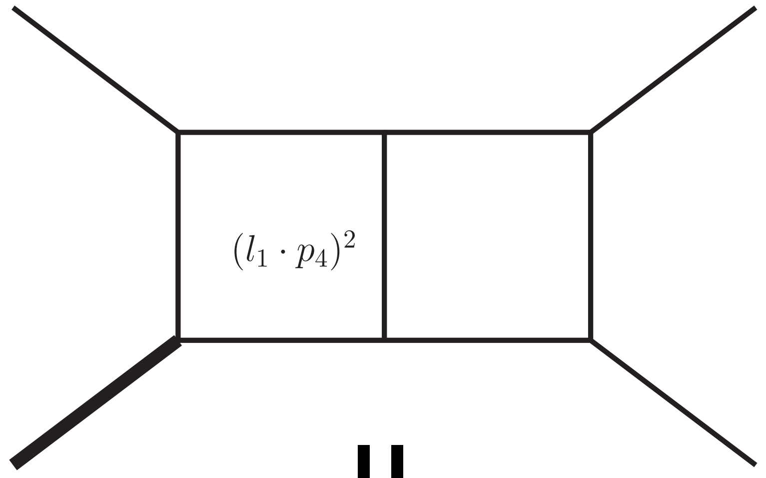
Integral reduction via Unitarity and Algebraic Geometry

