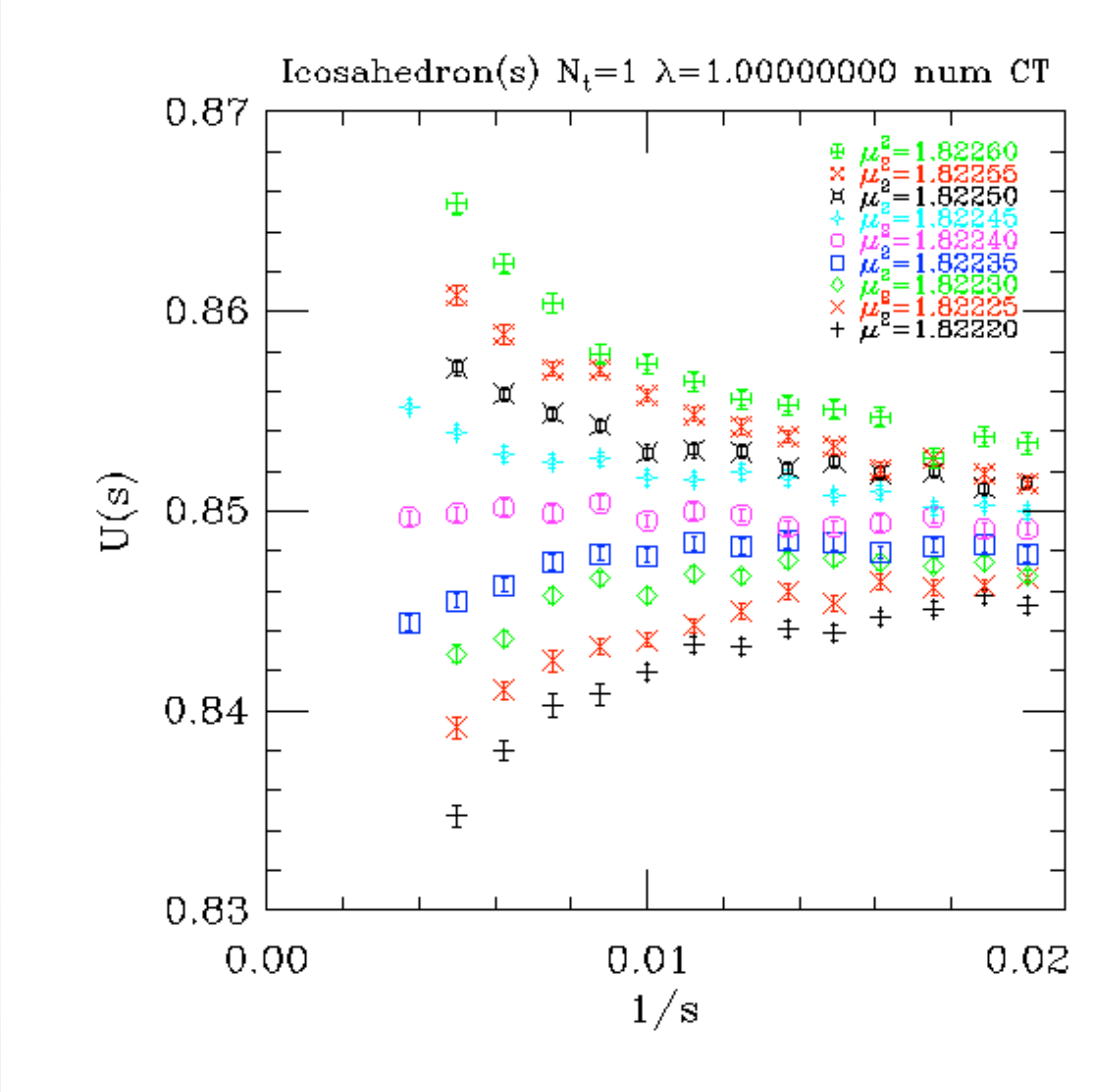
Very fast cluster algorithm:
Brower,Tamayo 'Embedded Dynamcis for phi 4th Theory" PRL 1989. Wolff single cluster + plus Improved Estimators etc

Conformal 2-pt functions on $\mathbb{S}^2$





BINDER CUMMULANT



s <= 267

.02 %

still some finite size effects

QFEM DIRAC EQUATION: MUCH HARDER

$$S = \frac{1}{2}\int d^D x \sqrt{g}\bar{\psi}[\mathbf{e}^\mu(\partial_\mu - \frac{i}{4}\boldsymbol{\omega}_\mu(x)) + m]\psi(x)$$

$\mathbf{e}^\mu(x) \equiv e^\mu_a(x)\gamma^a$ Verbein & Spin connection*

$\boldsymbol{\omega}_\mu(x) \equiv \omega^{ab}_\mu(x)\sigma_{ab}$, $\sigma_{ab} = i[\gamma_a, \gamma_a]/2$

(1) New spin structure "knows" about intrinsic geometry

(2) Need to avoid simplex curvature singularities at sites.

