

# Axion Dark Matter and lattice field theory

**Enrico Rinaldi**



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# Outline

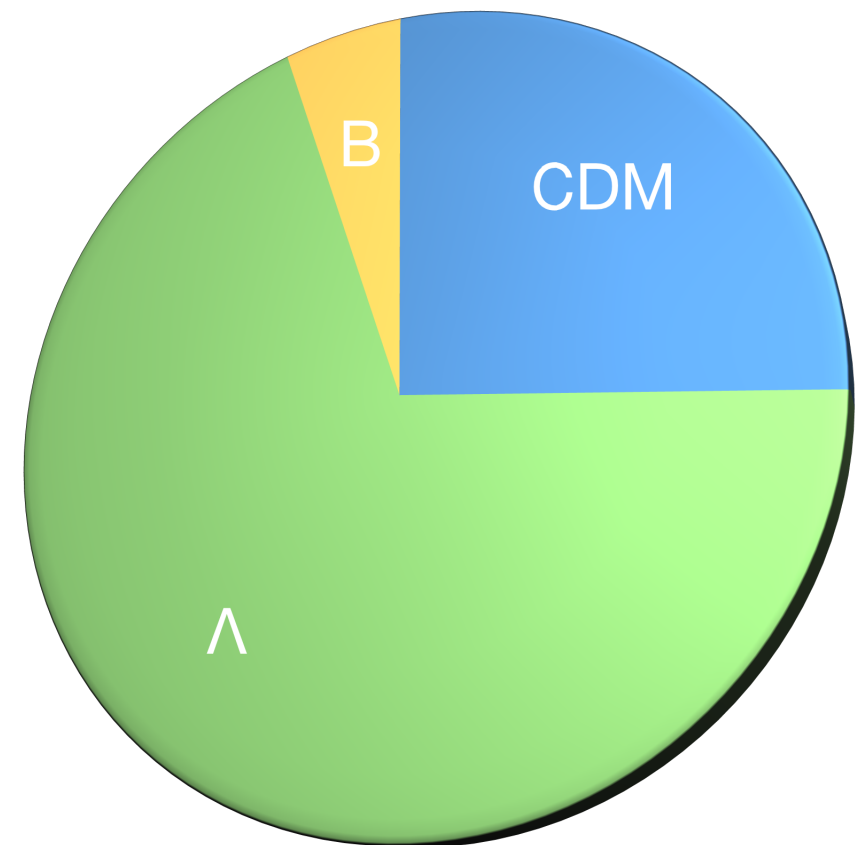
- General features of **axions** as a solution of the “Strong CP” problem
- Current physical **constraints** on axion models’ parameters based on dark matter interpretation
- **Lattice simulations** and input to axion cosmology

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- Current physical **constraints** on axion models’ parameters based on dark matter interpretation
- **Lattice simulations** and input to axion cosmology
- **GOAL:** Lower bound on the axion mass using lattice QCD as input to axion cosmology

# Axion dark matter

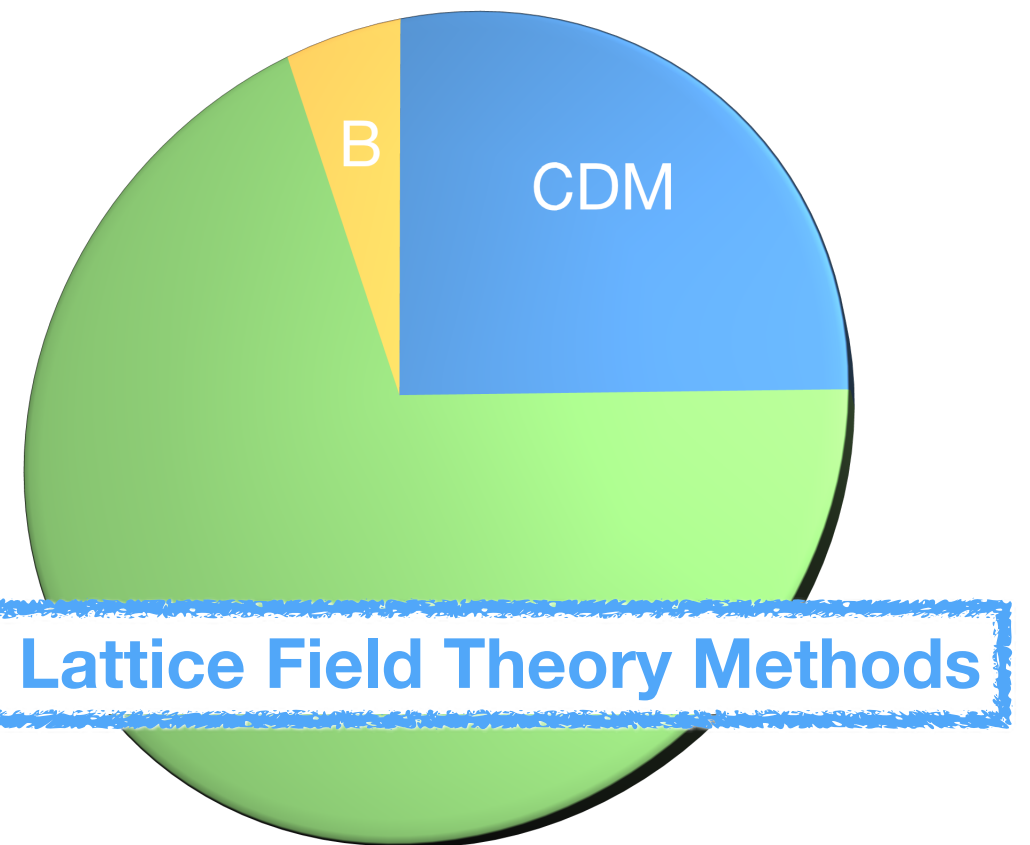
- Axions were originally proposed to deal with the Strong CP Problem
- They also form a plausible DM candidate
- The axion energy density requires non-perturbative QCD input
- Being sought in ADMX (LLNL, UW) & CAST-IAXO (CERN) with large discovery potential in the next few years
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# “Strong CP” problem

- QCD has a parameter,  $\theta$
- Controls QCD CP violation
- Topological
- $\theta$  can take **any** value in  $(-\pi, \pi]$

$$\mathcal{L}_{\text{QCD}} \ni \theta \frac{1}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

$$Q = \frac{1}{32\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \in \mathbb{Z}$$

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Why is  $\theta$  so small?



# Axions as a solution

- Couple to topological charge

$$\mathcal{L}_{\text{axions}} = \frac{1}{2} (\partial_\mu a)^2 + \left( \frac{a}{f_a} + \theta \right) \frac{1}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

- Otherwise have shift symmetry

$$a \rightarrow a + \alpha$$

- Amenable to effective theory treatment

$$V_{\text{eff}} \sim \cos(\theta + c\langle a \rangle)$$

- Axion mass from instantons effects

$$m_a^2 f_a^2 = \left. \frac{\partial^2 F}{\partial \theta^2} \right|_{\theta=0}$$

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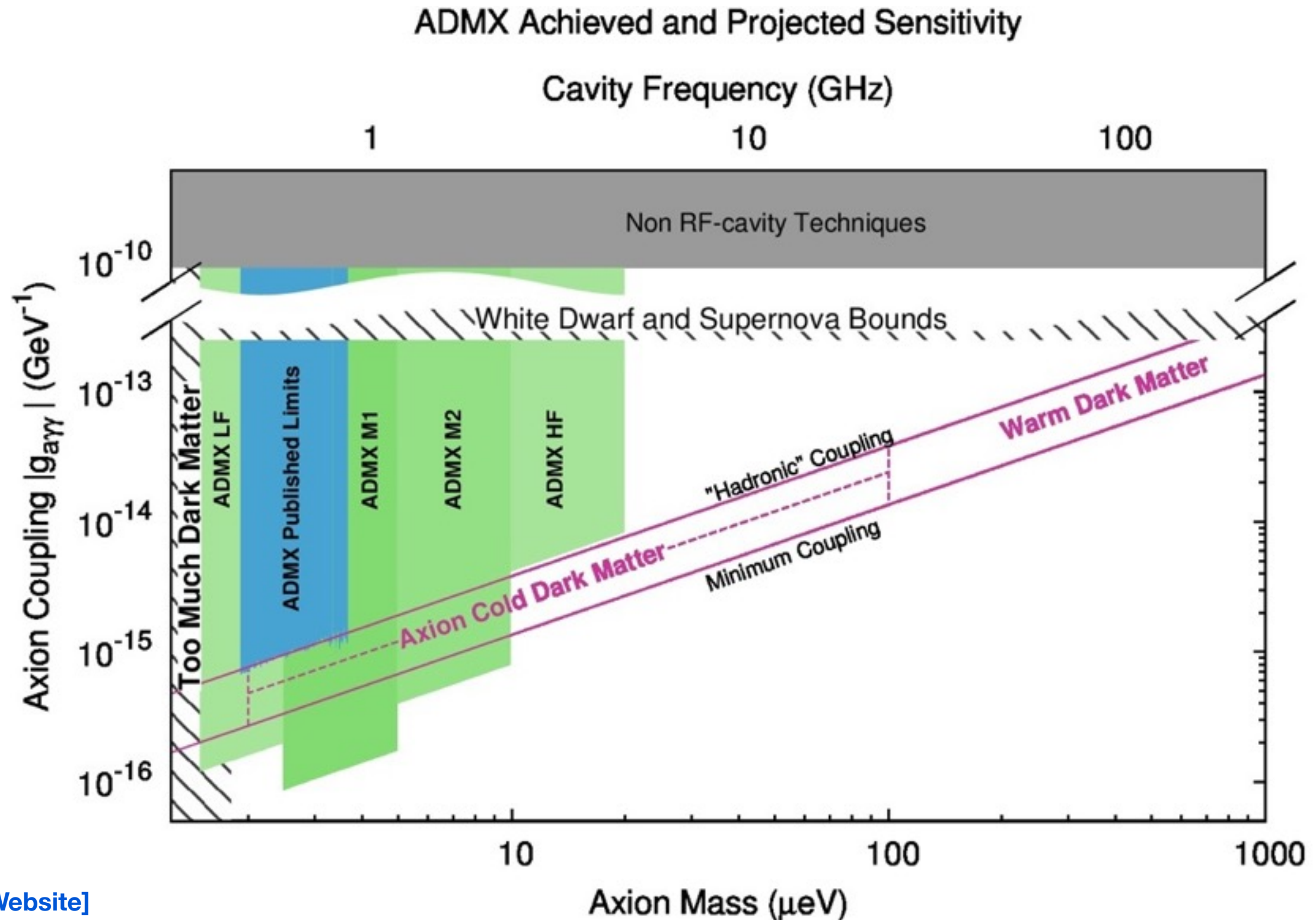
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QCD  
Topological  
Susceptibility

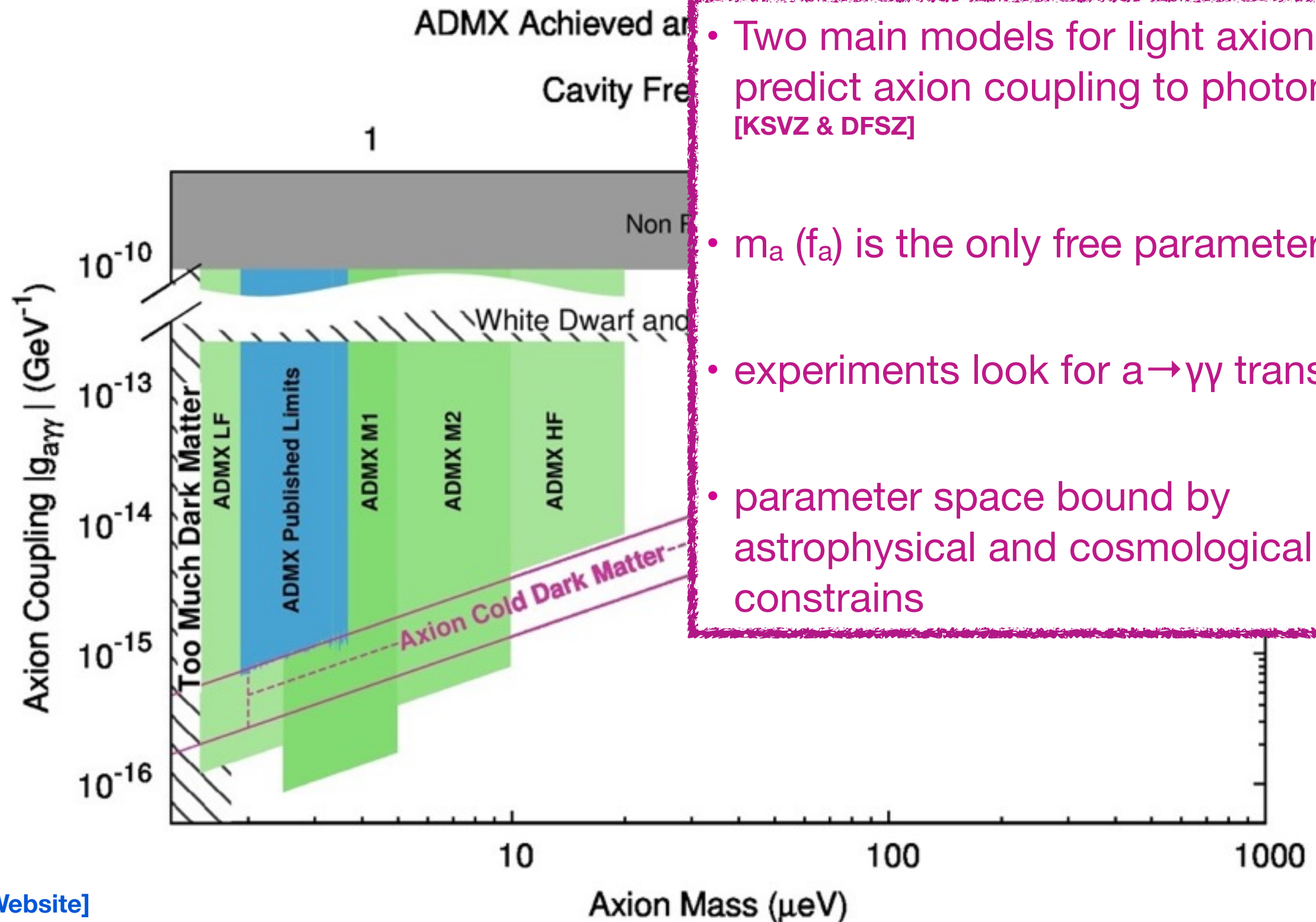
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# Current axion constraints

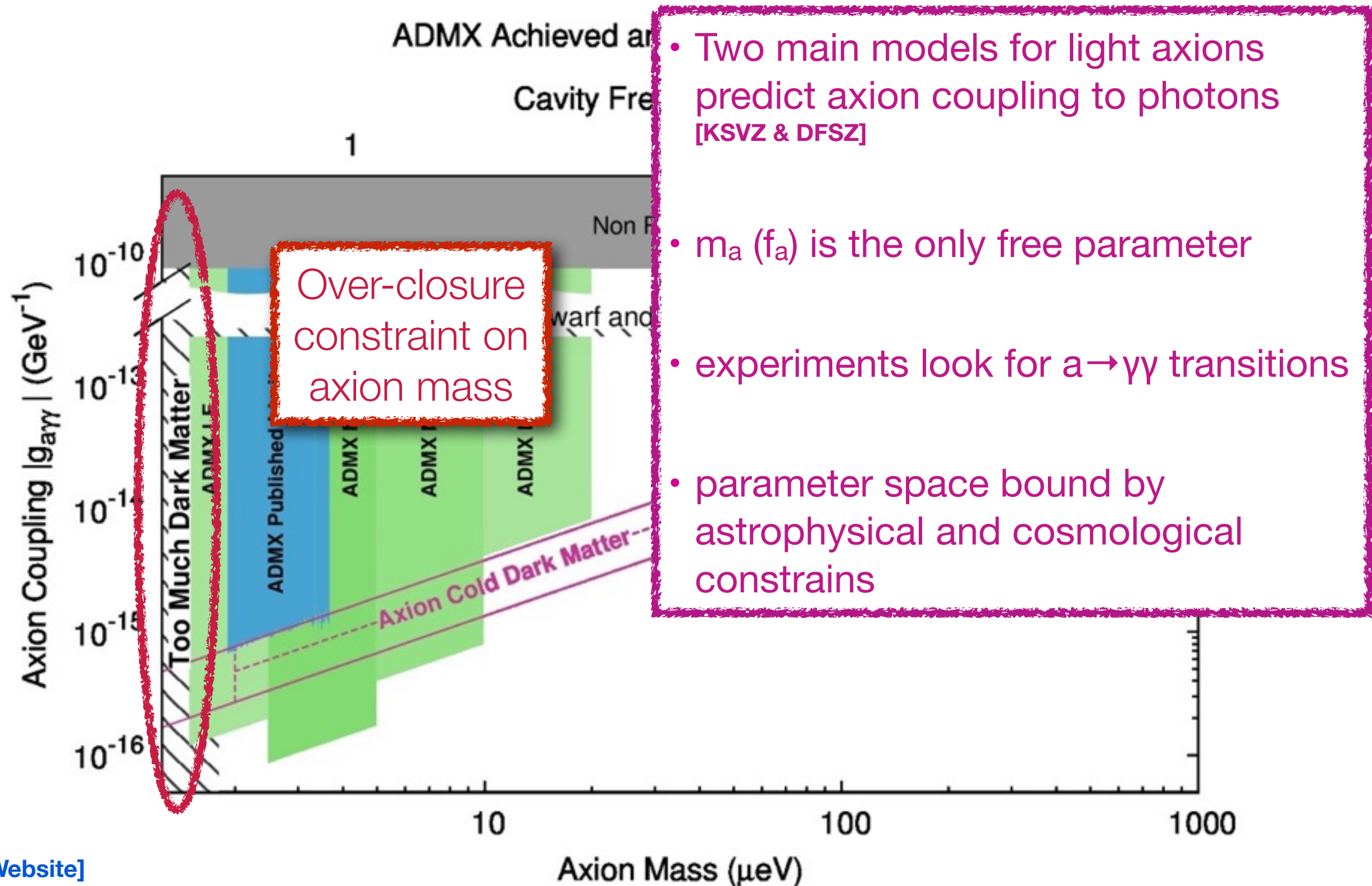


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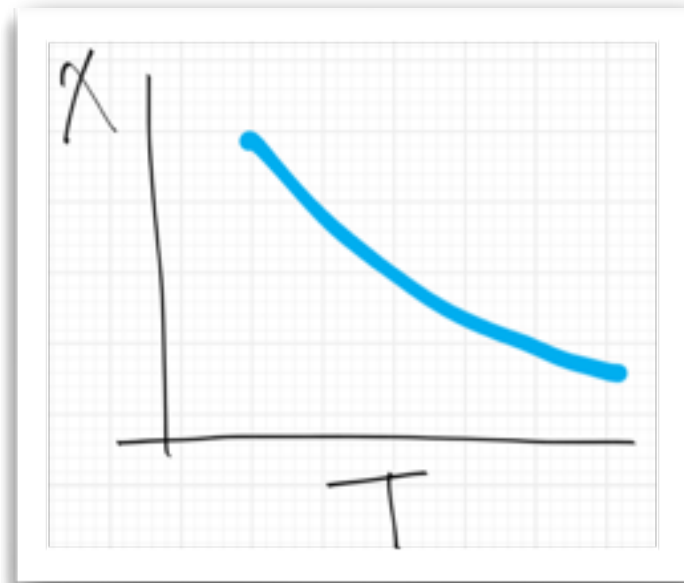
- Two main models for light axions predict axion coupling to photons [KSVZ & DFSZ]
- $m_a$  ( $f_a$ ) is the only free parameter
- experiments look for  $a \rightarrow \gamma\gamma$  transitions
- parameter space bound by astrophysical and cosmological constrains