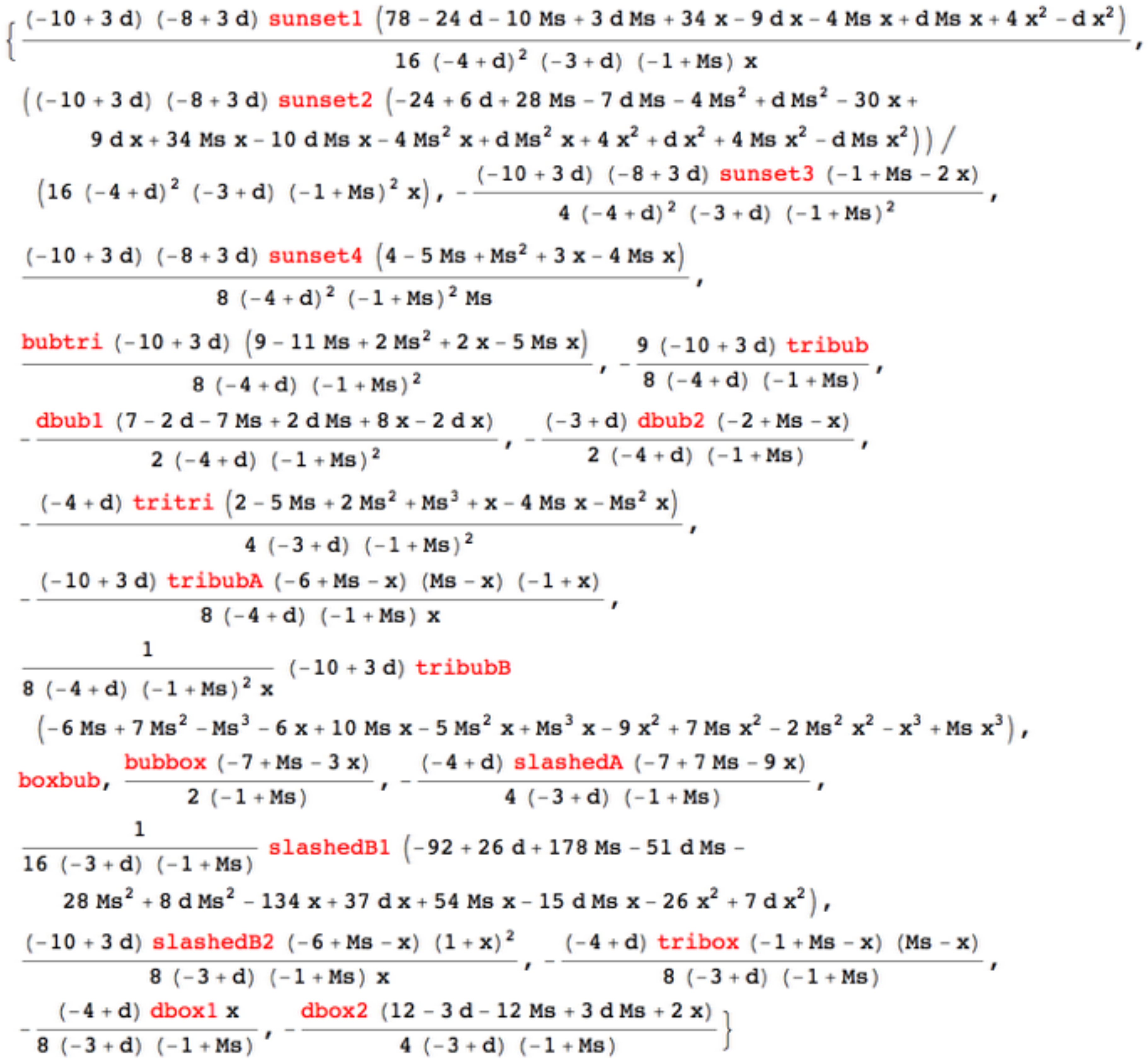
Yang Zhang

LHC Run II and the Precision Frontier
KITP, May. 17, 2016

Integration-by-parts (IBP) reduction

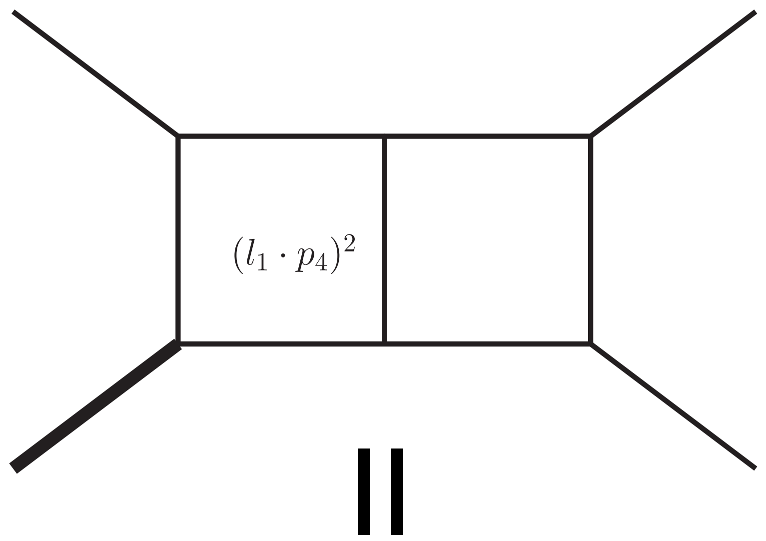


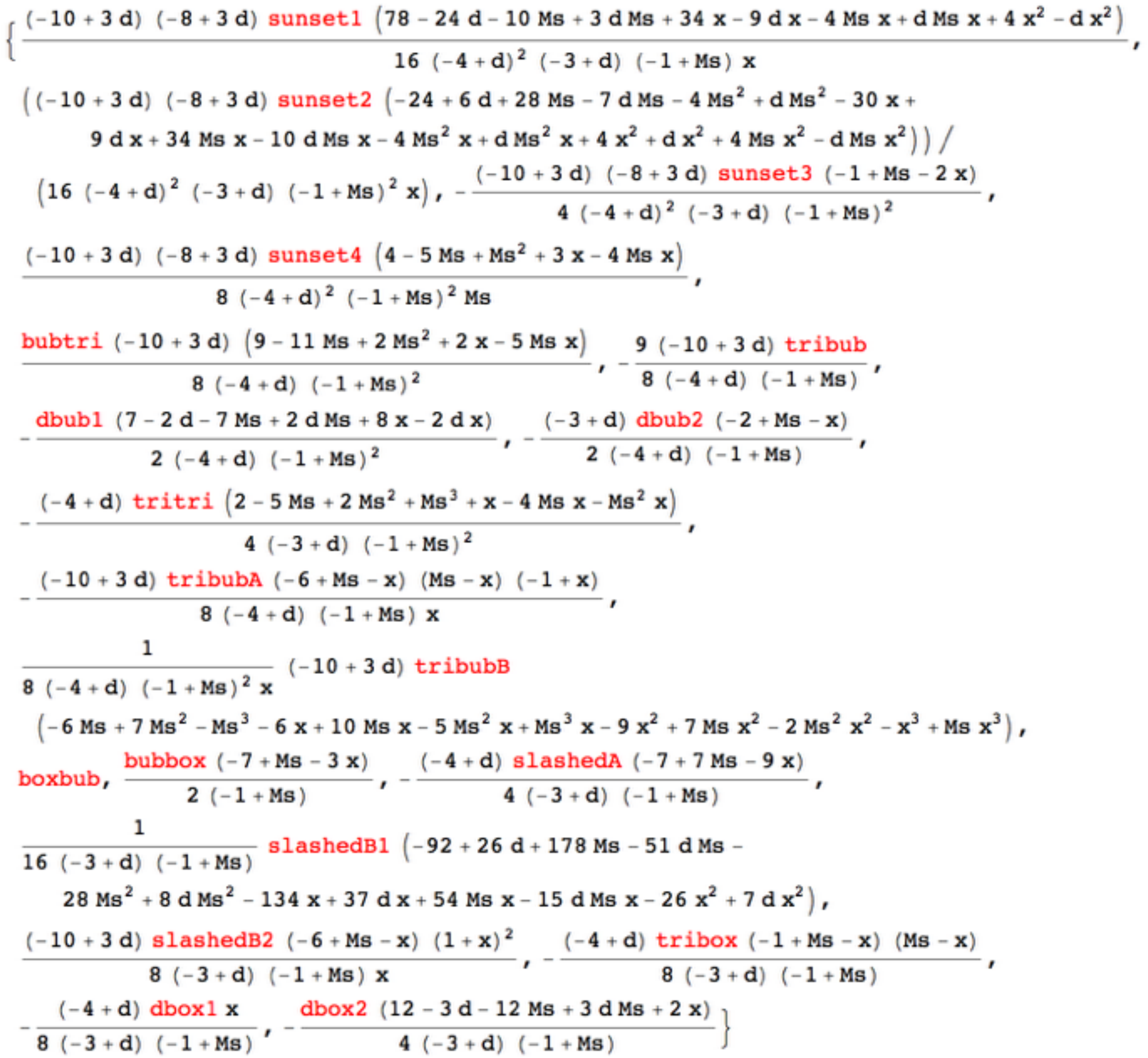
$$=$$



$$\begin{aligned}
\Bigg\{ &\frac{(-10+3d)\,(-8+3d)\,\mathrm{sunset1}\,(78-24d-10\,\mathrm{Ms}+3d\,\mathrm{Ms}+34x-9dx-4\,\mathrm{Ms}\,x+d\,\mathrm{Ms}\,x+4x^2-dx^2)}{16\,(-4+d)^2\,(-3+d)\,(-1+\mathrm{Ms})\,x}, \\
&\Big((-10+3d)\,(-8+3d)\,\mathrm{sunset2}\,\big(-24+6d+28\,\mathrm{Ms}-7d\,\mathrm{Ms}-4\,\mathrm{Ms}^2+d\,\mathrm{Ms}^2-30x+ \\
&\quad 9dx+34\,\mathrm{Ms}\,x-10d\,\mathrm{Ms}\,x-4\,\mathrm{Ms}^2x+d\,\mathrm{Ms}^2x+4x^2+dx^2+4\,\mathrm{Ms}\,x^2-d\,\mathrm{Ms}\,x^2\big)\Big)\Big/ \\
&\big(16\,(-4+d)^2\,(-3+d)\,(-1+\mathrm{Ms})^2\,x\big),\ -\frac{(-10+3d)\,(-8+3d)\,\mathrm{sunset3}\,(-1+\mathrm{Ms}-2x)}{4\,(-4+d)^2\,(-3+d)\,(-1+\mathrm{Ms})^2}, \\
&\frac{(-10+3d)\,(-8+3d)\,\mathrm{sunset4}\,(4-5\,\mathrm{Ms}+\mathrm{Ms}^2+3x-4\,\mathrm{Ms}\,x)}{8\,(-4+d)^2\,(-1+\mathrm{Ms})^2\,\mathrm{Ms}}, \\