(3) Spinors rotations (Lorentz group) is double of O(D).

$$e^{i(\theta/2)\sigma_3/2} \to -1 \qquad \text{as} \quad \theta \to 2\pi$$

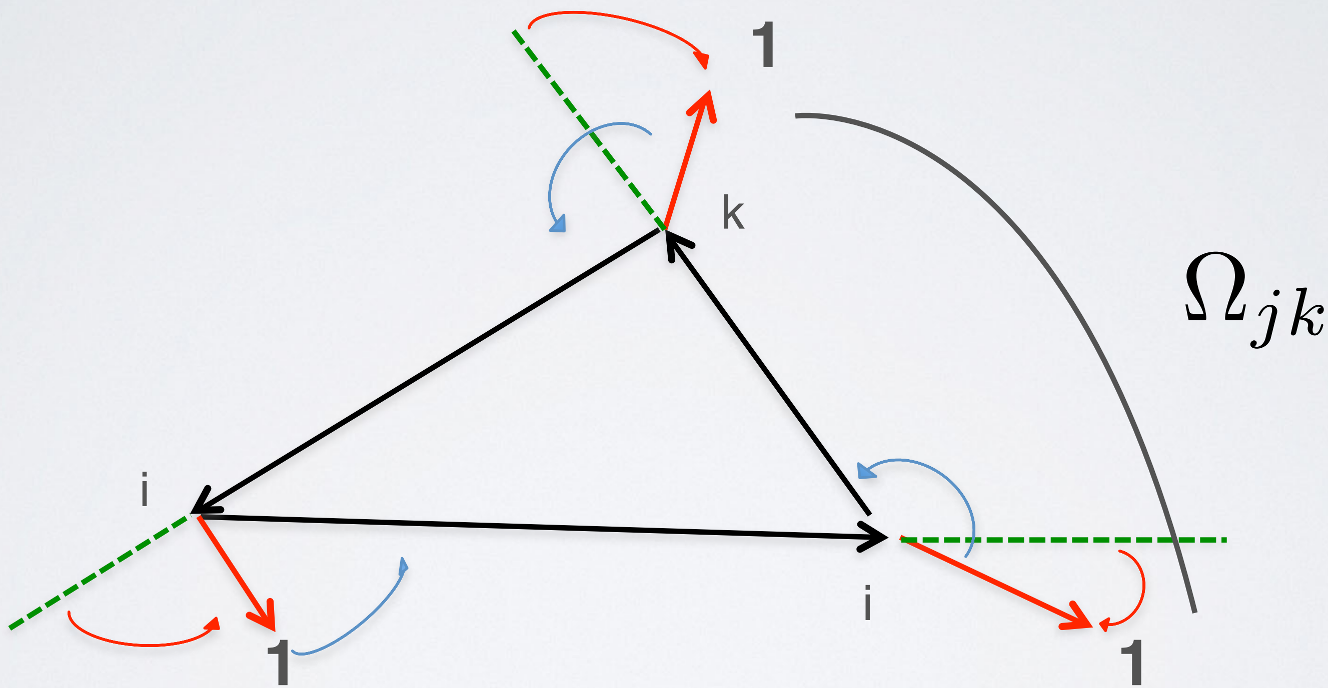
---

* Must satisfy the tetrad postulate!

$$\omega^{ab}_\mu = \frac{1}{2}e^{\nu[a}(e^{b]}_{\nu,\mu} - e^{b]}_{\mu,\nu} + e^{b]\sigma}e^c_\mu e_{\nu c,\sigma})$$

2D Solution to Lattice Dirac on simplicial lattice

$$S_{QFEM} = K_{ij}\bar{\psi}_i[\vec{t}_{ij}\cdot\vec{\sigma}]\Omega_{ij}\psi_j + K_{ij}|t_j|(\bar{\psi}_i - \bar{\psi}_j\Omega^\dagger_{ij})(\psi_i - \Omega_{ij}\psi_j)$$



The spin connection is gauge field whose curl gives the local curvature or deficit angle of the 2D simplex

Construction Procedure for Discrete Spin connection

(1) Assume Elements with Spherical Triangles (i,jk) (see Andy's talk) or boundaries give by geodesics on an 2D manifold

(Angles at each vertex add to 2 pi exactly)

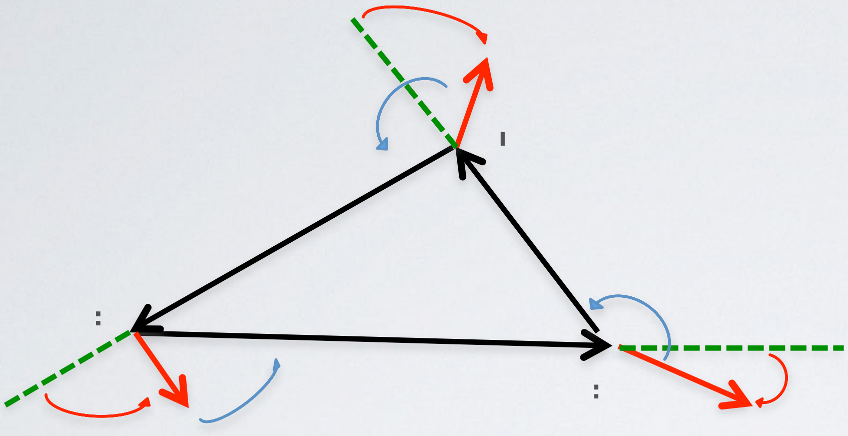
(2) Rotate spinor at (i) to have axes **1** on tangent vectors to (j) and parallel transport to j rotate back ==> Ω_{ij}