# Current axion constraints



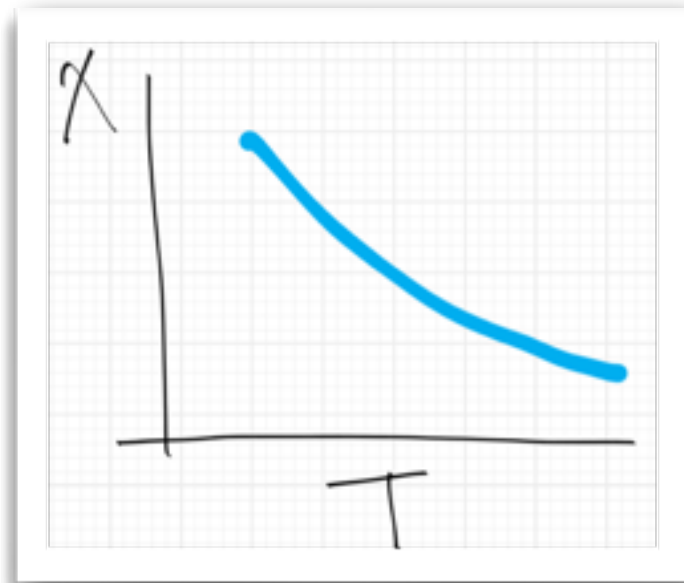


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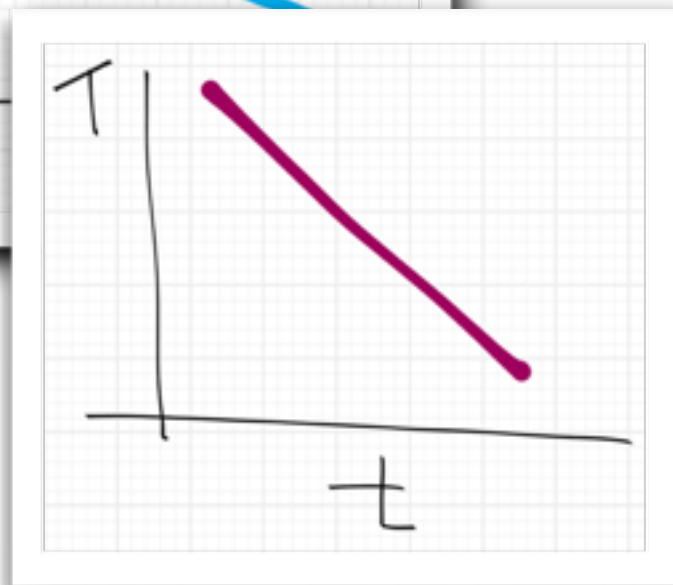
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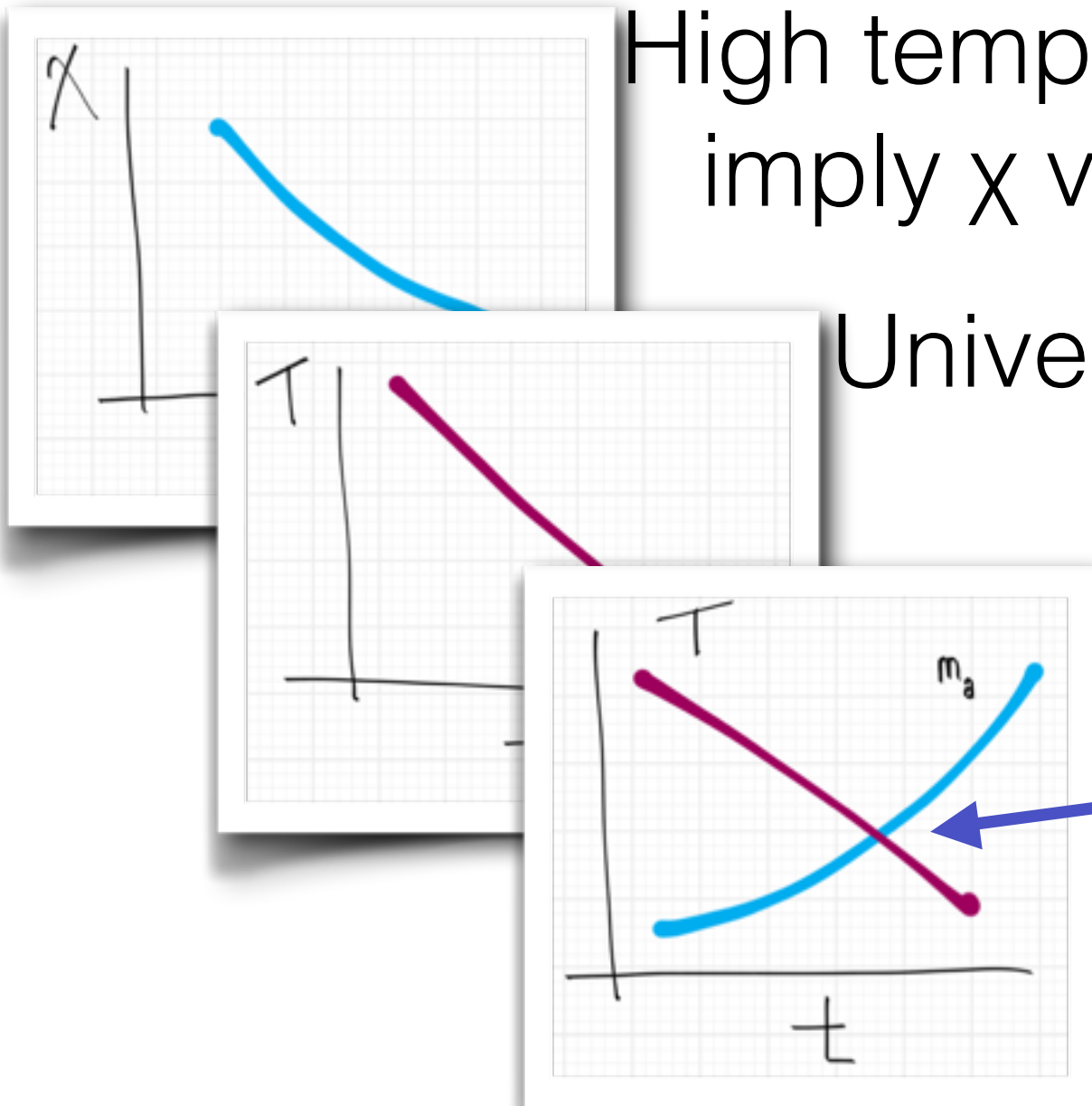
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$$3H \sim m_a$$

H: Hubble constant



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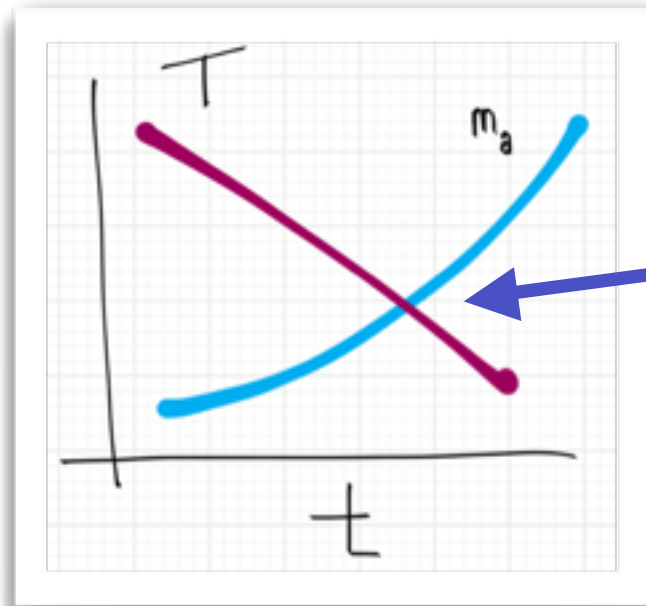
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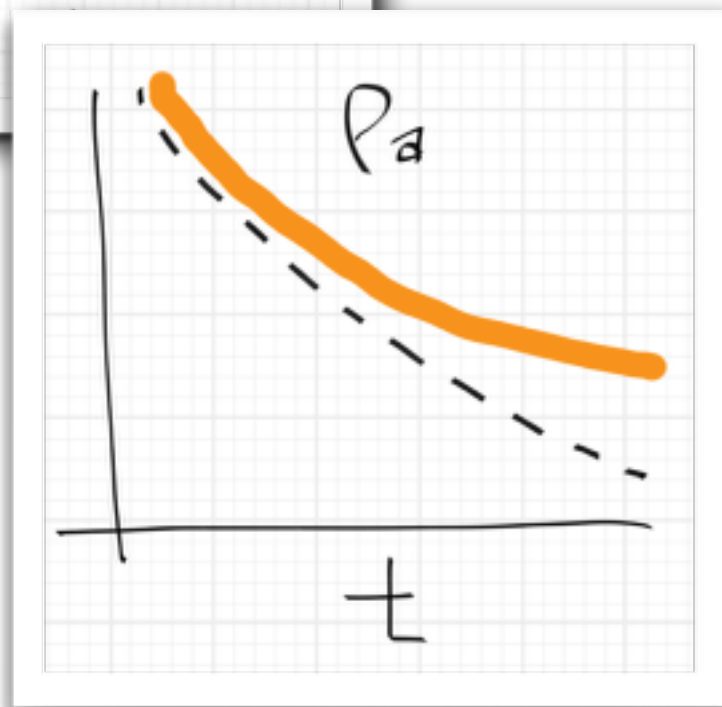
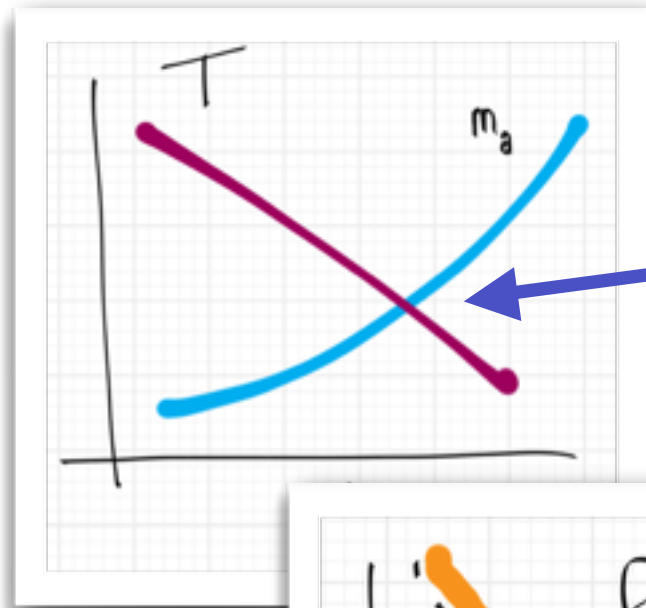
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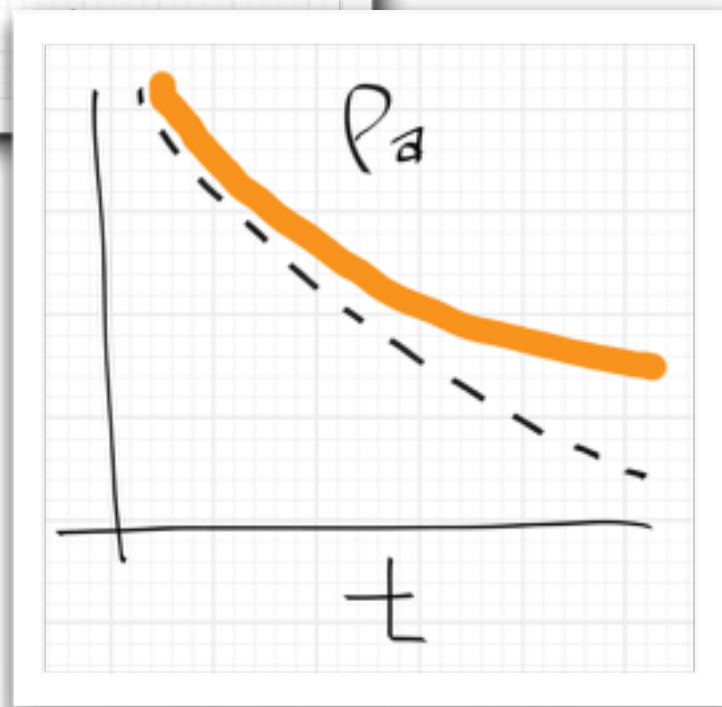
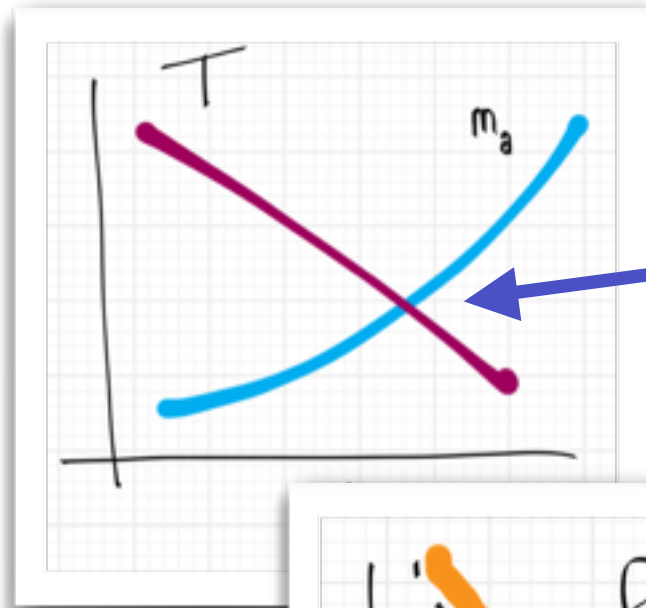
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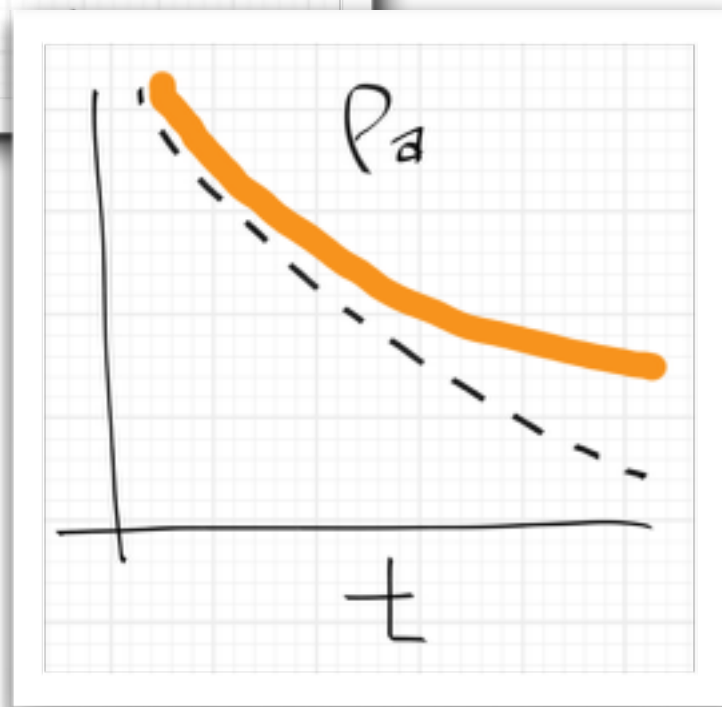
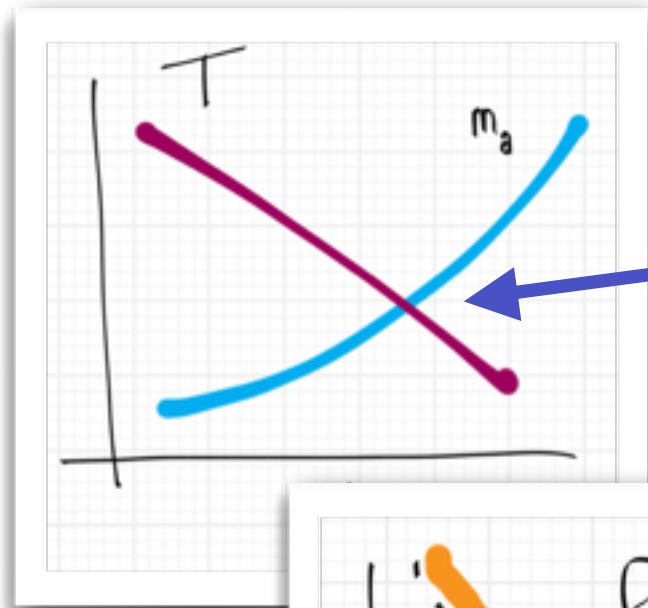
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energy density per co-moving volume  
is NOT invariant because the mass  
changes with time!

# Axion energy density

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$$\rho(T_\gamma) = \rho(T_1) \frac{m_a(T_\gamma)}{m_a(T_1)} \left( \frac{R(T_1)}{R(T_\gamma)} \right)^3 \quad T_\gamma = 2.73\text{K}$$



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How do we get this?

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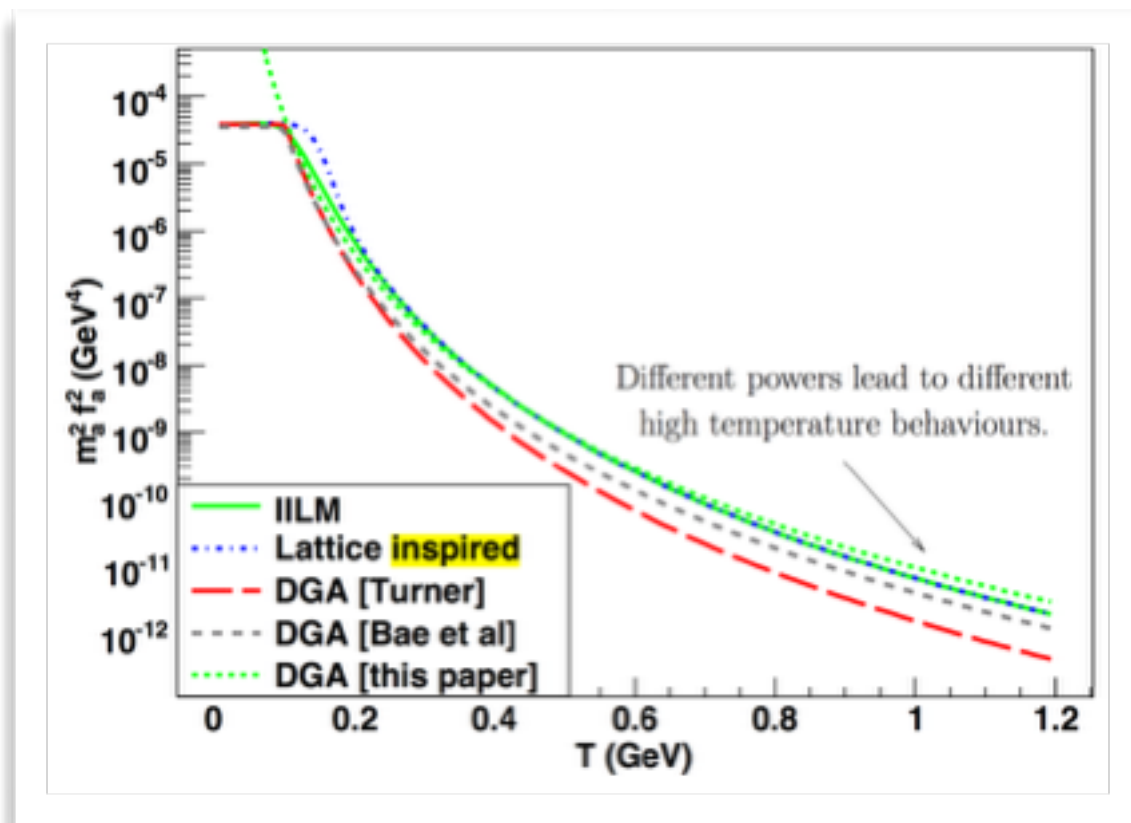
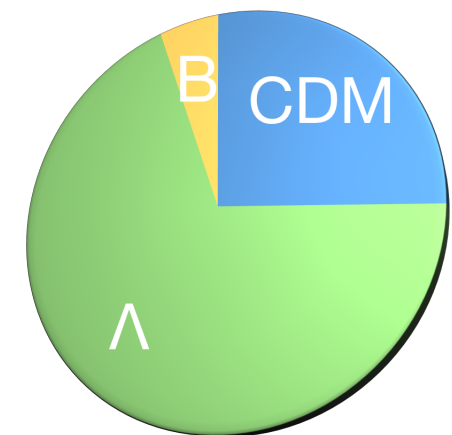
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$$m_a = \frac{\sqrt{m_u m_d}}{m_u + m_d} \frac{f_\pi m_\pi}{f_a}$$

- $m_a(T)$  is provided by models (DIGM, IILM)

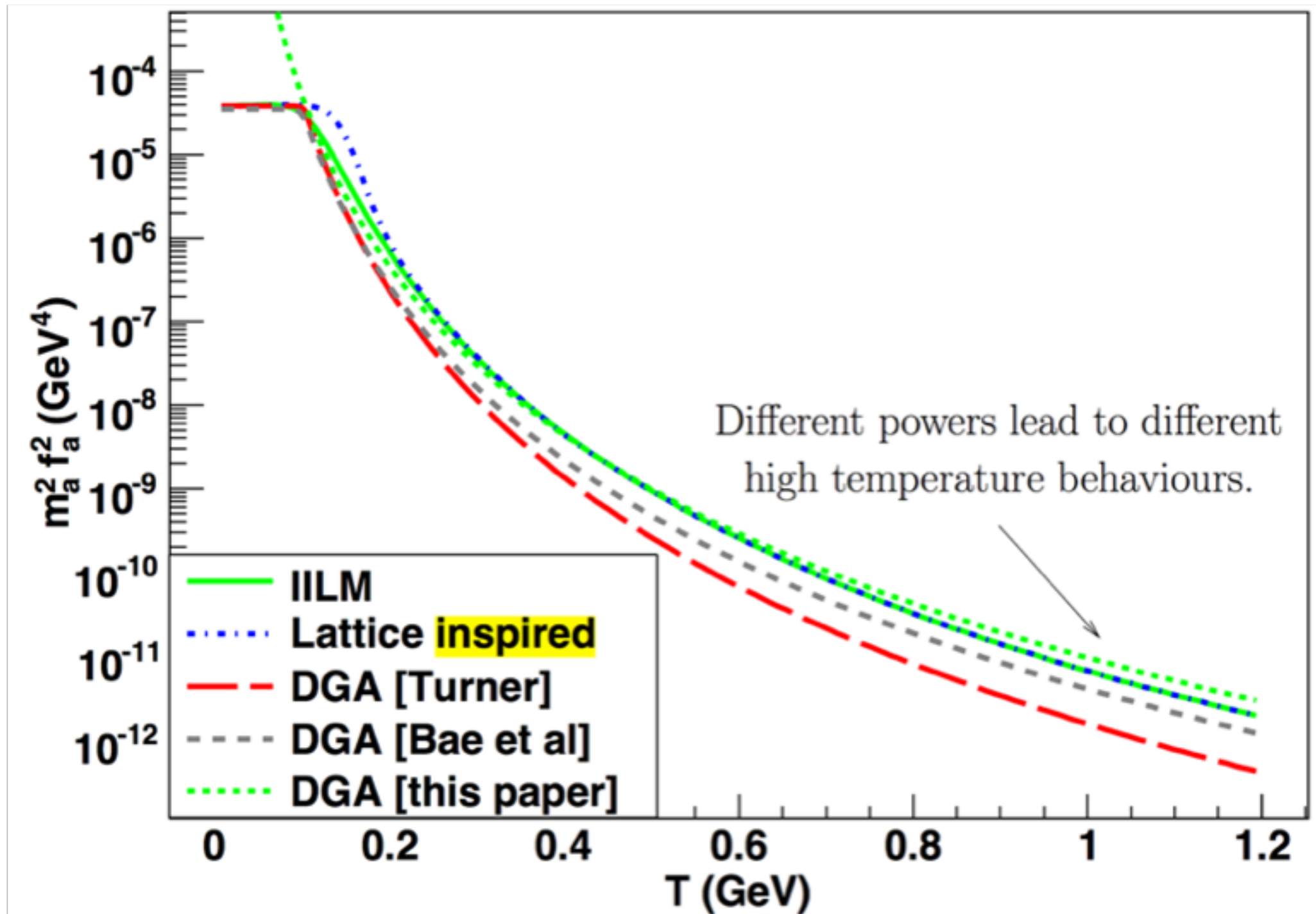


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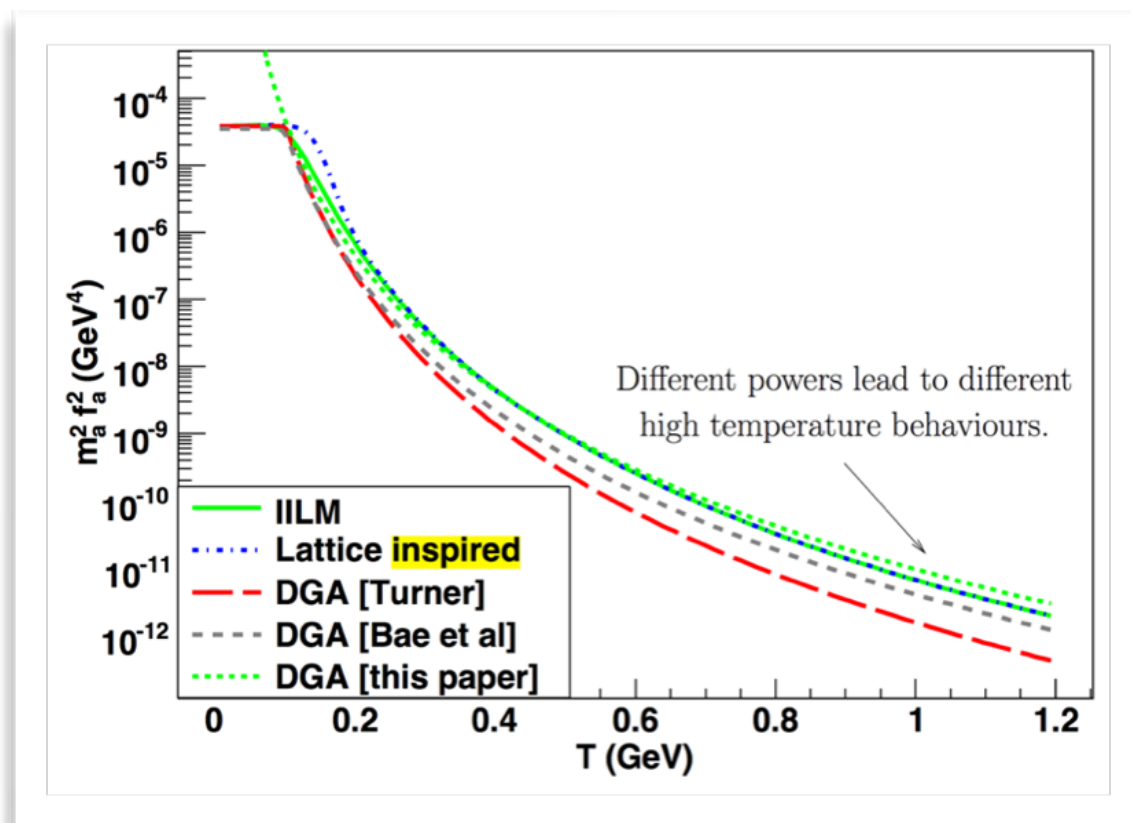
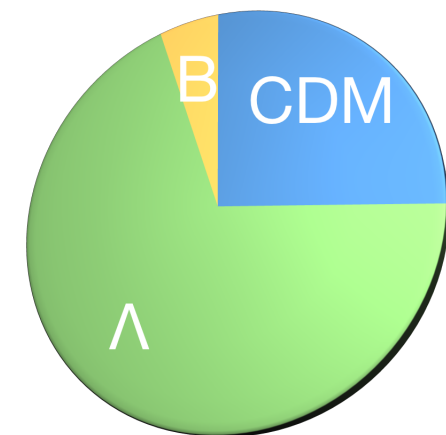
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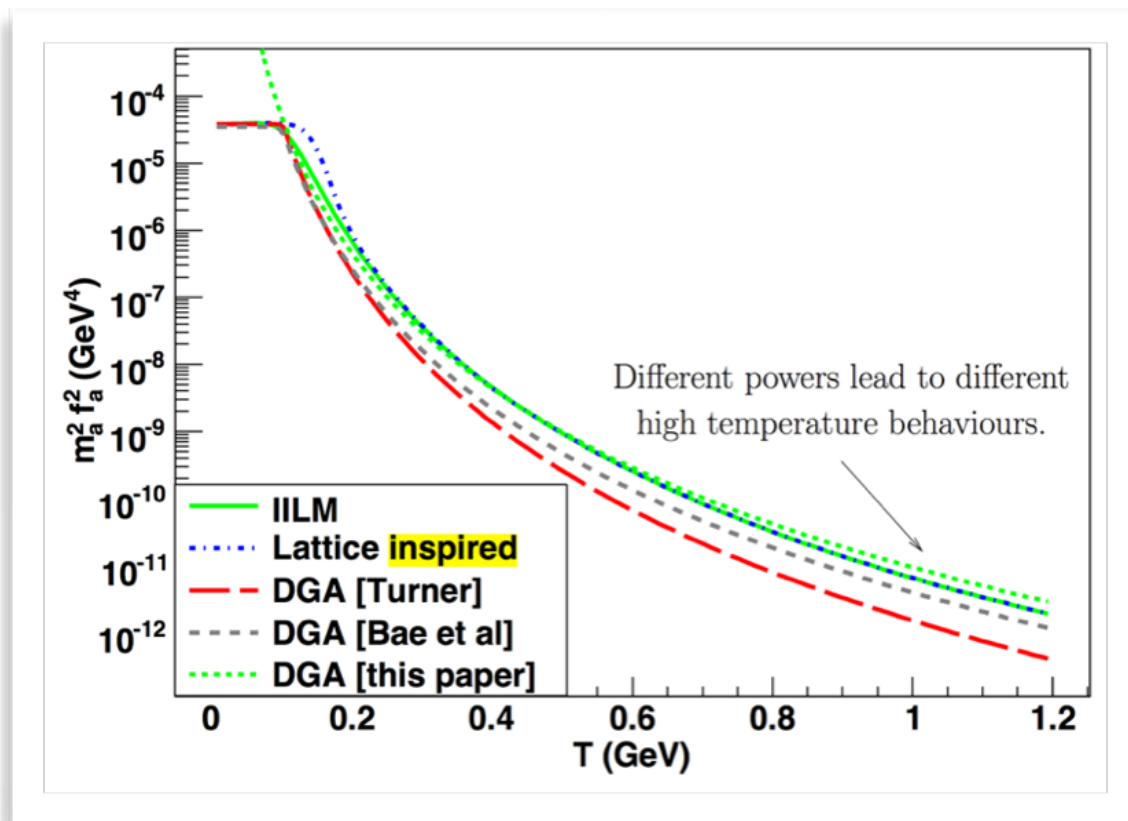
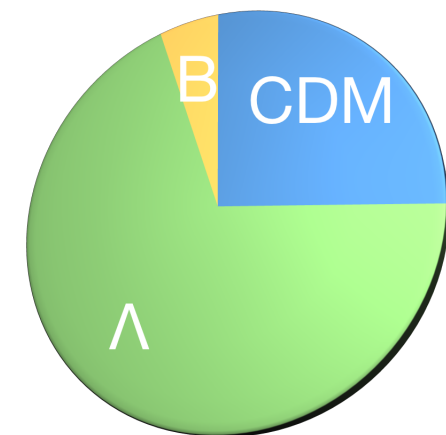
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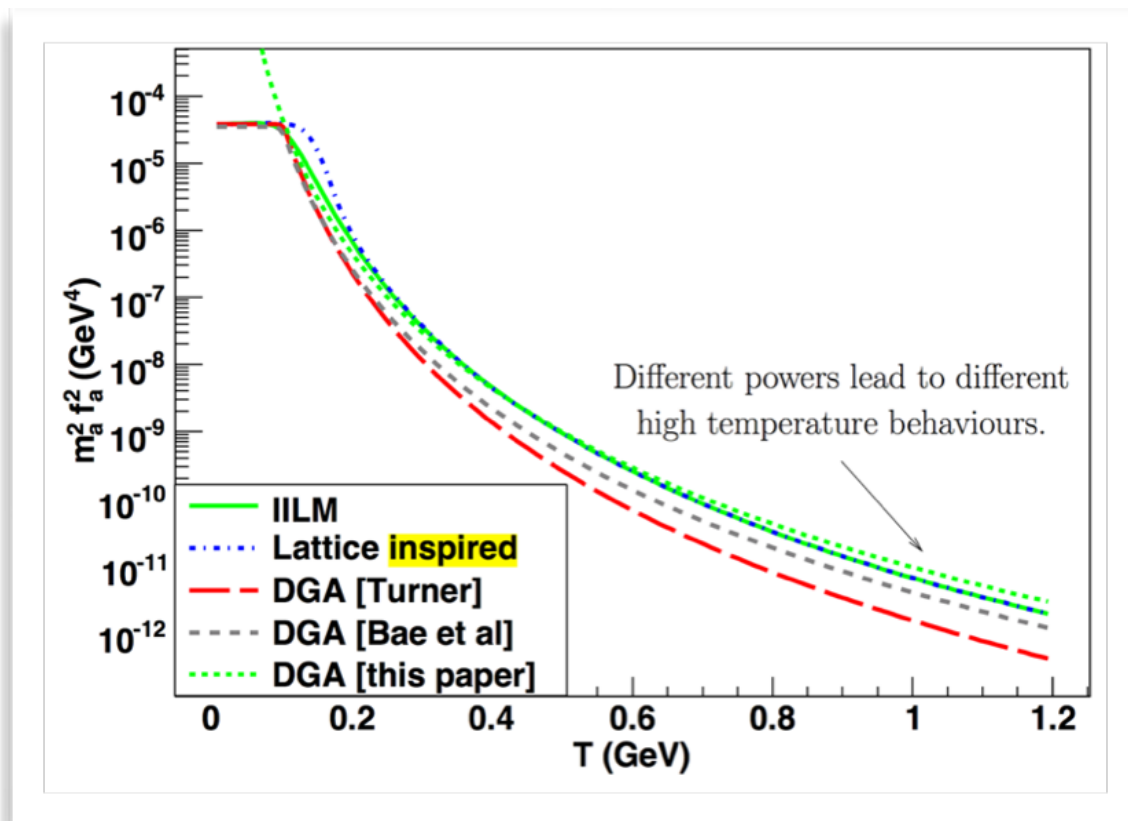
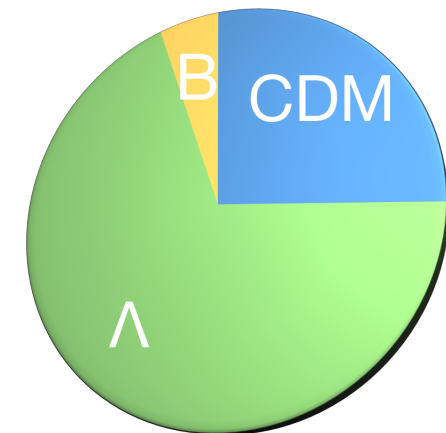
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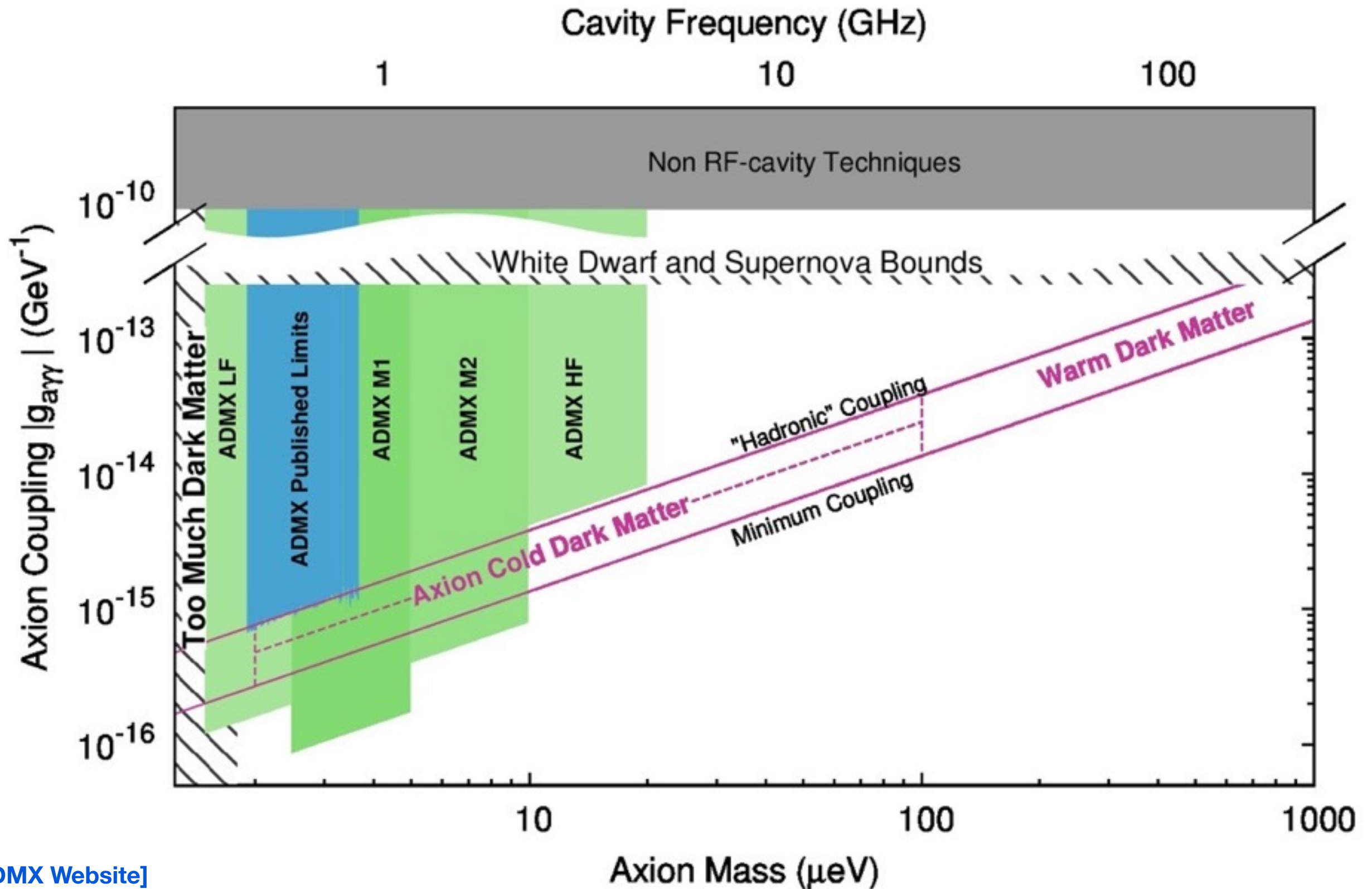
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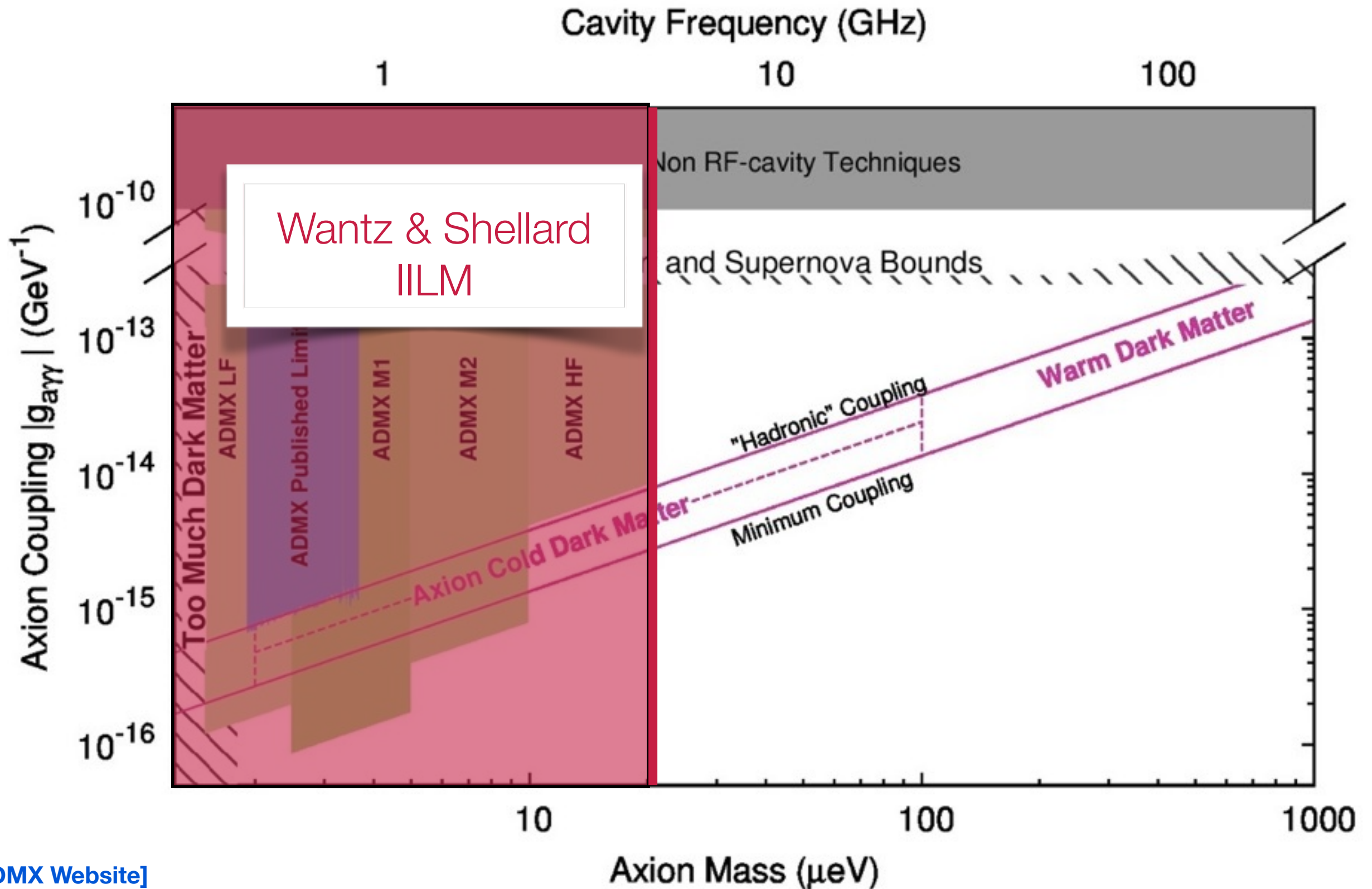
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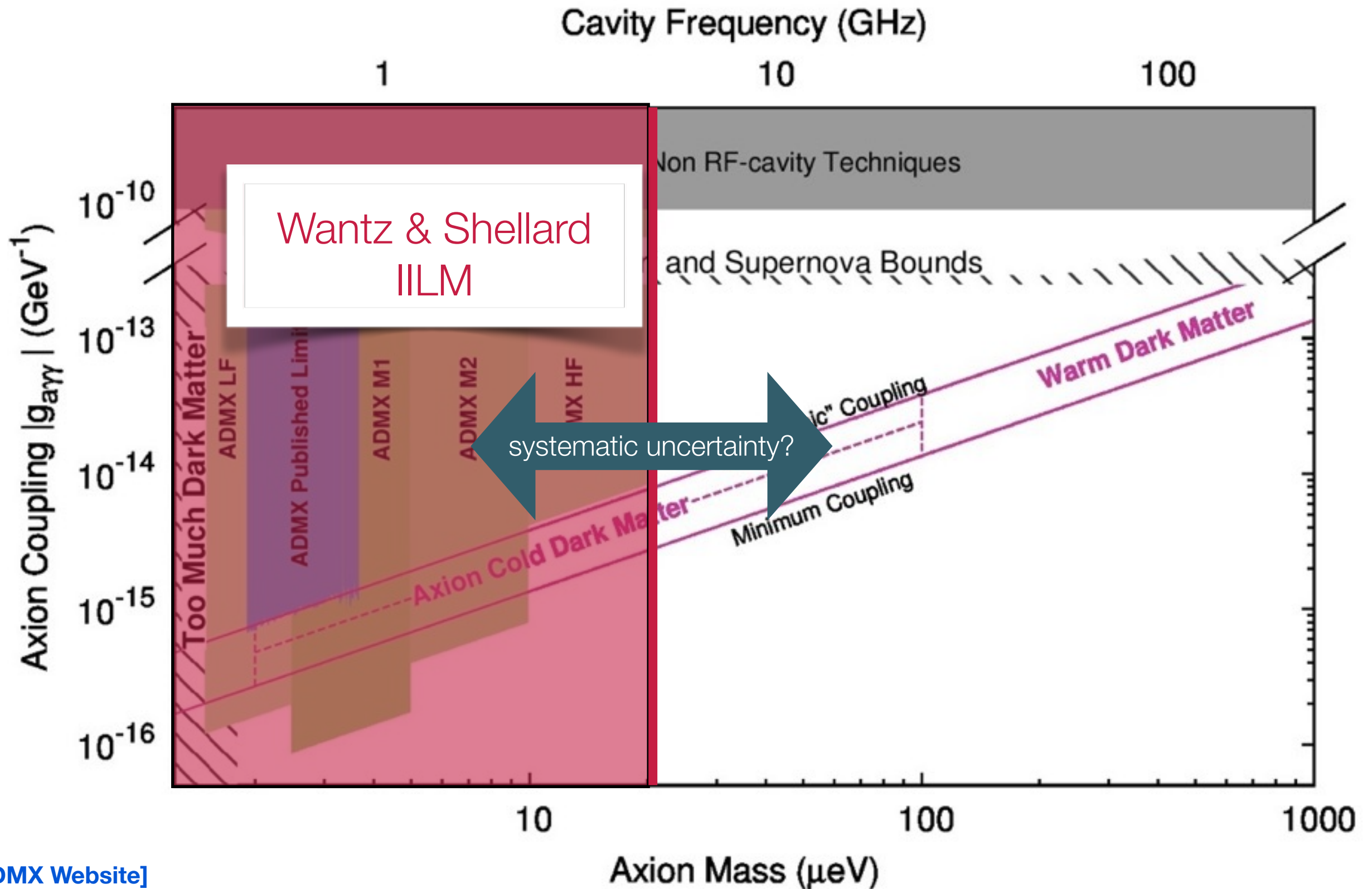