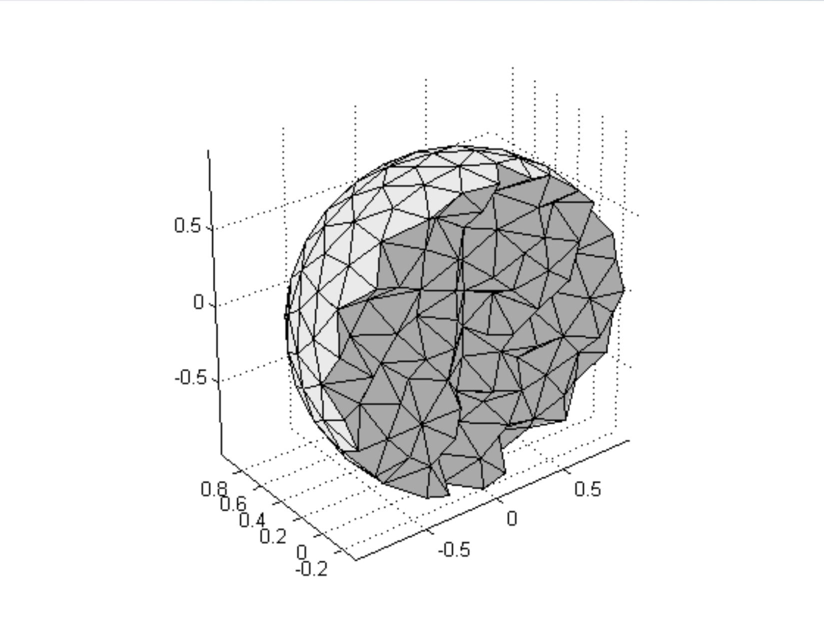
(3) Calculate discrete "curl" around the triangle

$$\Omega_{ij}\Omega_{jk}\Omega_{ki} = e^{i(2\pi - \delta_\triangle)\sigma_3/2}$$

(4) Fix $\Omega_{ij} \to \pm\Omega_{ij}$ so $\delta_\triangle \sim A_{ijk}/4\pi R$

THIS IS ALWAYS POSSABLE IF A SPIN CONNECTION EXISTS





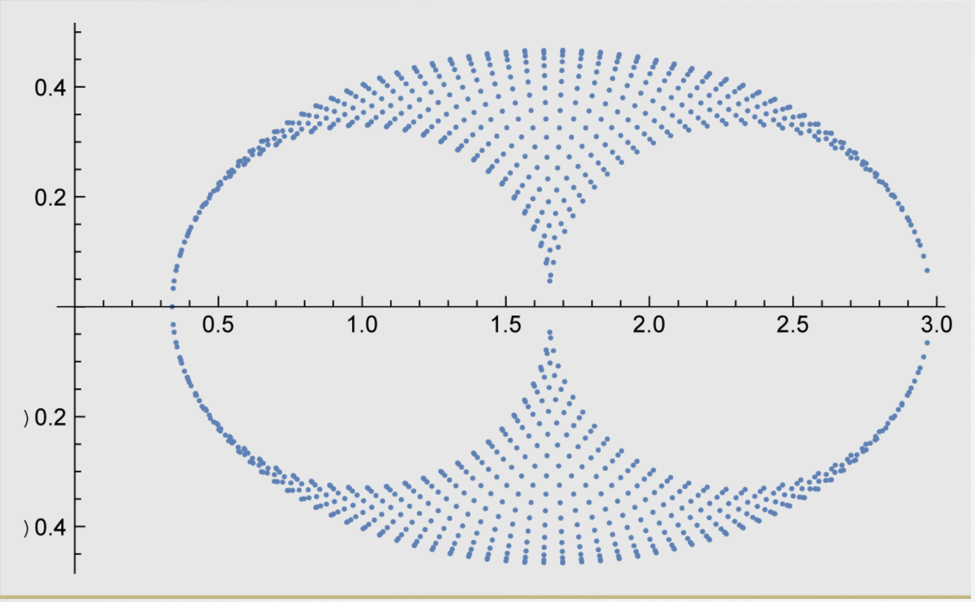
Sphere: or any manifold with this topology has a unique lattice spin connection upto gauge Lorentz transformation on spinors