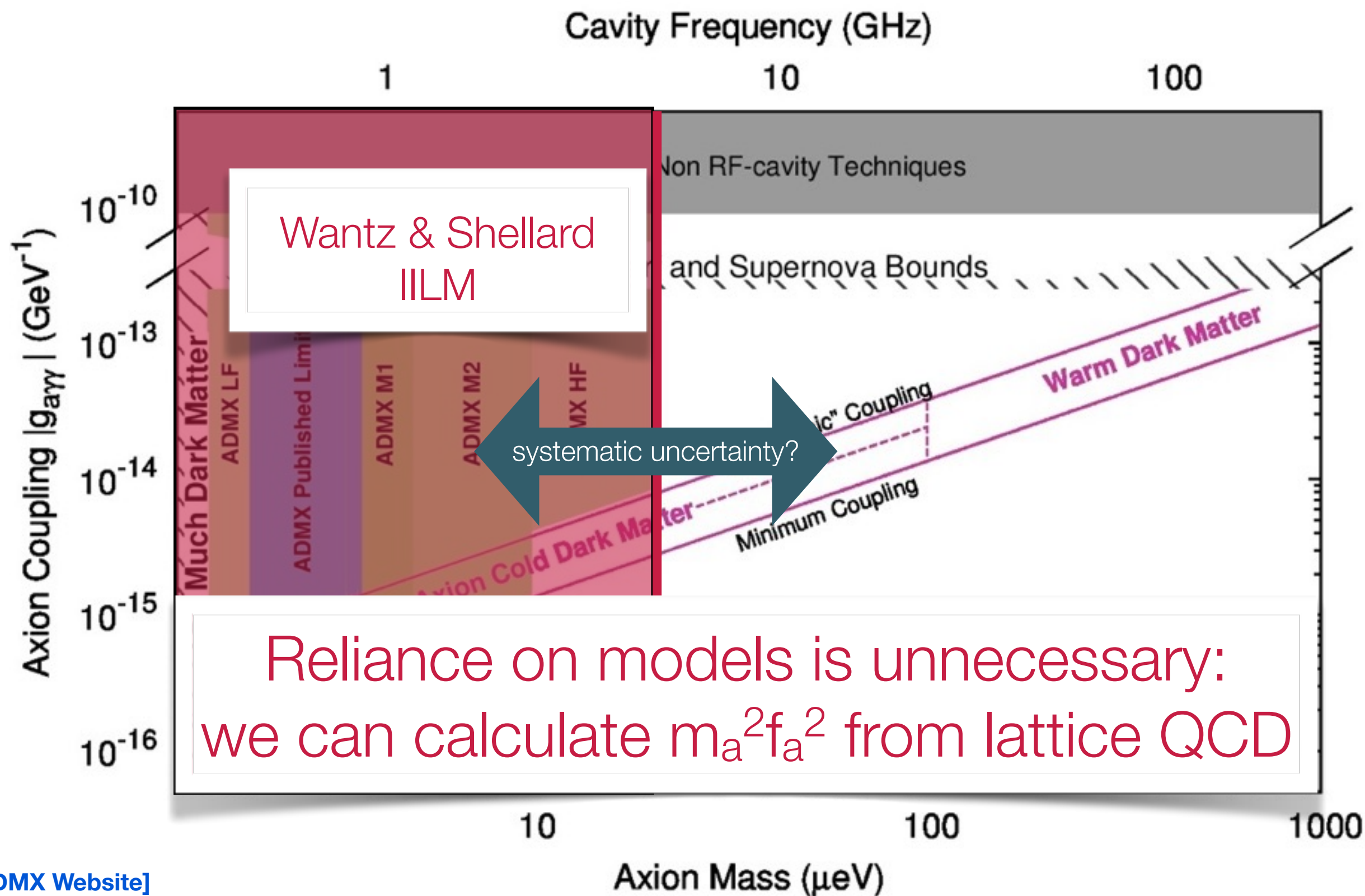
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# Lattice simulations

- SU(3) YM with **Wilson plaquette action**
- Temperature between **1.2 and 2.5  $T_c$**
- $N_\sigma$  between 48 and 144 (large aspect ratio)
- $N_\tau$  chosen to be 4, 6 or 8
- Between 14K and 52K measurements
  - Combined **hot** & **cold** starts
  - Cut of 3000 cfg.s for thermalization
  - Negligible autocorrelations
  - Gauge smoothing via “**cooling**”
- Different **bosonic** definitions of the topological charge  $Q_i$  on the lattice (and susceptibility)

$$Q_{\mathbb{R}} = \frac{1}{32\pi^2} \sum_x \epsilon^{\mu\nu\rho\sigma} \square_{\mu\nu} \square_{\rho\sigma}$$

raw measurement

$$Q_{\mathbb{Z}}$$

naïve rounding

$$Q_a$$

artifact corrected  
Lucini & Teper, hep-lat/0103027

$$Q_f$$

globally fit  
del Debbio *et al.*, hep-th/0204125

$$\chi_i = \lim_{V \rightarrow \infty} \frac{\langle Q_i^2 \rangle}{V}$$

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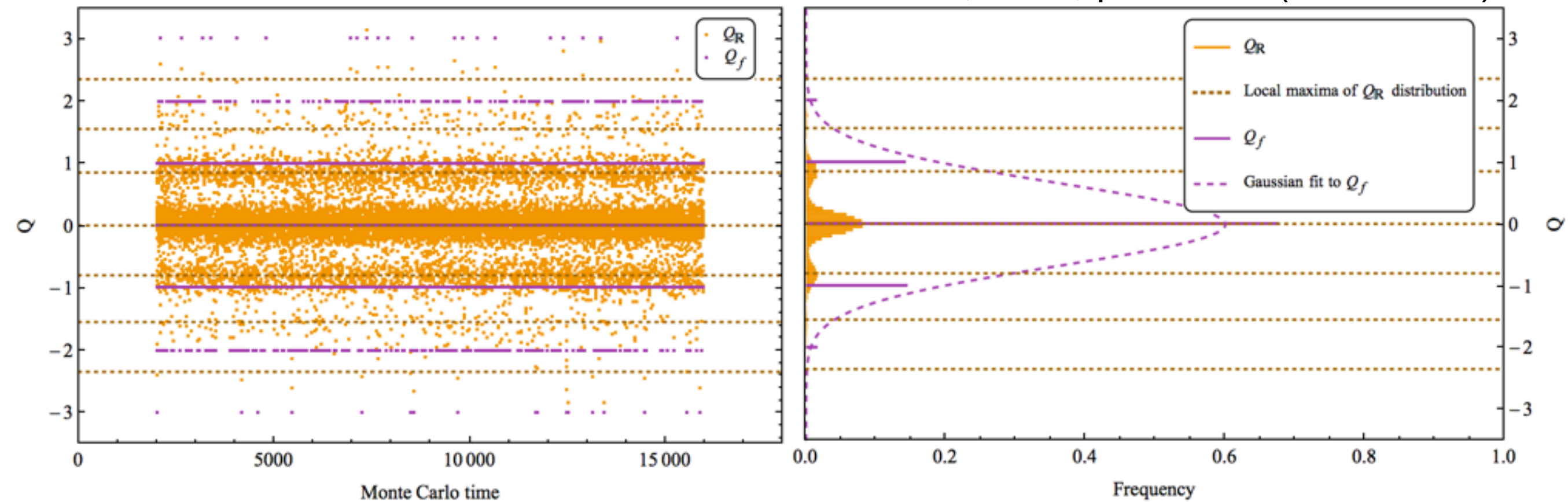
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Essentially no  
discretization or finite  
volume corrections

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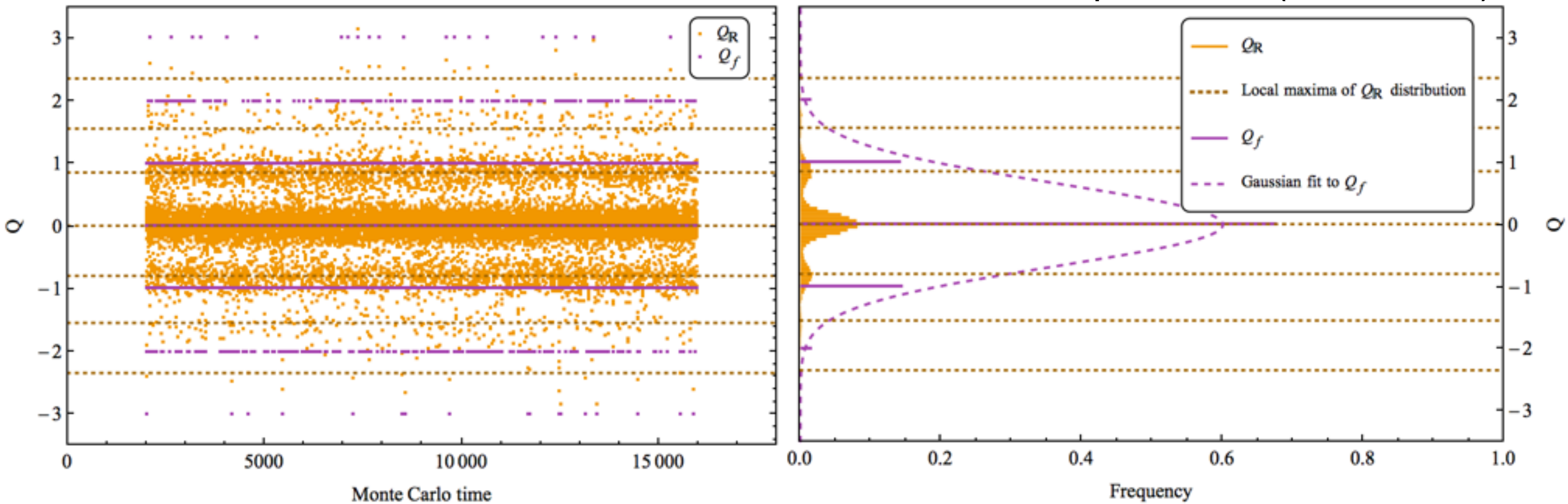
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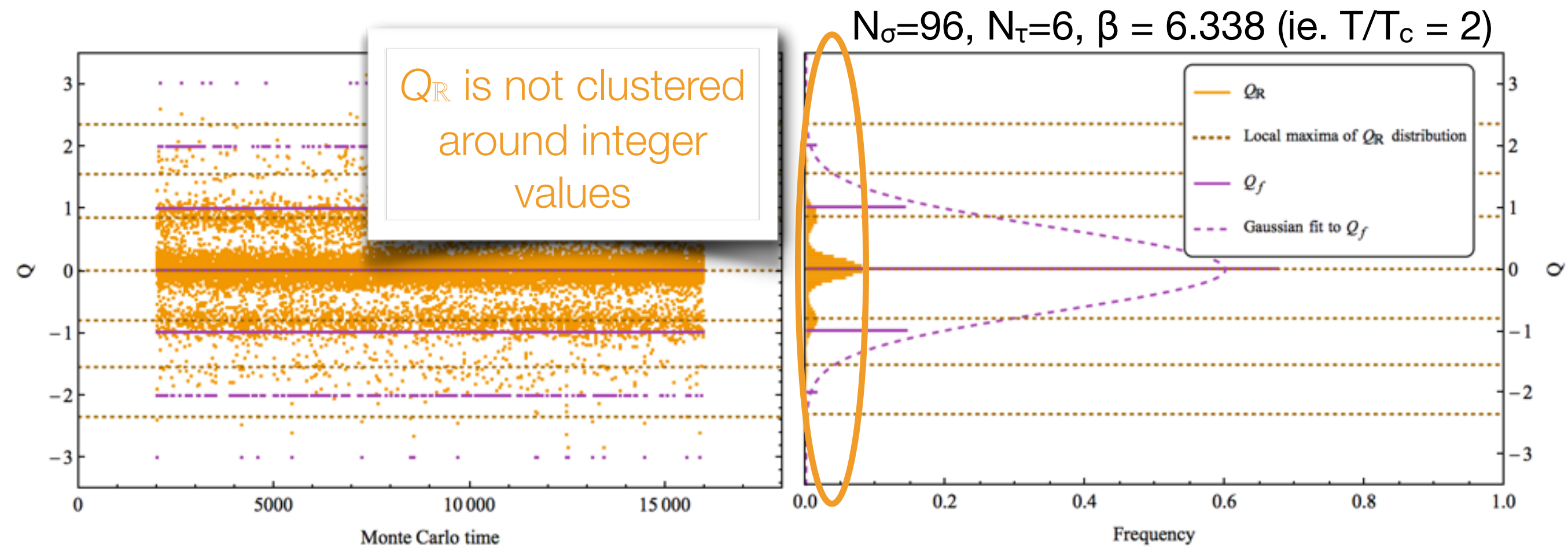
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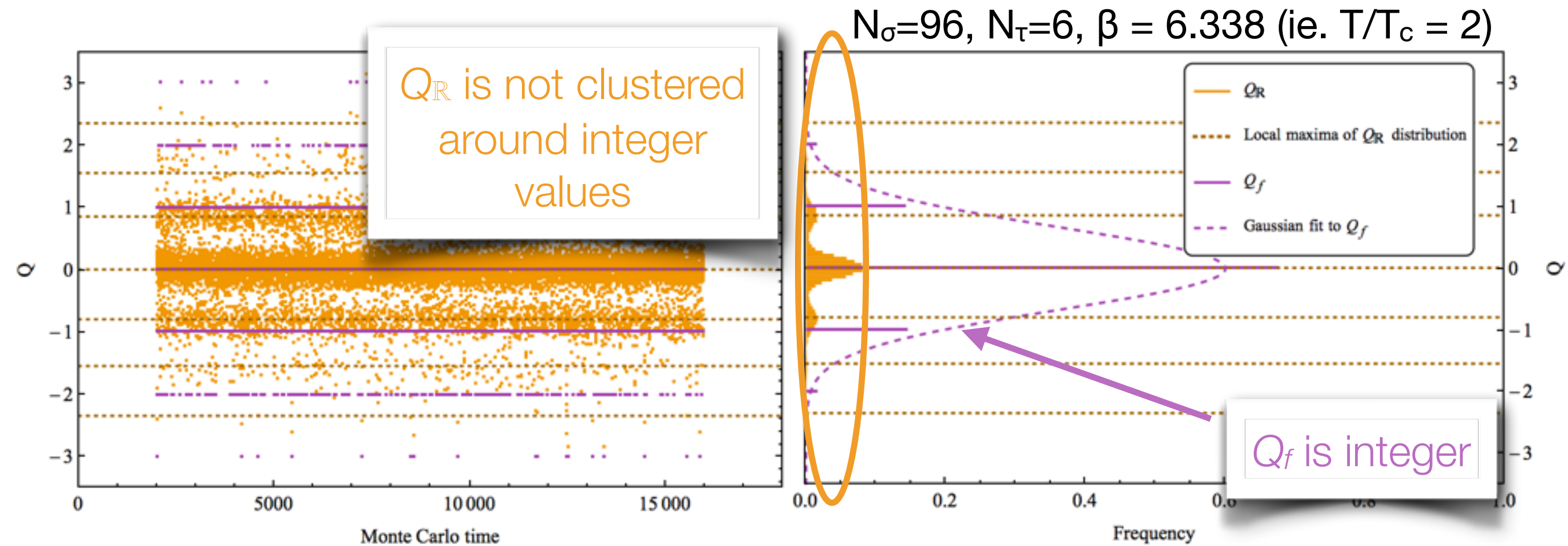
# Fluctuations of topology



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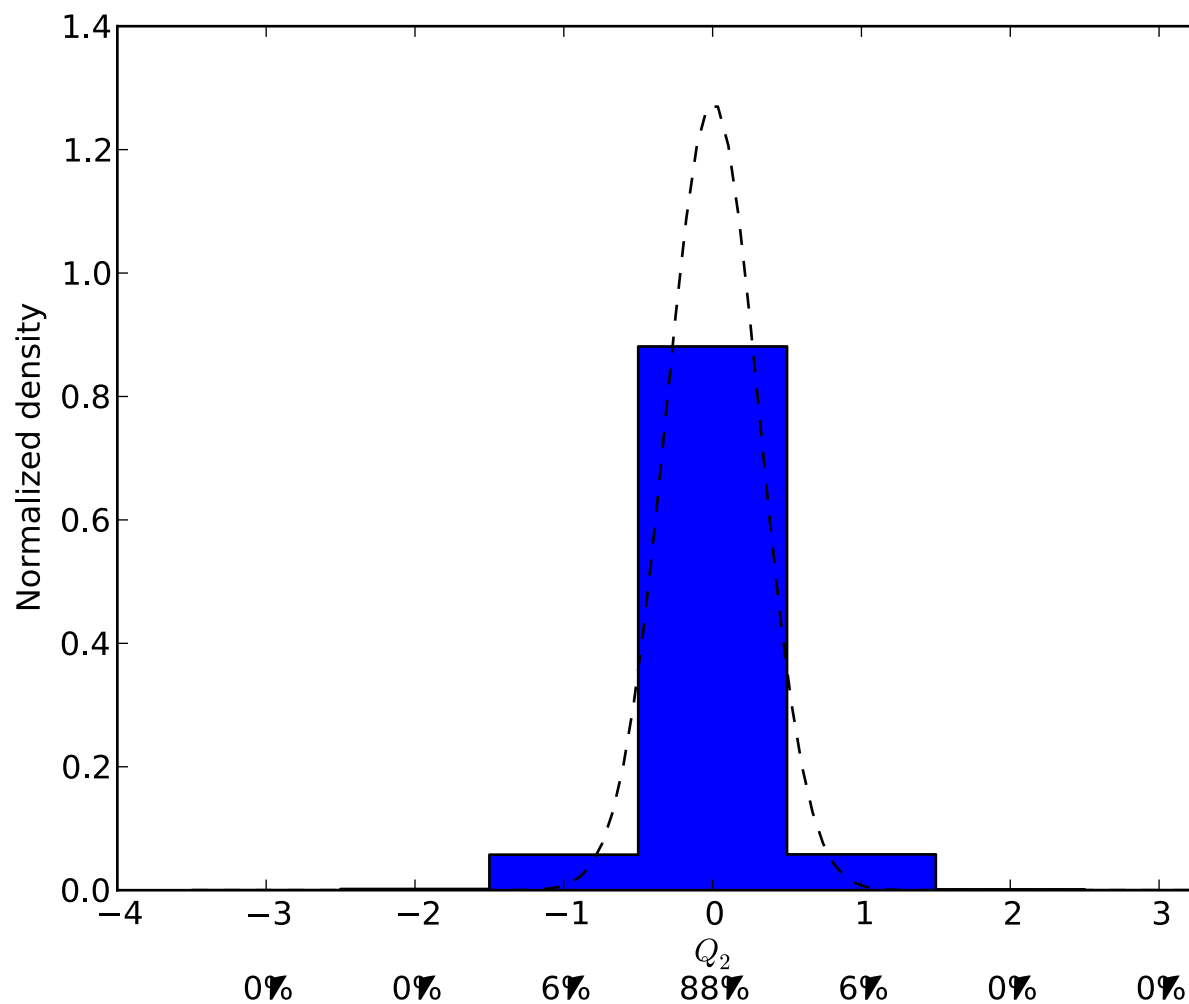
# Fluctuations of topology



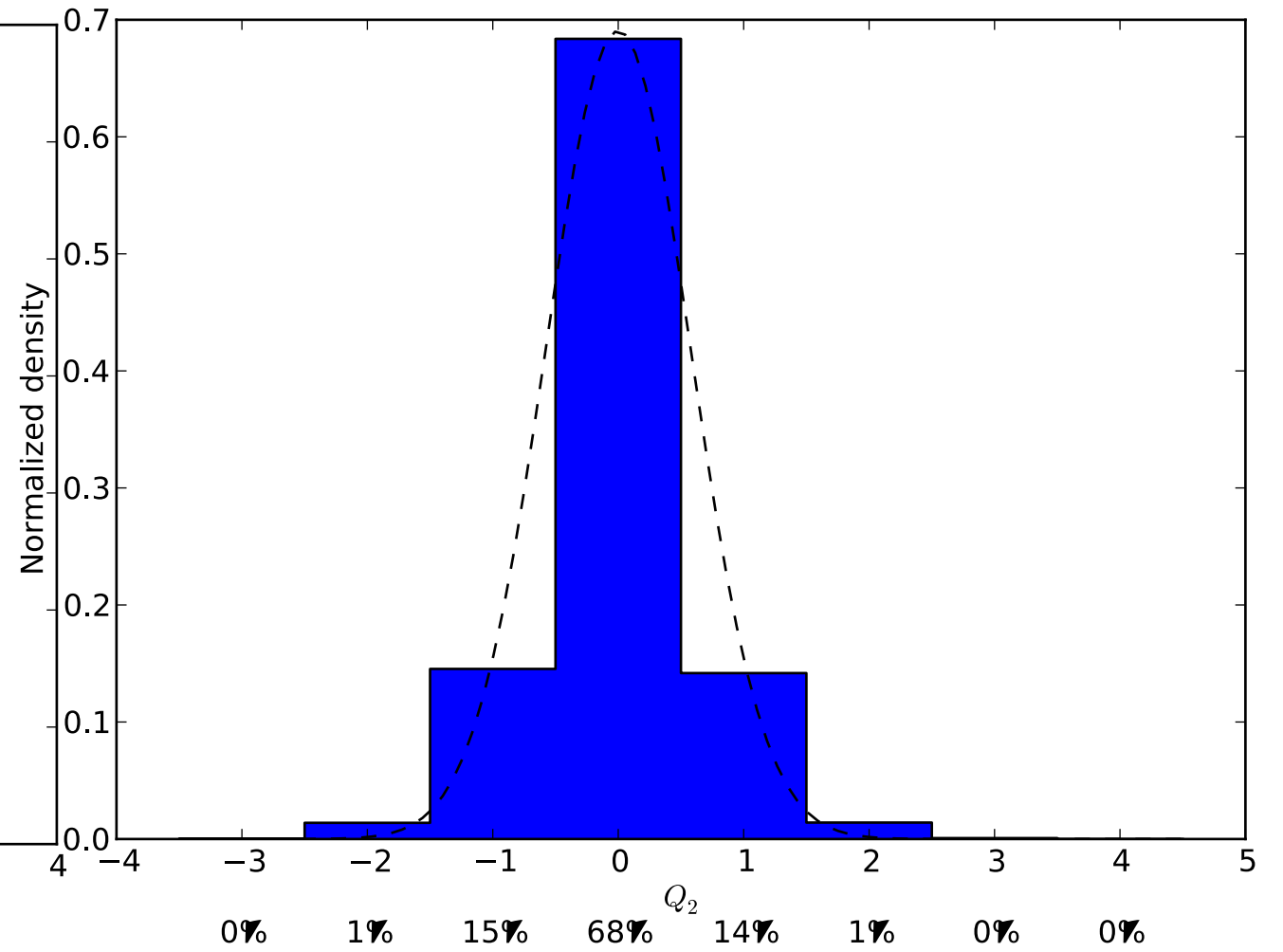
- Topology fluctuates sufficiently for measuring  $\chi$

# Fluctuations of topology

$N_\sigma=64, N_\tau=6, T/T_c=2$

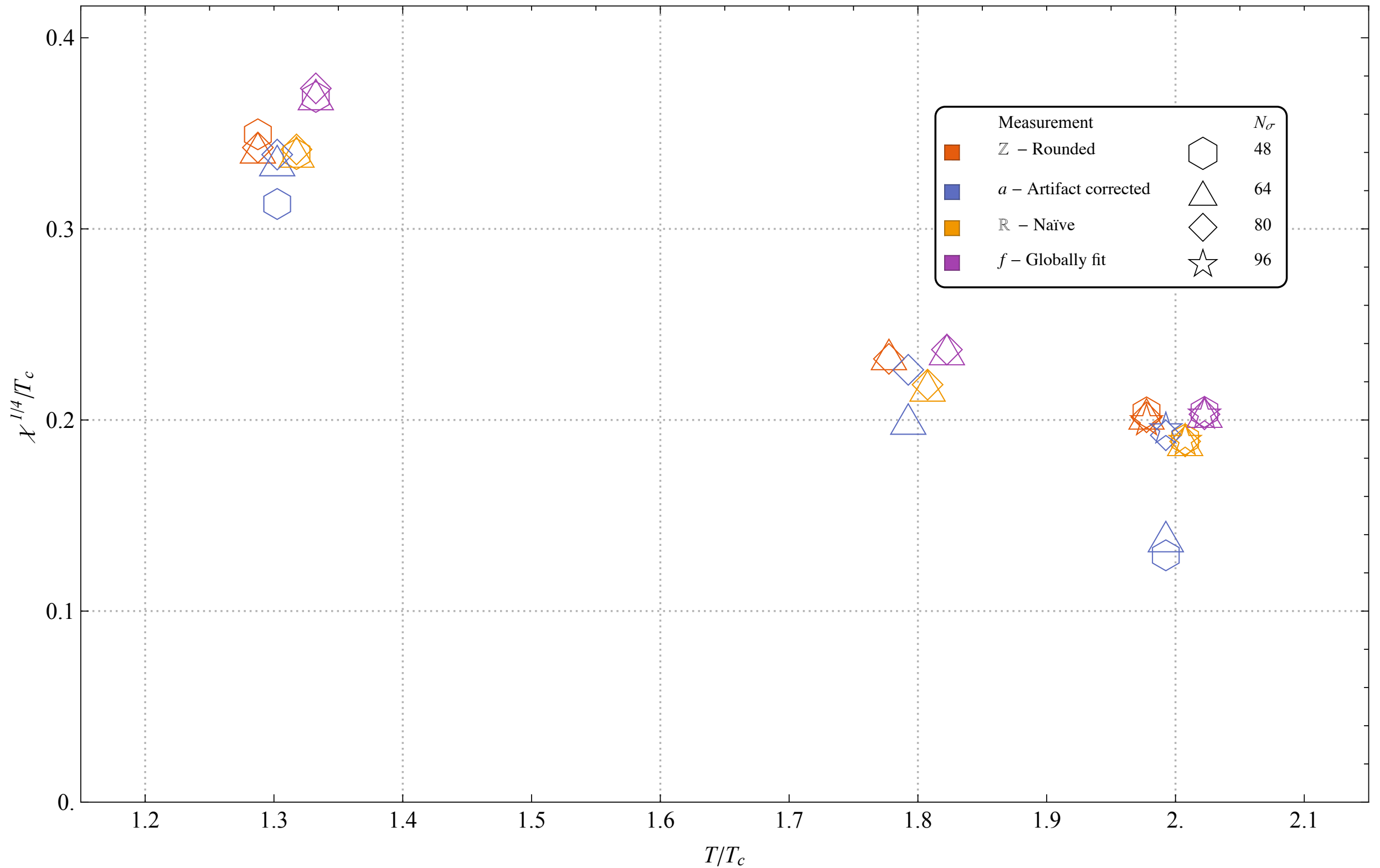


$N_\sigma=96, N_\tau=6, T/T_c=2$



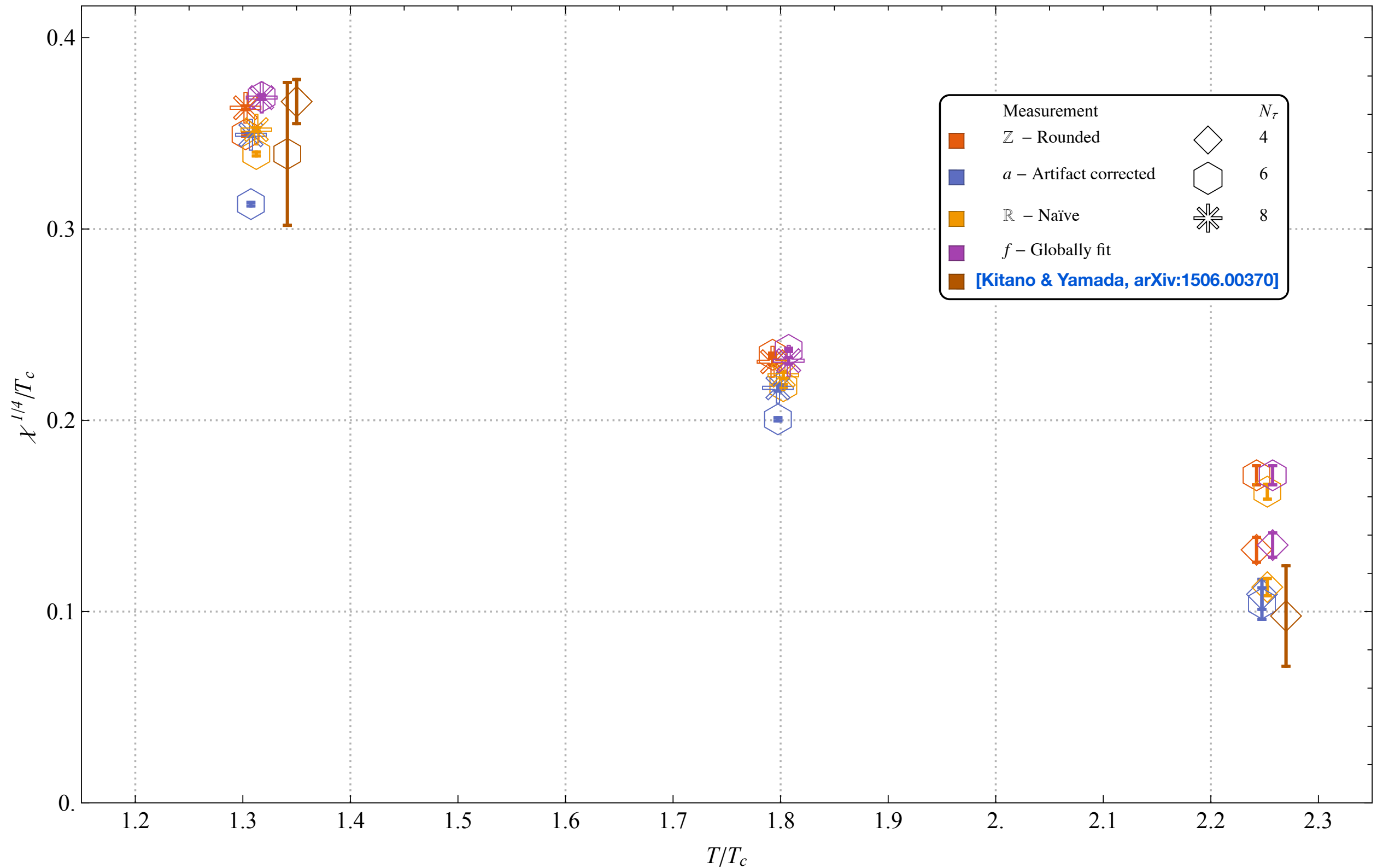
- Topology fluctuates sufficiently for measuring  $\chi$
- Large volumes help obtaining a more reliable  $\chi$