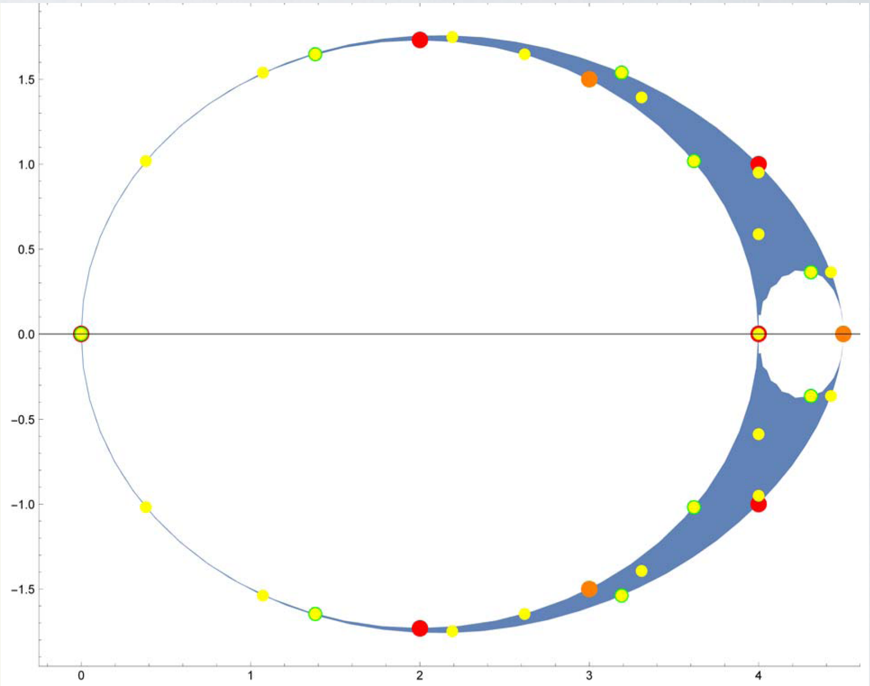
Torus: There are 4 solutions: (periodic/anti-periodic): Non-contractible loops

Proof: Use Euler's Theorem.

This is completely general for any smooth orientable Reimann surfaces

2D DIRAC SPECTRA ON TORUS





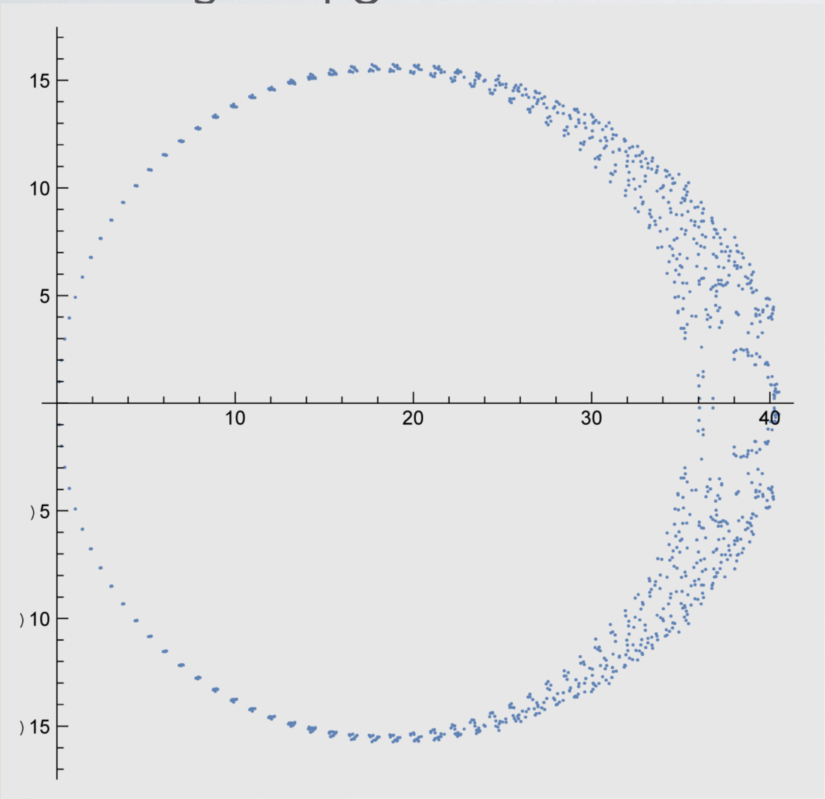
Torus: Square Lattice

Torus: Triangular Lattice

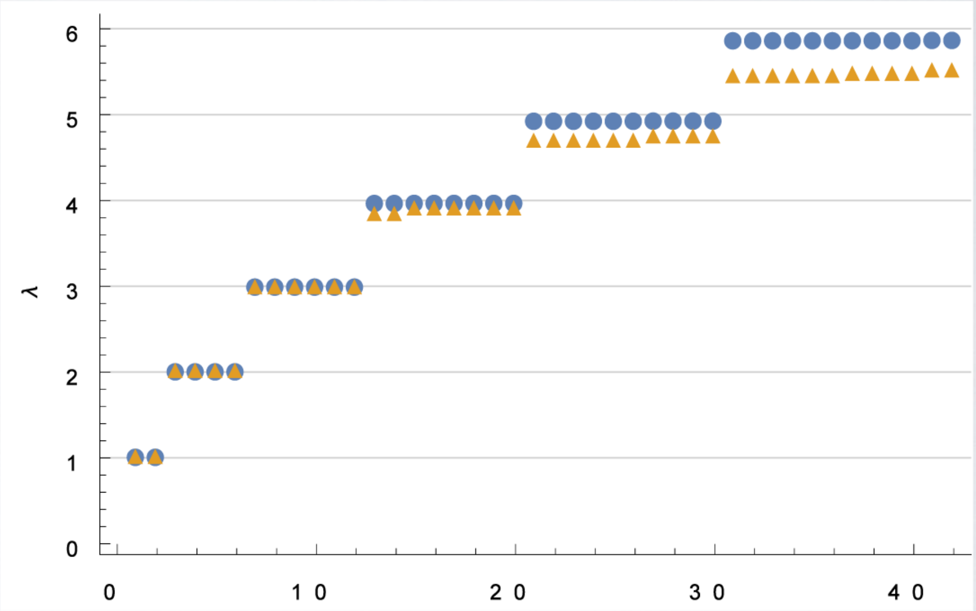
9 pts (orange) 16 pts (red) 25 pts(green) 100 pts (yellow)