# Finite Volume Effects

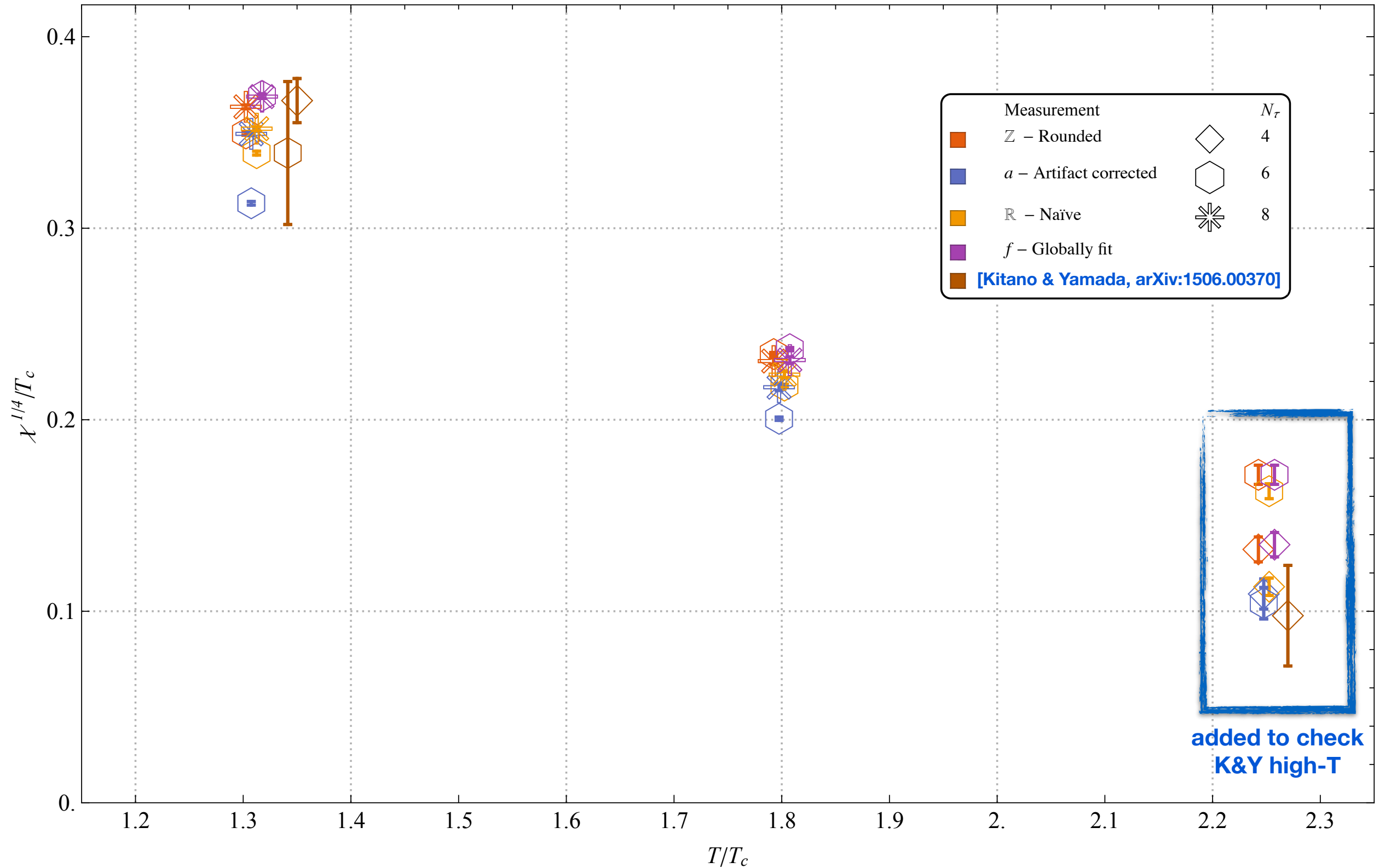




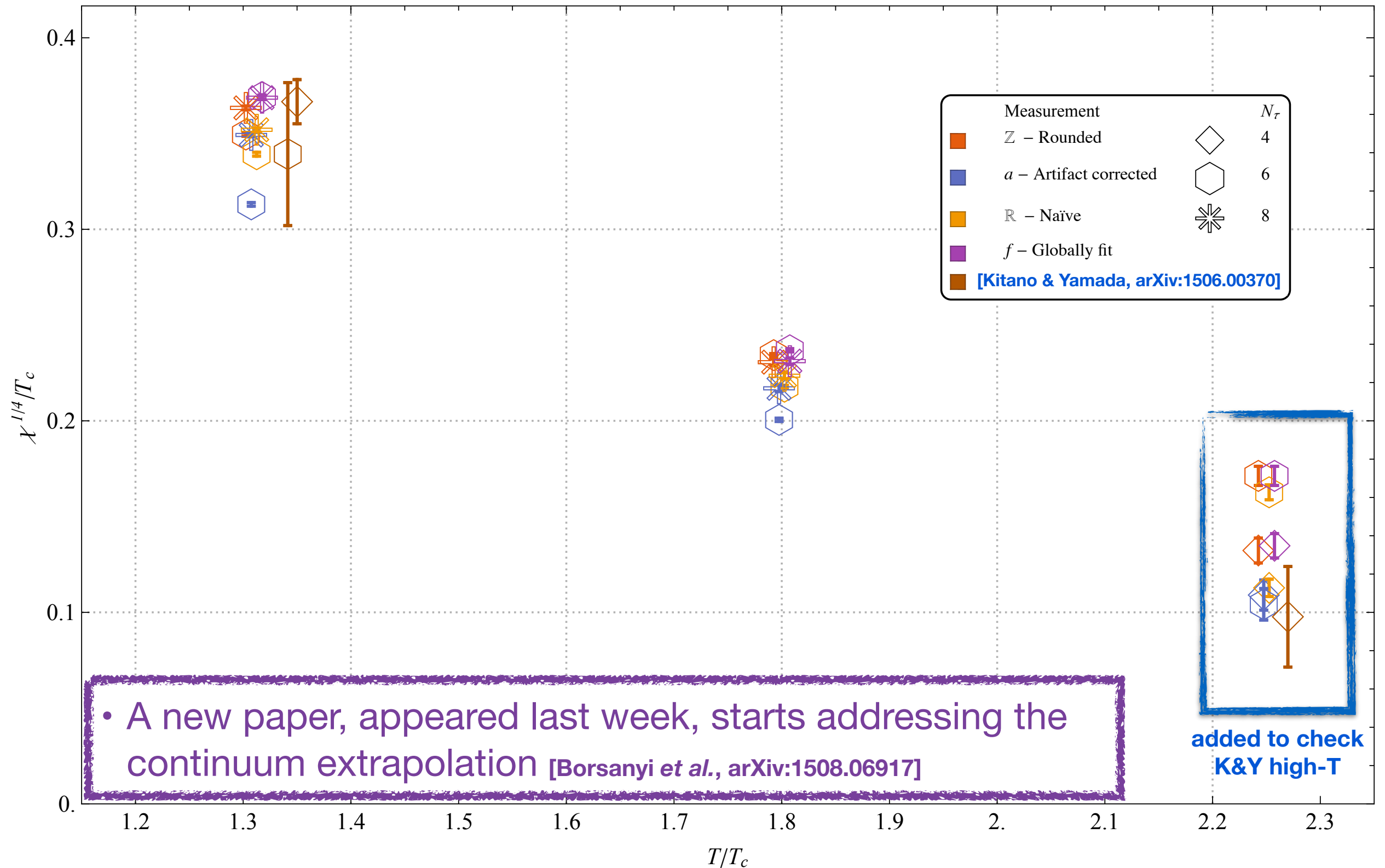
# Discretization Effects



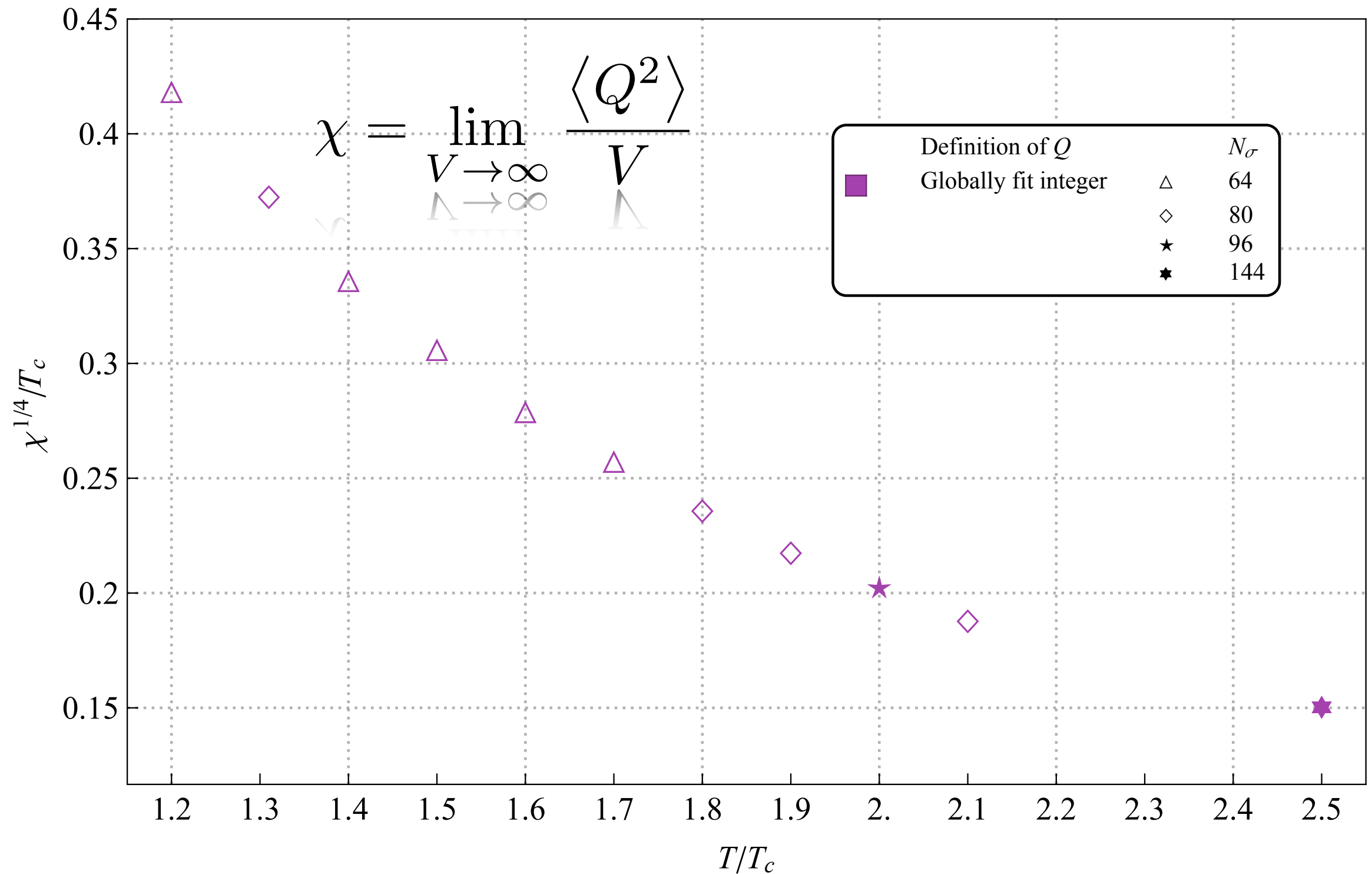
# Discretization Effects



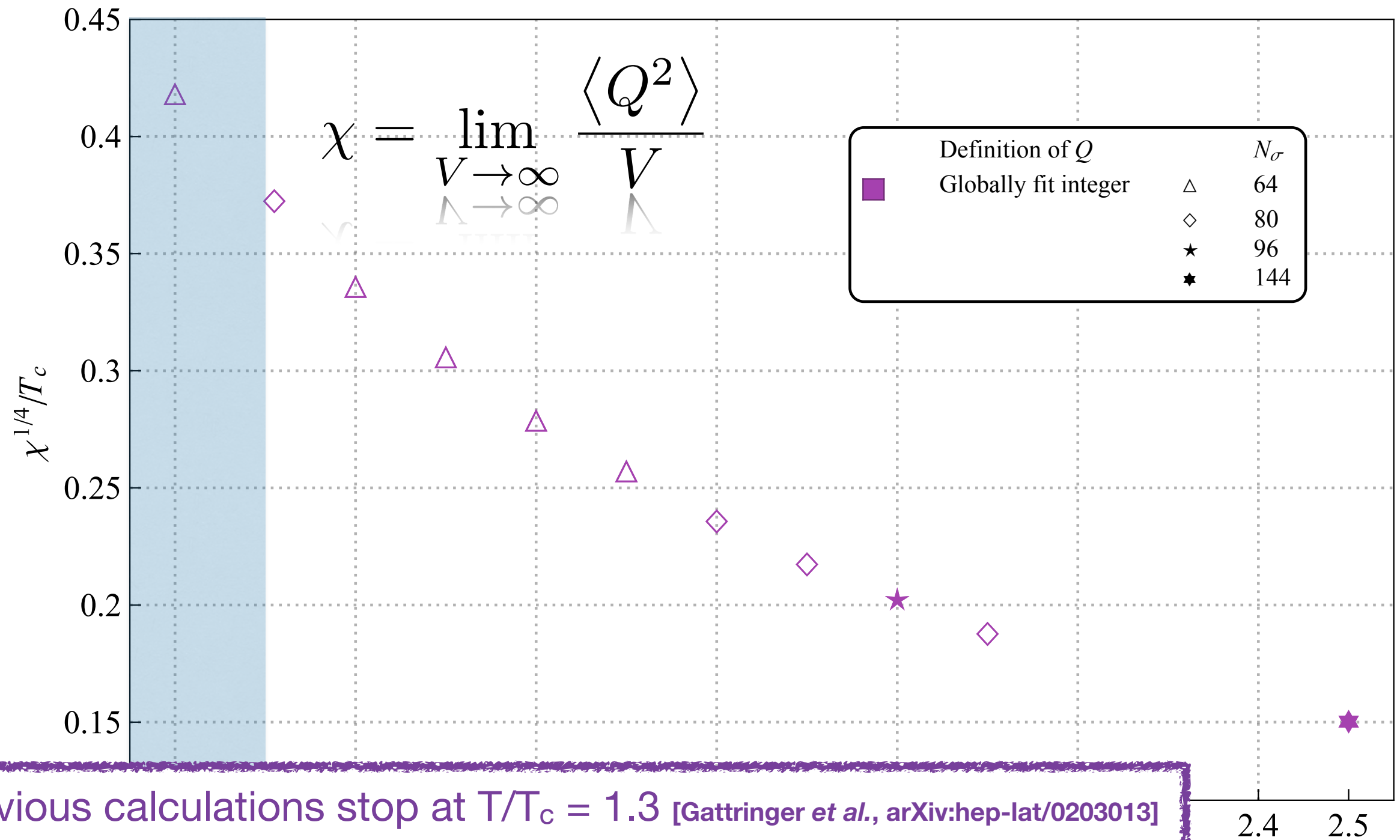
# Discretization Effects



# Lattice results



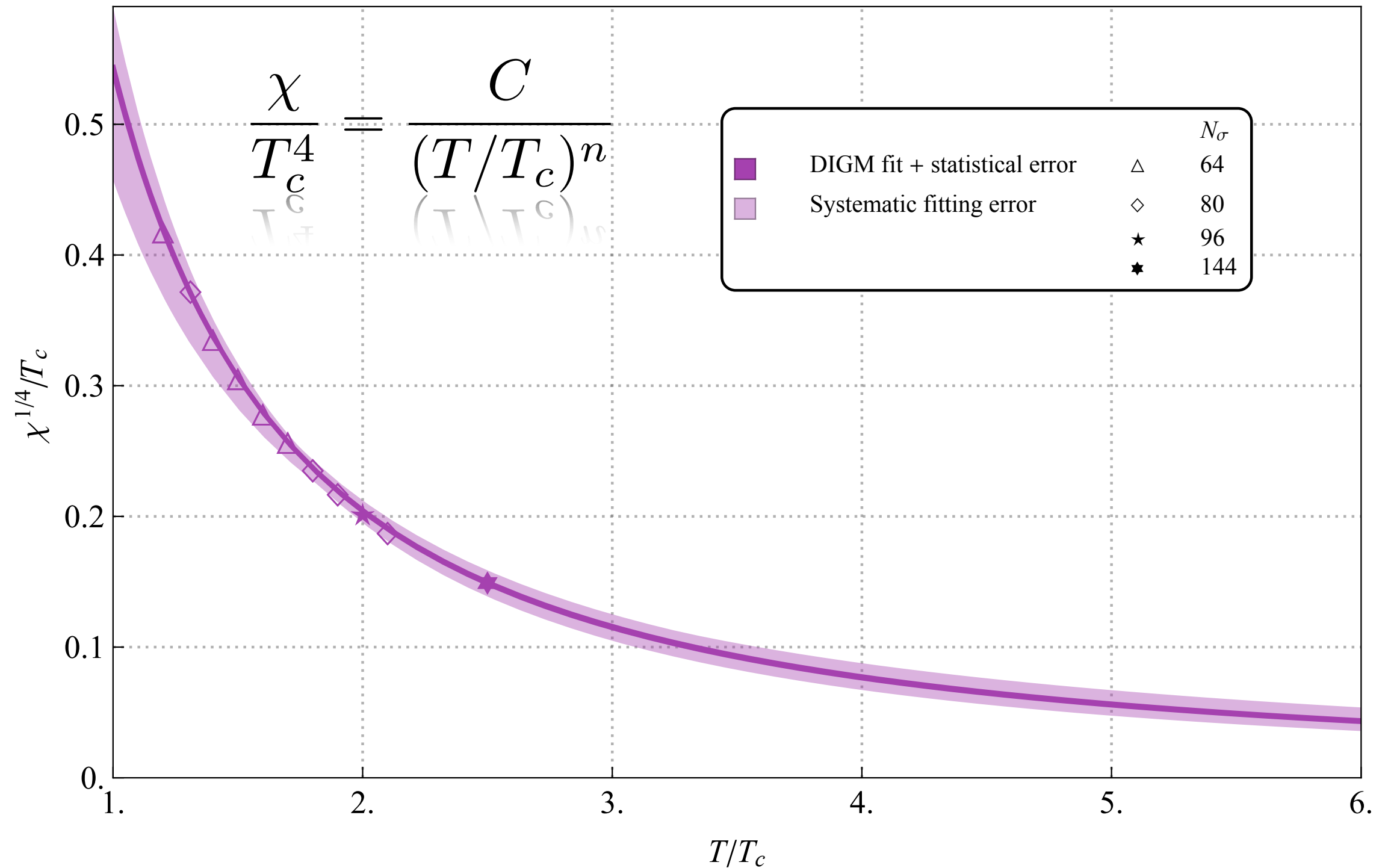
# Lattice results



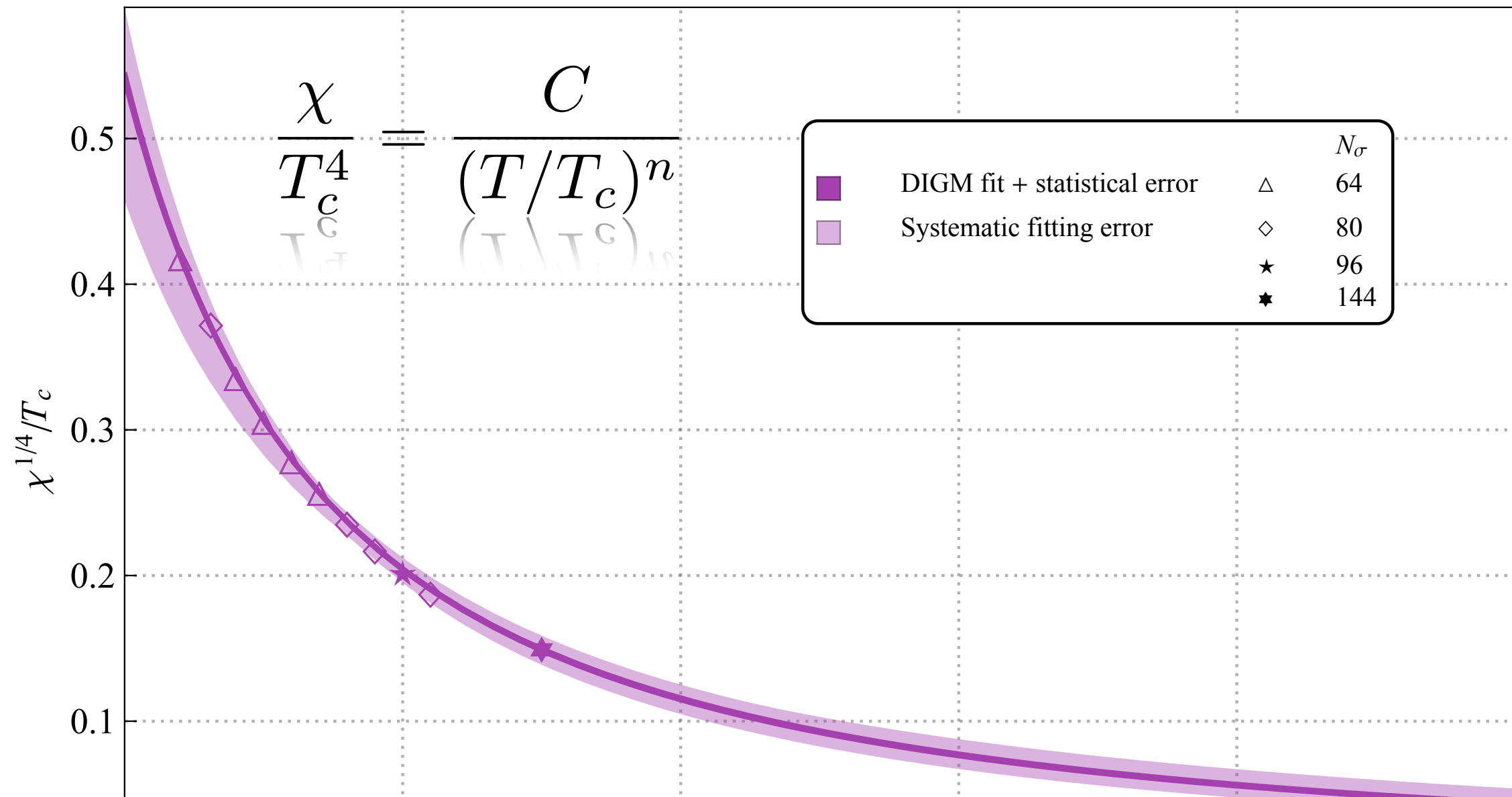
• previous calculations stop at  $T/T_c = 1.3$  [Gattringer *et al.*, arXiv:hep-lat/0203013]