2D DIRAC SPECTRA ON SPHERE

$s = 16$



$s = 8$ vs $s = 16$

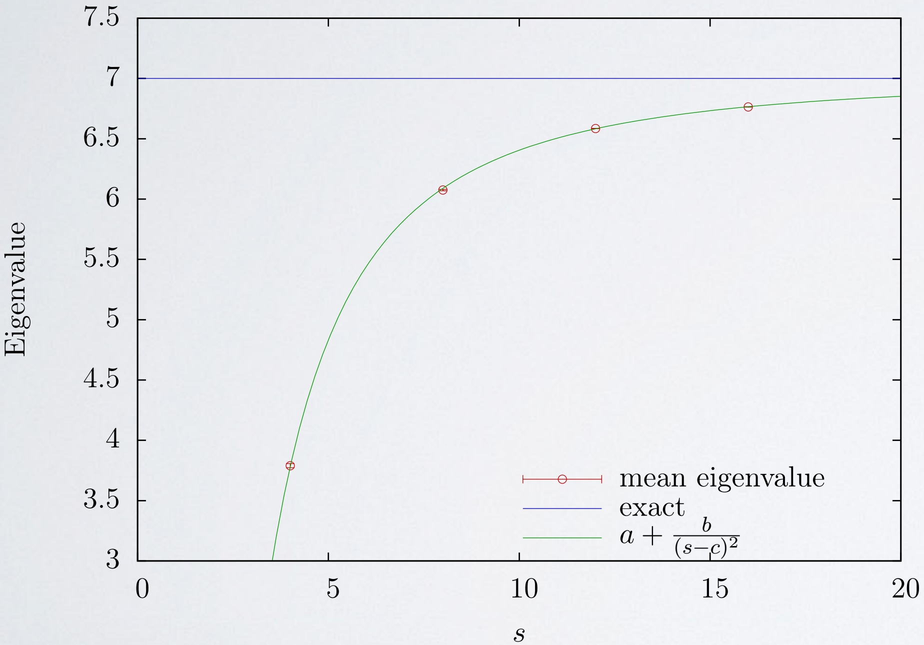


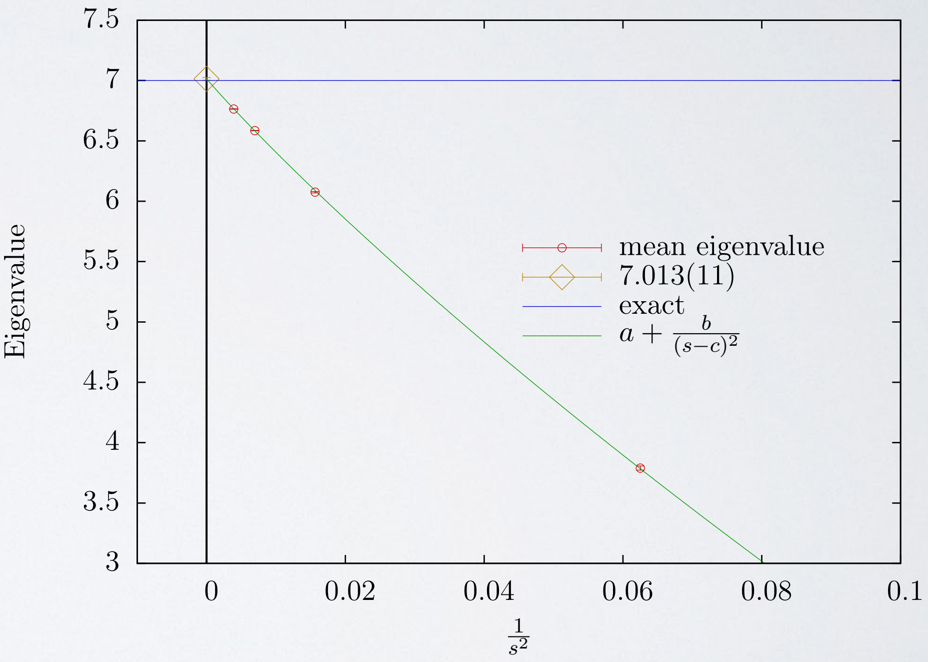
E = j + 1/2

Exact is integer spacing for $j = 1/2, 3/2, 5/2$ ...