• newest calculation stops at  $T/T_c = 4.0$  [Borsanyi *et al.*, arXiv:1508.06917]

# Model and extrapolation

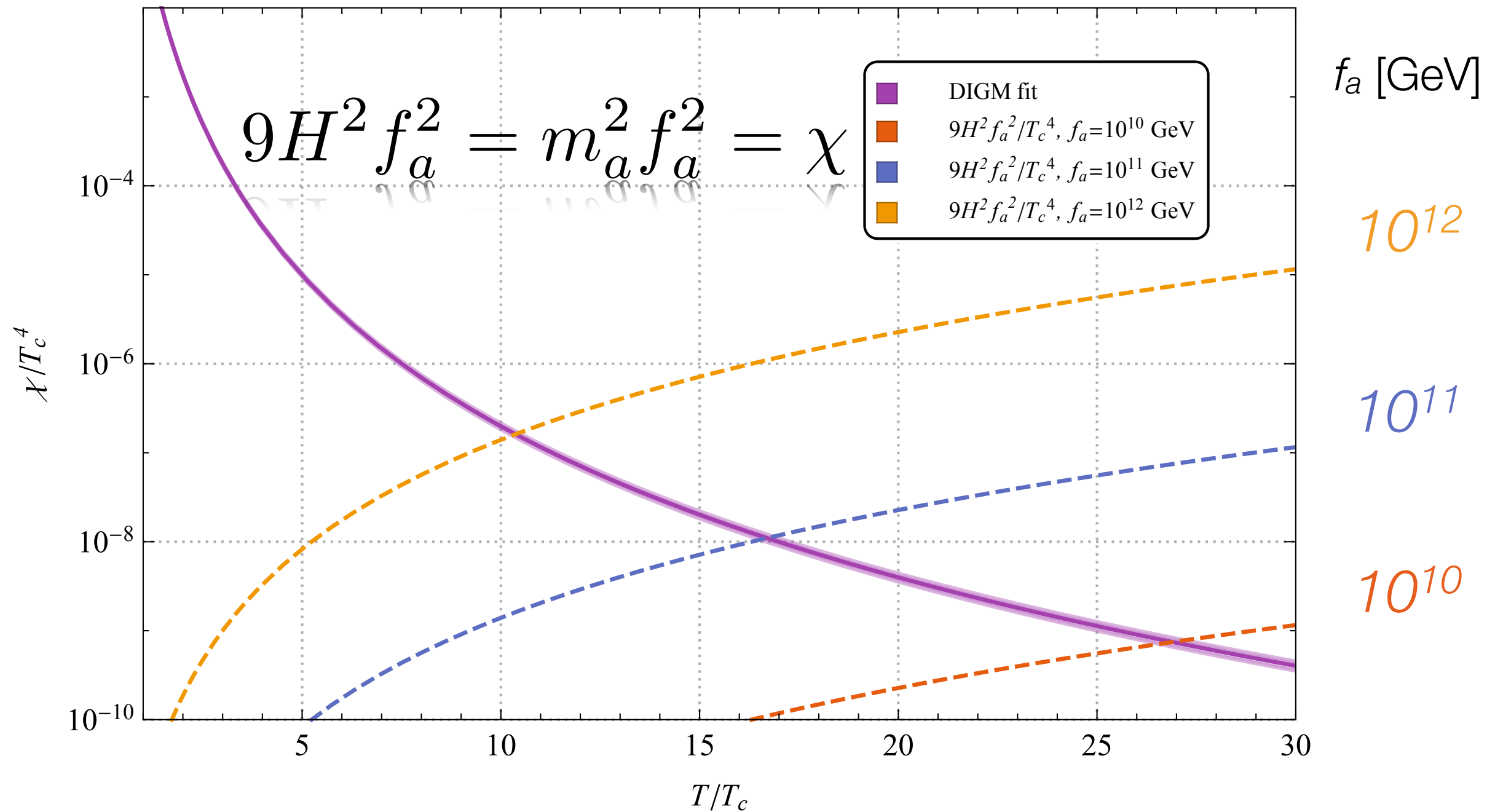


# Model and extrapolation



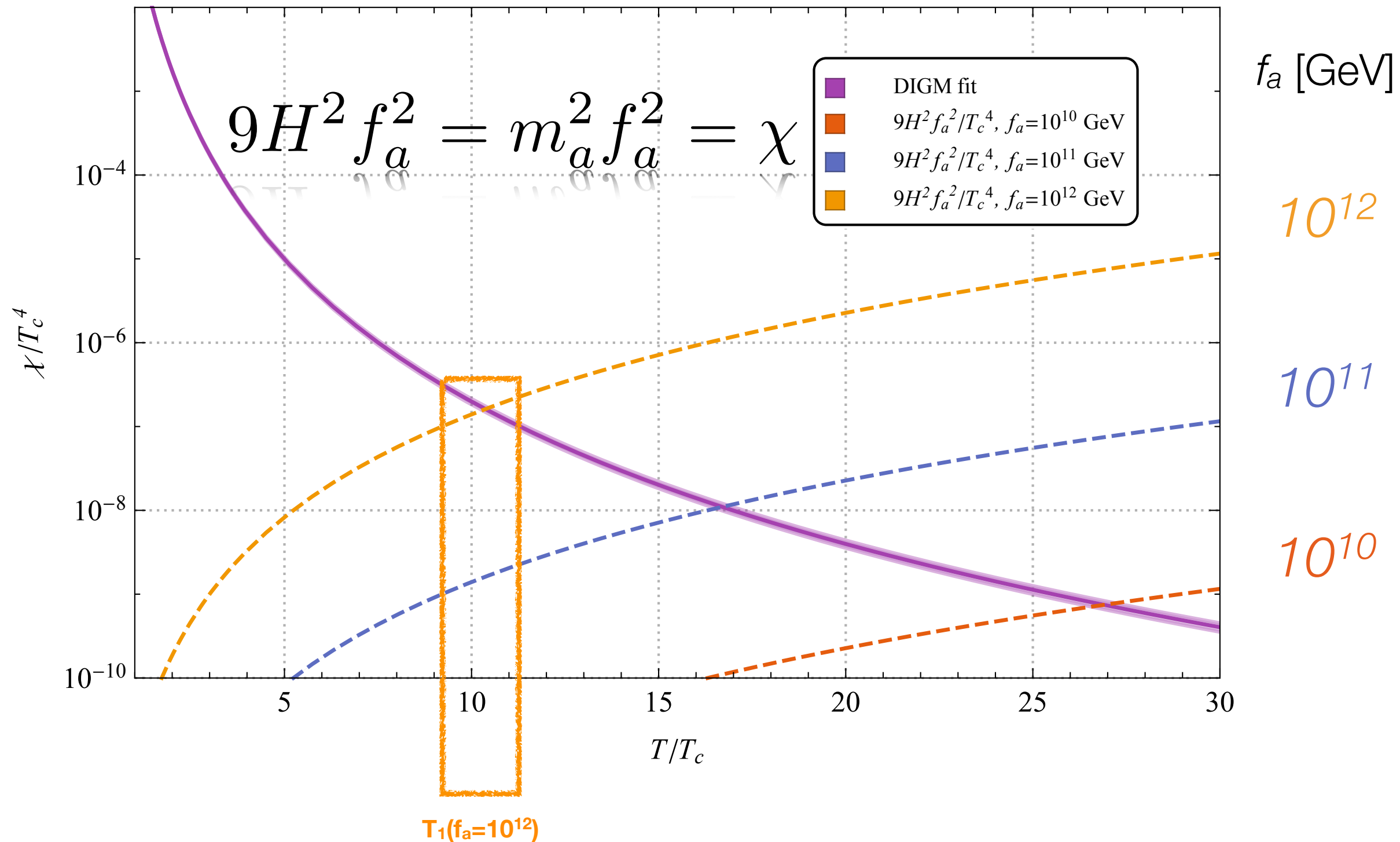
- asymptotic large-T susceptibility in the DIG model [Gross&Yaffe, RevModPhys.53.43]
- fits lattice data of  $\chi$  remarkably well for all temperatures: first high-temperature comparison of DIGM to lattice data!

# Axion oscillations

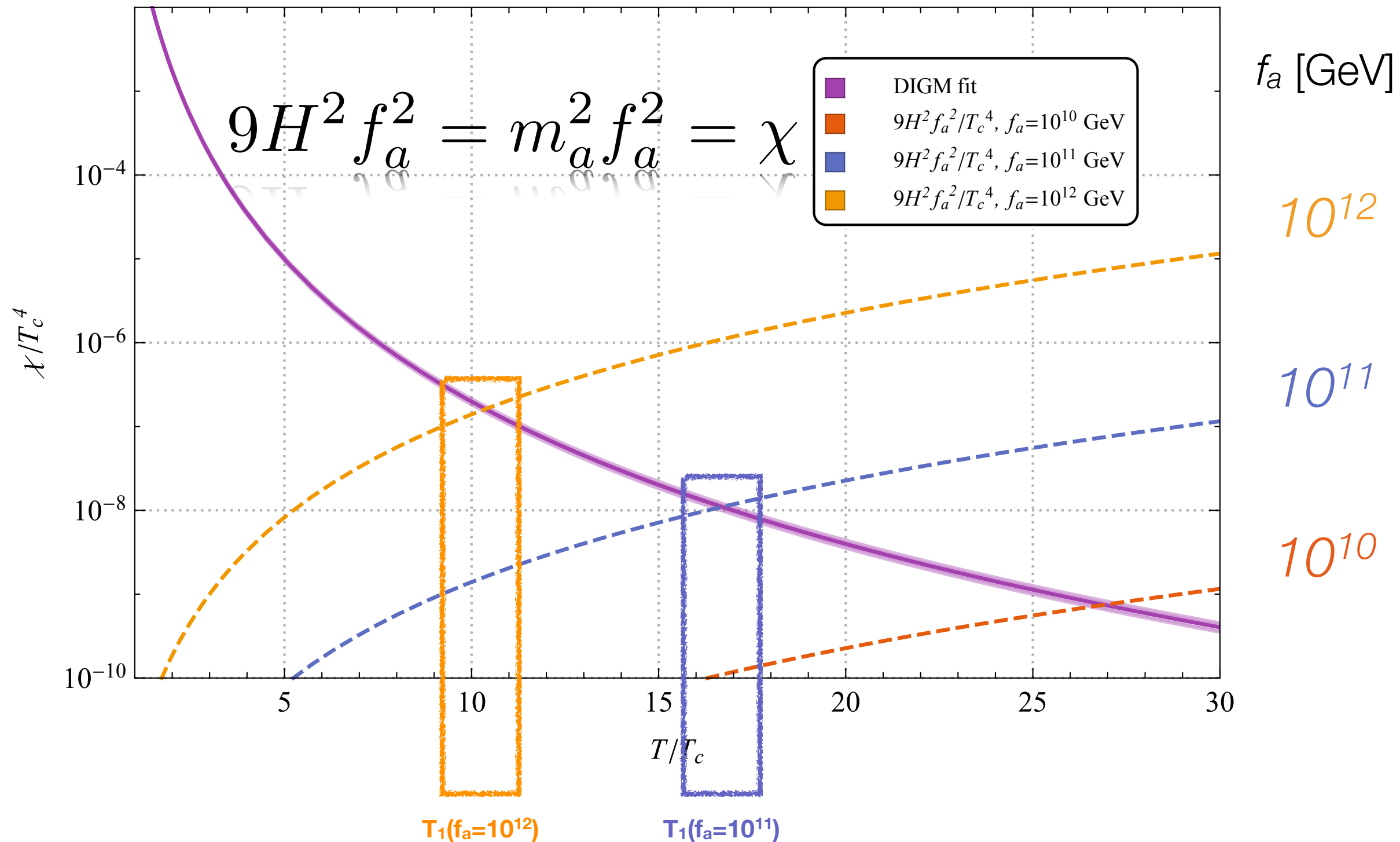




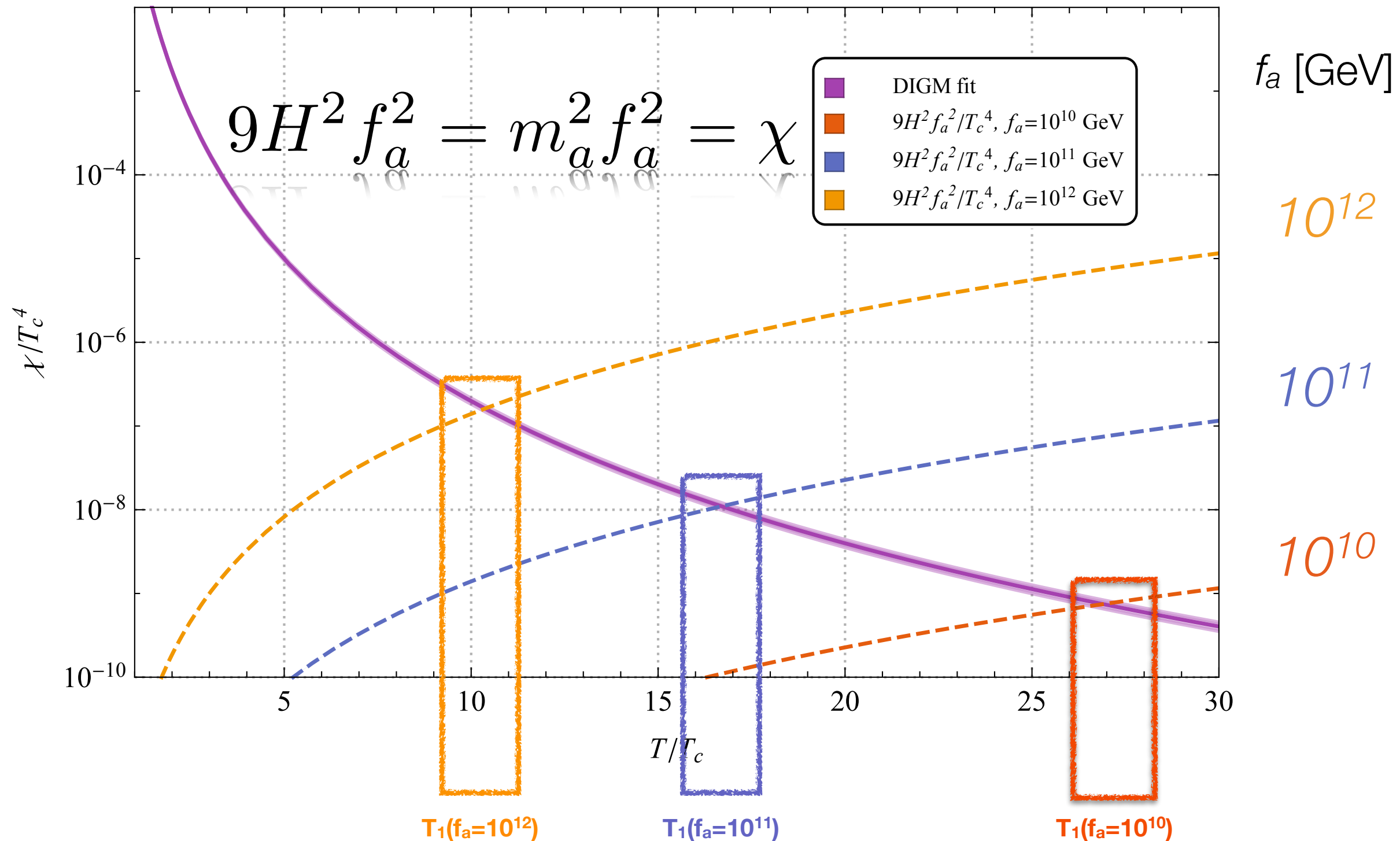
# Axion oscillations



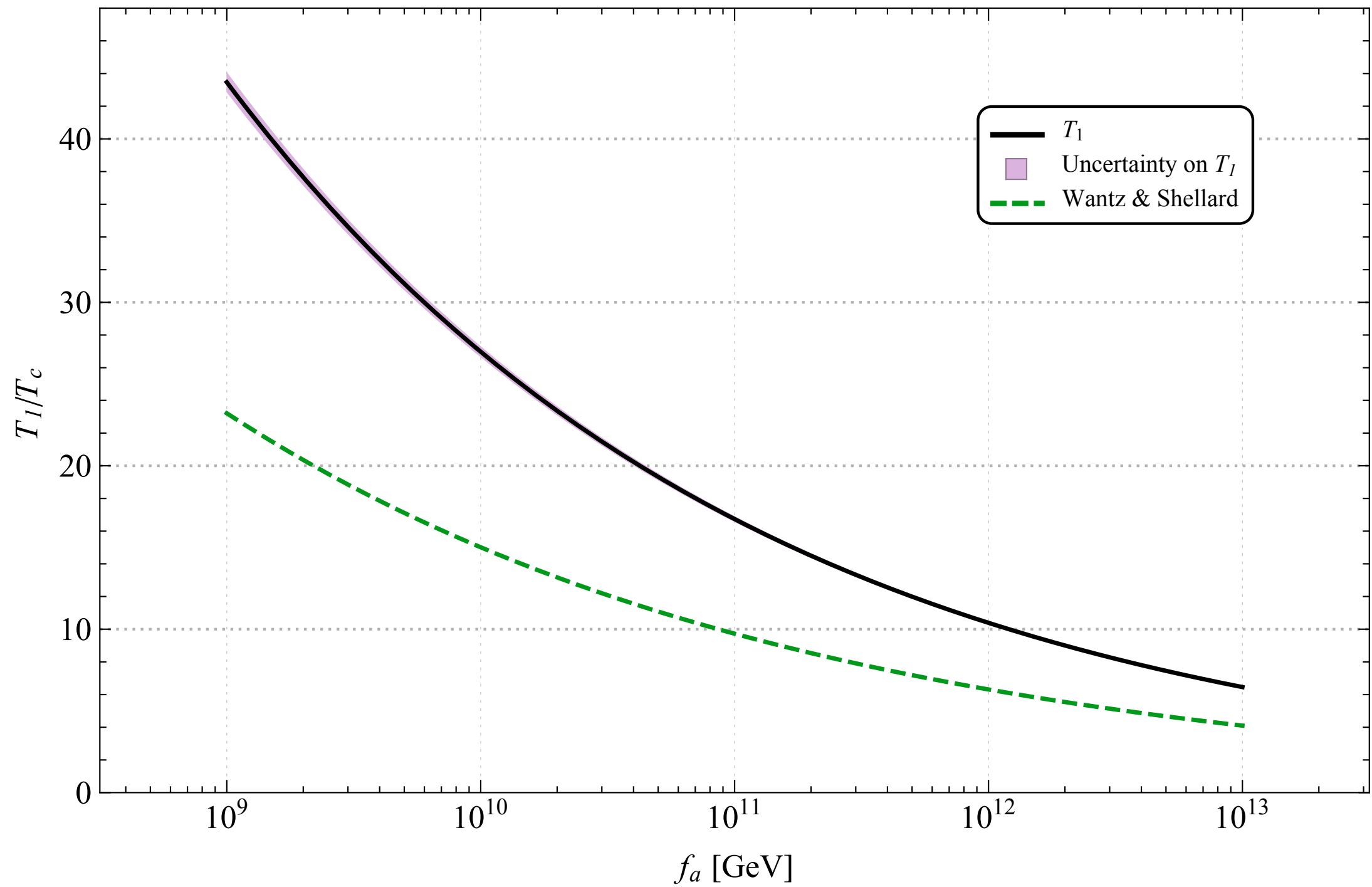
# Axion oscillations



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# Axion energy density

$$\rho(T_\gamma) = \rho(T_1) \frac{m_a(T_\gamma)}{m_a(T_1)} \left( \frac{R(T_1)}{R(T_\gamma)} \right)^3 \quad T_\gamma = 2.73\text{K}$$


$$T_1 = T_1(f_a) \quad m_a(T_1) = \frac{\sqrt{\chi(T_1)}}{f_a}$$

$$m_a(T_\gamma) = \frac{1}{f_a} \frac{\sqrt{m_u m_d}}{m_u + m_d} f_\pi m_\pi \quad \text{from } \chi\text{PT}$$

$R(T)$  from cosmology

$$\rho(T_1) = \frac{1}{2} m_a^2 f_a^2 \theta_1^2$$

$\theta_1$  random: PQ breaks after inflation

$$\langle \theta_1^2 \rangle = \frac{\pi^2}{3}$$


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Rely on our  
lattice calculation!