Exact degeneracy $2j + 1$: No zero mode in chiral limit!.

CONVERGENCE RATE



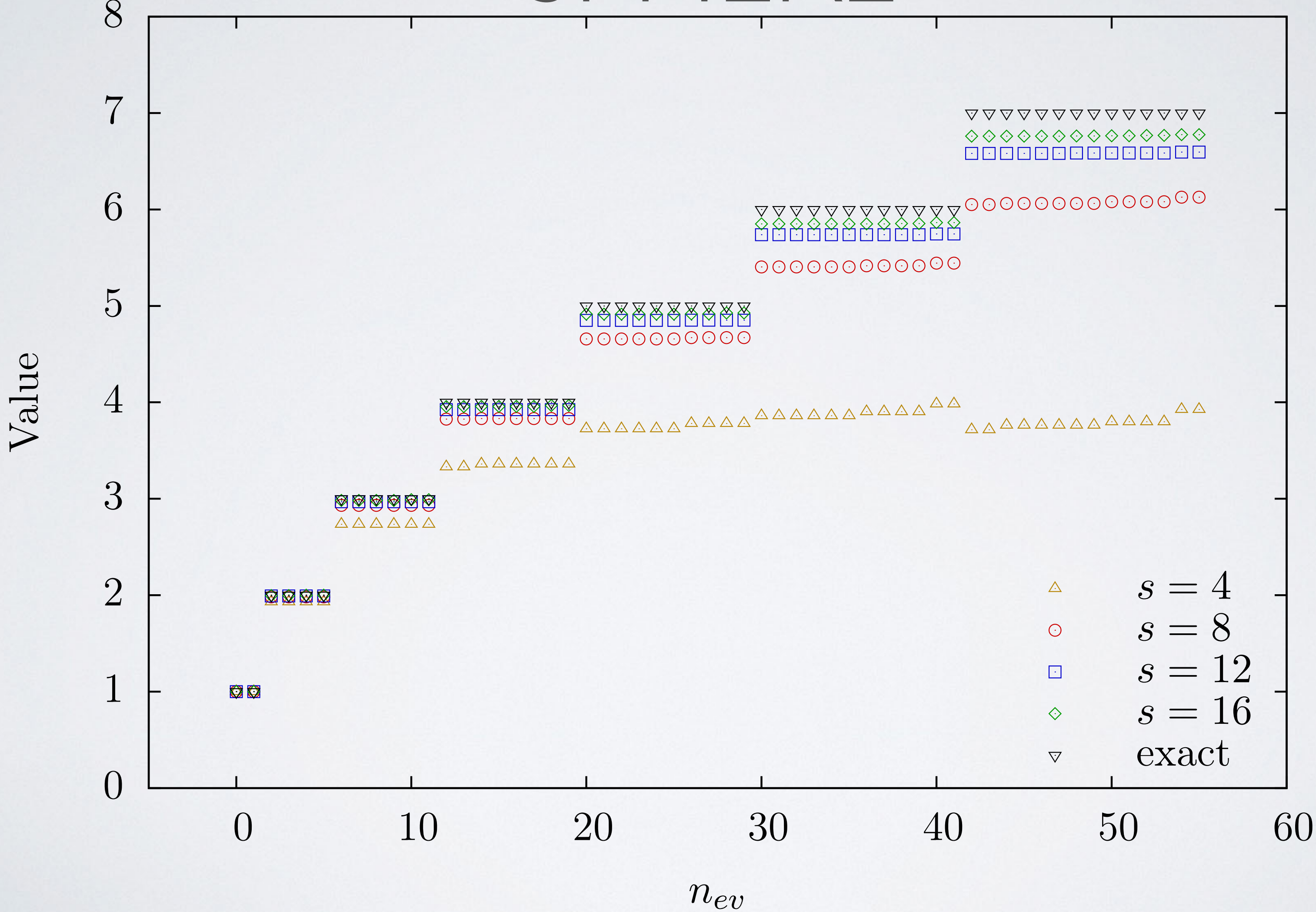


COMMENTS:
QFE IS NOT JUST FEM+REGGE

- *Quantum Field Theory requires renormalized QFEM counter terms (position dependent) (George's talk on 2D)*
- *FEM introduces a co-ordinate system breaking diffeomorphism invariance: recovered in continuum*
- *Our Dirac weights are NOT linear FEM element but they can be constructed by 3 linear elements meeting at the circumcenter.*
- *Dirac lattice action can be generalized to 3 and 4 dimensions.*
- *Gauge fields can be done a la Christ et al or 'tHooft et al?*