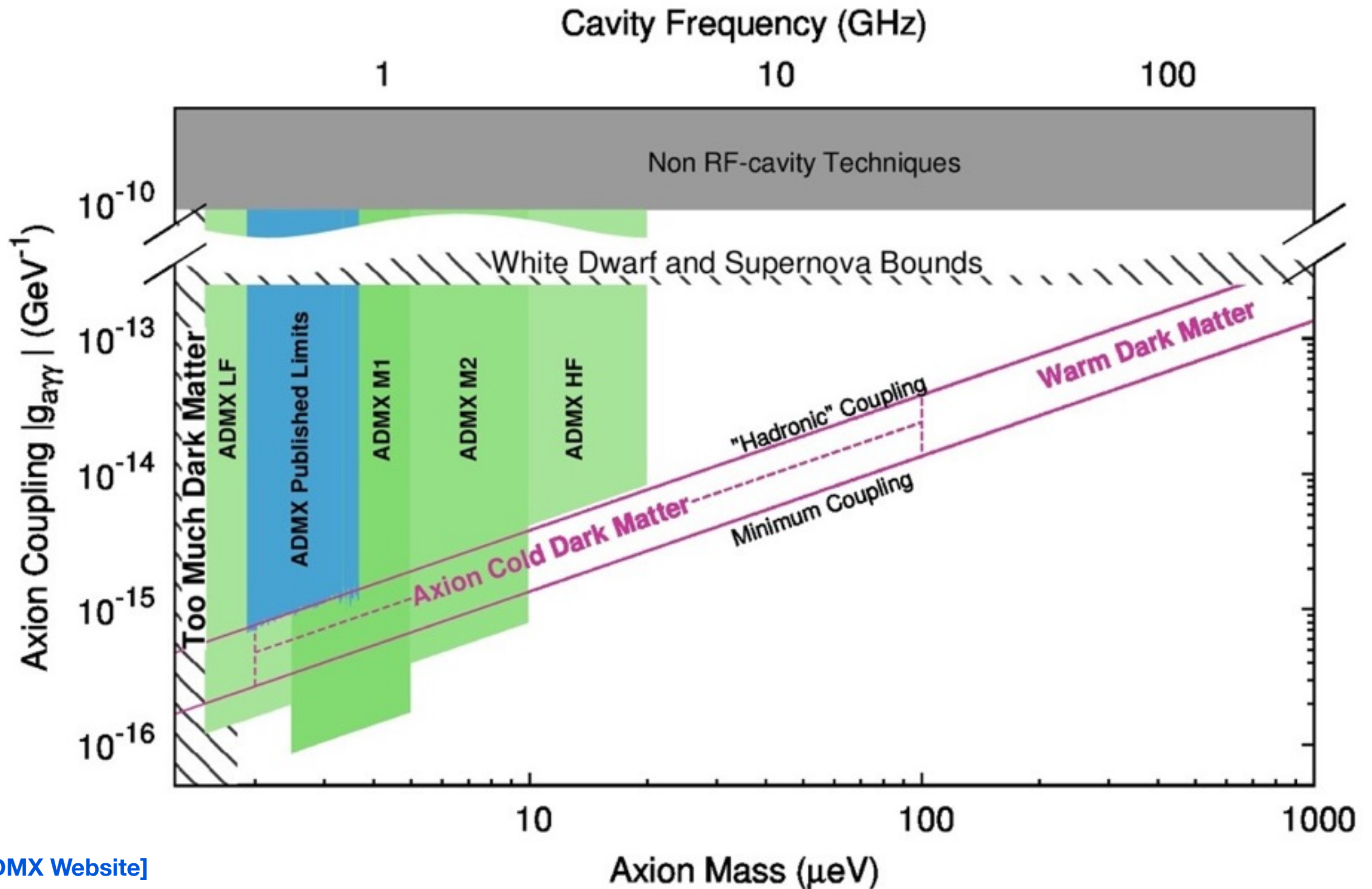
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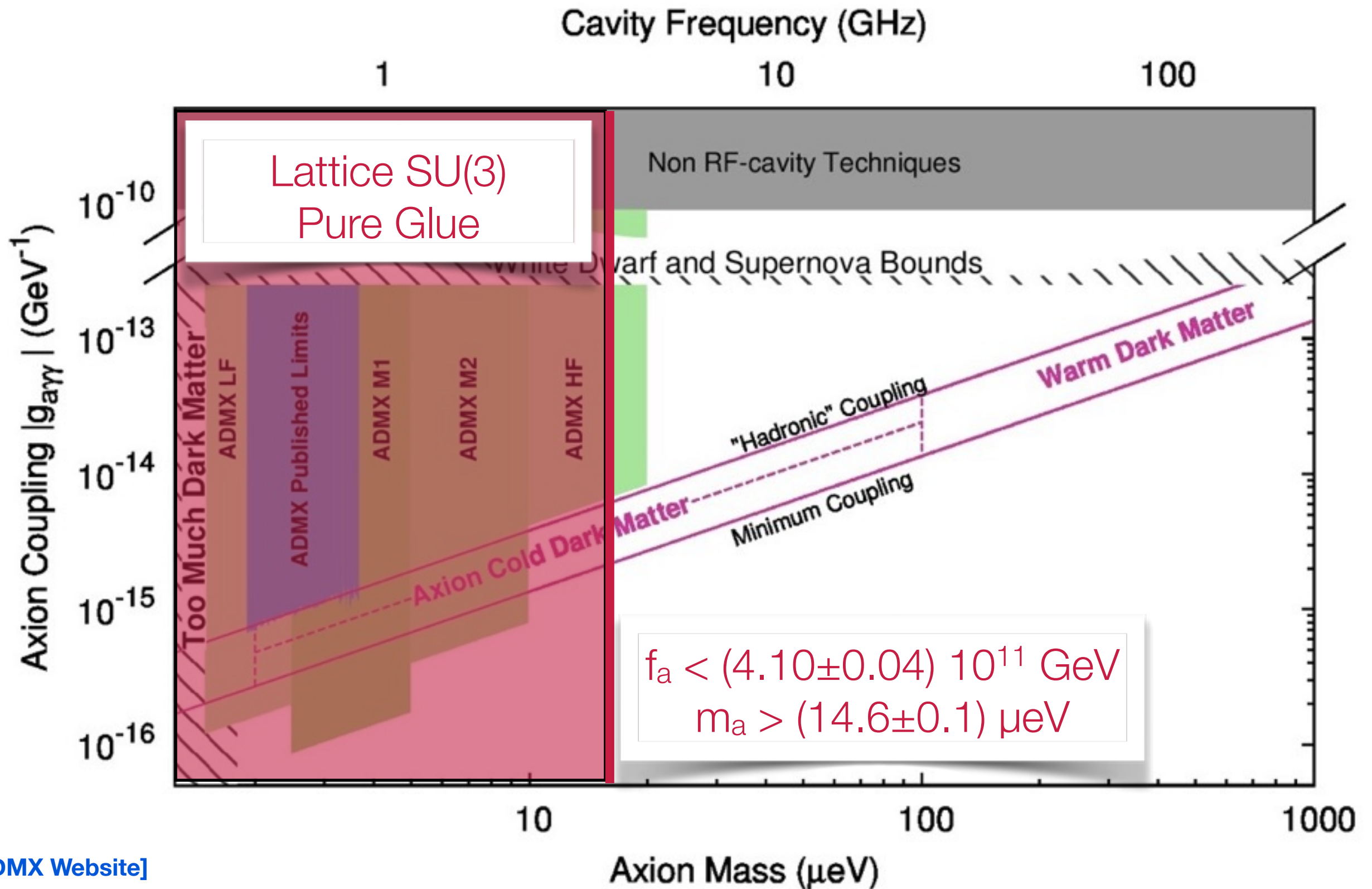
$$\langle \theta_1^2 \rangle = \frac{\pi^2}{3}$$


# Lattice lower bound



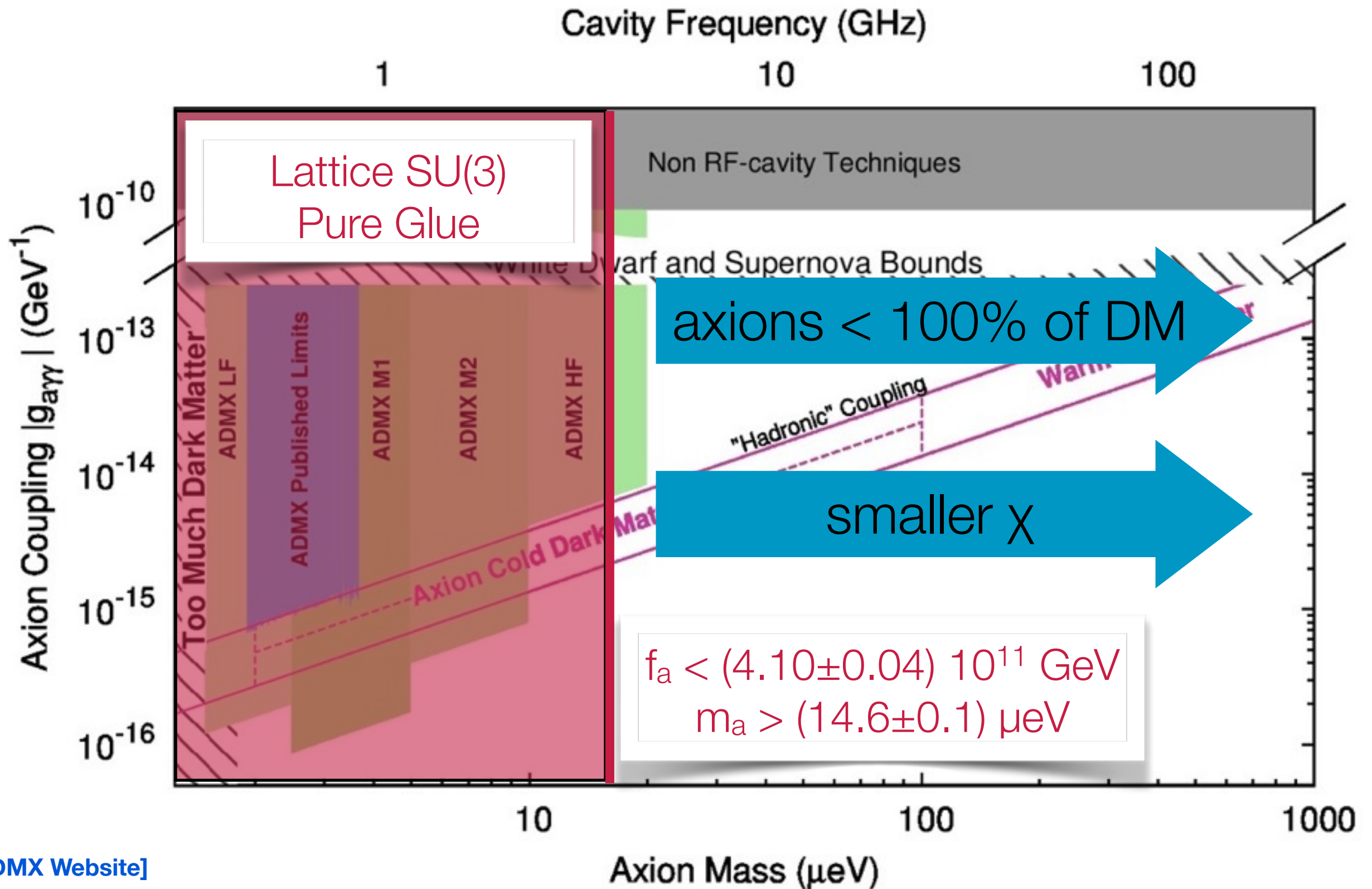


# Lattice lower bound

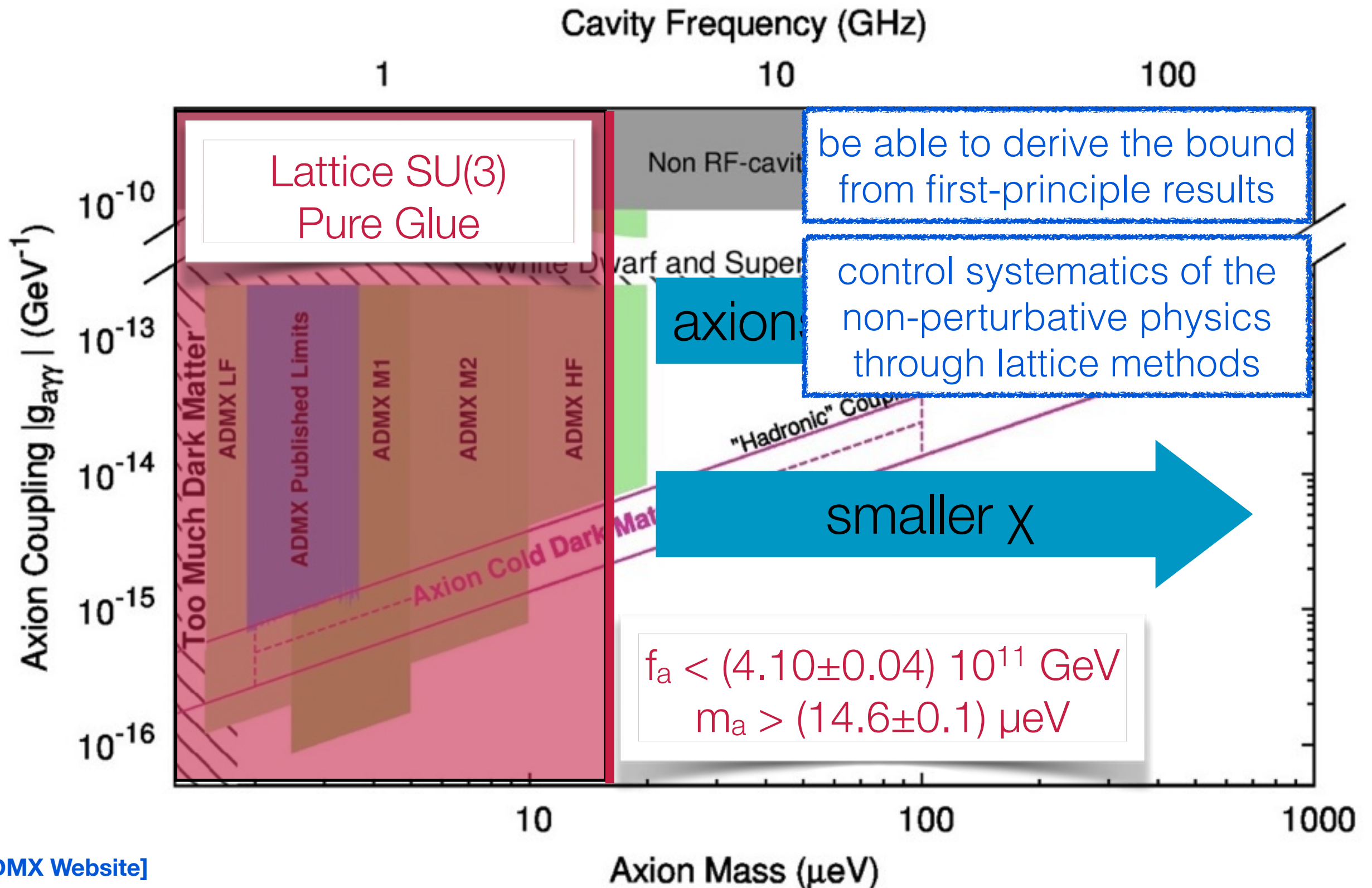




# Lattice lower bound



# Lattice lower bound



# Conclusions

- Peccei-Quinn symmetry:
  - cleans up the Strong CP problem
  - provides a plausible, largely unconstrained DM candidate: [the axion](#).
- Axion searches will probe interesting parameter space soon:
  - lattice QCD can provide [important non-perturbative input](#) for calculating the axion energy density
- [DIGM](#) fits outstandingly to SU(3) YM data at high temperature.
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# Future Steps

- Measure higher moments? Possible to get  $\chi_4$  at zero temperature and at finite temperature. [\[Cé et al., arXiv:1506.06052\]](#)  
[\[Borsanyi et al., arXiv:1508.06917\]](#)
- Incorporate quarks: use heavy quarks? match with Yang-Mills?
- Fermionic definition of Q (Overlap) to reduce misidentification  
[\[Kitano & Yamada, arXiv:1506.00370\]](#)
- Wilson Flow smoothing to address a robust continuum limit  
[\[Borsanyi et al., arXiv:1508.06917\]](#)
- Move to anisotropic lattices to alleviate finite-volume effects at high T and slow tunneling [\[Xiong et al., arXiv:1508.07704\]](#)
- Fixed topology methods / open boundary conditions at high T.  
[\[Aoki et al., arXiv:0707.0396v2\]](#) [\[Lüscher & Schaefer, arXiv:1105.4749\]](#)
- Finite  $\theta$ :
  - Imaginary- $\theta$  has no sign problem
  - Real, finite  $\theta$  may be amenable to Langevin methods