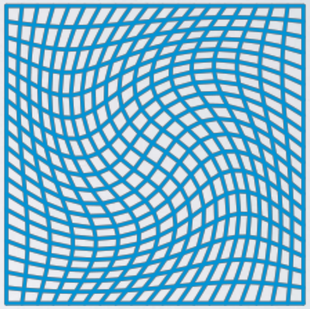
Q & AL EXTRAS

SPECTRUM OF QFE DIRAC ON SPHERE



WEYL VS CONFORMAL DIFFEOMORPHISMS

- Weyl (change the manifold) $g_{\mu\nu}(x) \to \Omega(x) g_{\mu\nu}(x)$
- Diffeomorphism (change the co-ordinates) $x \to \xi = f(x)$
- Conformal Diffeomorphism (largest subgroup)
- Primary field: $\phi(\xi) = b^{-\Delta}(x)\phi(x)$



$$ds^2 = g_{\mu\nu}(\xi) d\xi^\mu d\xi^\nu$$

$$ds^2 = b^2(x) g_{\mu\nu}(x) dx^\mu dx^\nu$$

Theorem: $|\xi_1 - \xi_2|^2 = b(x_1)|x_1 - x_2|^2 b(x_2)$

SIMPLICIAL LATTICE INDUCES A DIFFEOMORPHISM

- We have found the counter term of the FEM Lagrangian
- Our sequence of simplicies induce a (conformal?) deffeomorphism
- BUT now need to find correct for primary operator.

$$\phi(x) \to \Phi(x) = b(x)^{-\Delta}\phi(x)$$

- This is now being implement (Stay tuned)
- CT for 3D are being computed and will be tested

Proof of one loop QFEM Counter Term:

Try: $$\delta m^2 = q(x)\log(1/a^2m^2) + c(x) + O(a^2)$$

- Step #1: FEM: Cea's lemma (no Gauss' law)

$$||\phi(x) - \phi_h^{fem}(x)|| < ||\phi(x) - v_h(x)||$$

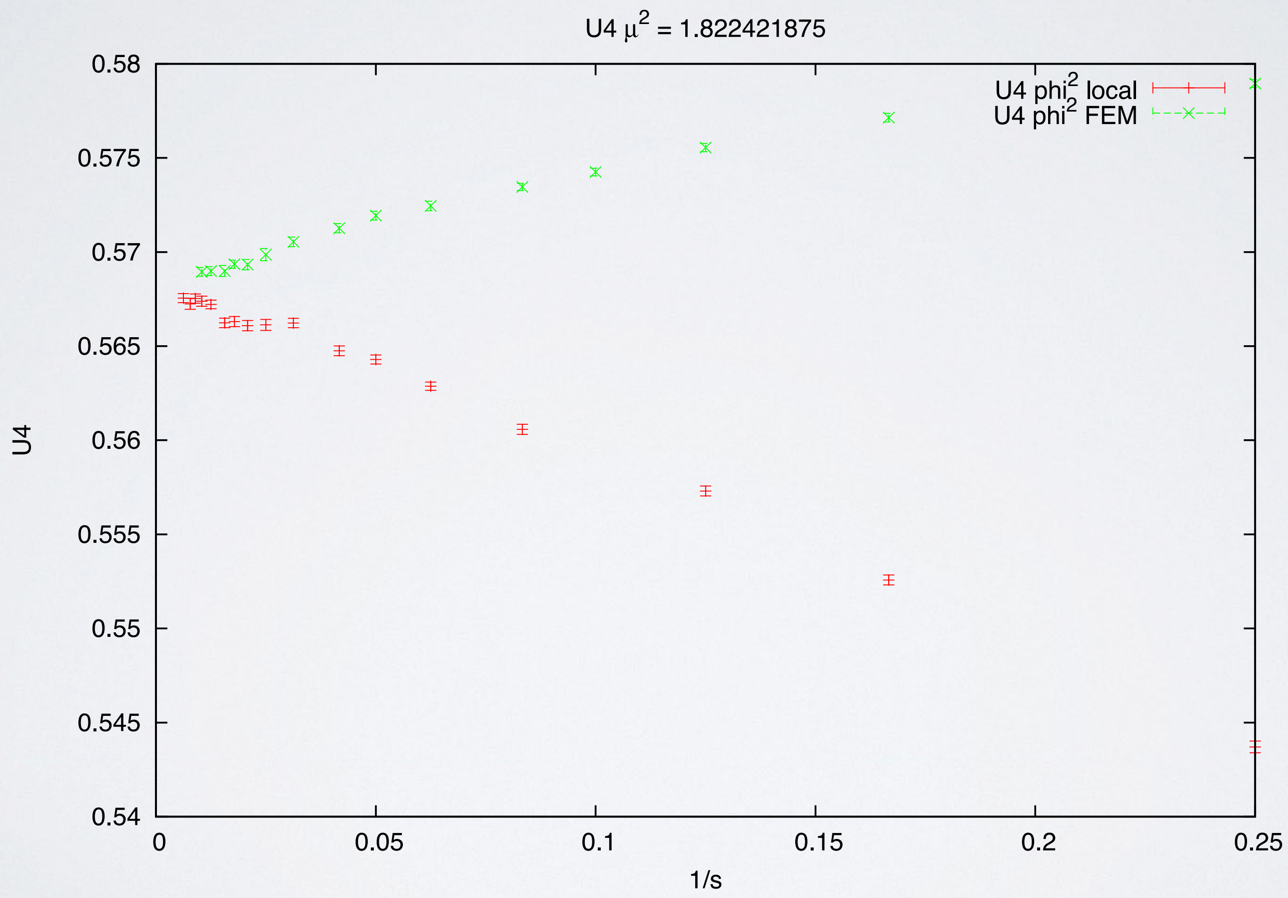
$$\text{where} \quad ||v(x)|| = \int_\Omega dx[\nabla v(x)\nabla v(x) + m^2v^2(x)]$$

$$\implies q(x) = Q$$

- Step #2: Regge Gravity: diffeomorphisms

$$dz d\bar{z} = |f'(w)|^2 dw d\bar{w} \implies Qlog(|w|^2) = Q\log(|z|^2/\sqrt{g(z)})$$

LOCAL VS FEM MASS TERM



FEM HAVE "SPECTRAL FIDELITY"

- Taylor expansion on hypercubic lattice:

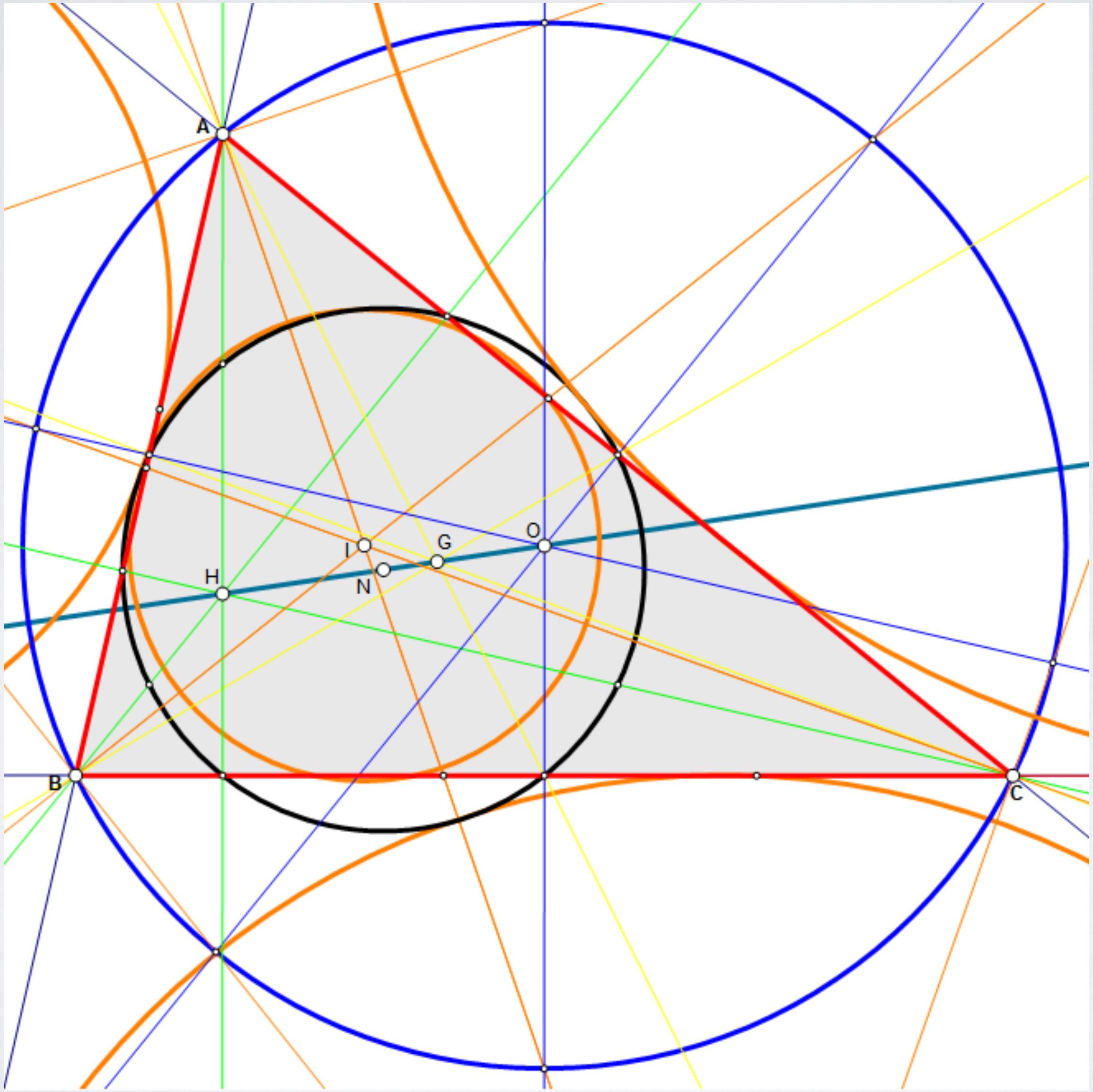
$$a^{-1}\sum_{\pm\mu}(\phi(x)-\phi(x+a\mu)^2 \simeq (\nabla\phi)^2 + O(a^2)$$

- Taylor series for FEM does not work!

$$a^2\sum_{y}K(x,y)(\phi(x)-\phi(y))^2 \simeq c_{\mu\nu}\partial_\mu\phi\partial_\nu\phi + O(a^2)$$

- FEM theorems: error & spectra < cut-off converges O(a^2) if triangles are "shape regular" and "uniformly" refined.

CENTERS OF A TRIANGLE



K-SIMPLICIES