&\frac{\mathrm{bubtri}\,(-10+3d)\,(9-11\,\mathrm{Ms}+2\,\mathrm{Ms}^2+2x-5\,\mathrm{Ms}\,x)}{8\,(-4+d)\,(-1+\mathrm{Ms})^2},\ -\frac{9\,(-10+3d)\,\mathrm{tribub}}{8\,(-4+d)\,(-1+\mathrm{Ms})}, \\
&-\frac{\mathrm{dbub1}\,(7-2d-7\,\mathrm{Ms}+2d\,\mathrm{Ms}+8x-2dx)}{2\,(-4+d)\,(-1+\mathrm{Ms})^2},\ -\frac{(-3+d)\,\mathrm{dbub2}\,(-2+\mathrm{Ms}-x)}{2\,(-4+d)\,(-1+\mathrm{Ms})}, \\
&-\frac{(-4+d)\,\mathrm{tritri}\,(2-5\,\mathrm{Ms}+2\,\mathrm{Ms}^2+\mathrm{Ms}^3+x-4\,\mathrm{Ms}\,x-\mathrm{Ms}^2x)}{4\,(-3+d)\,(-1+\mathrm{Ms})^2}, \\
&-\frac{(-10+3d)\,\mathrm{tribubA}\,(-6+\mathrm{Ms}-x)\,(\mathrm{Ms}-x)\,(-1+x)}{8\,(-4+d)\,(-1+\mathrm{Ms})\,x}, \\
&\frac{1}{8\,(-4+d)\,(-1+\mathrm{Ms})^2\,x}\,(-10+3d)\,\mathrm{tribubB} \\
&\big(-6\,\mathrm{Ms}+7\,\mathrm{Ms}^2-\mathrm{Ms}^3-6x+10\,\mathrm{Ms}\,x-5\,\mathrm{Ms}^2x+\mathrm{Ms}^3x-9x^2+7\,\mathrm{Ms}\,x^2-2\,\mathrm{Ms}^2x^2-x^3+\mathrm{Ms}\,x^3\big), \\
&\mathrm{boxbub},\ \frac{\mathrm{bubbox}\,(-7+\mathrm{Ms}-3x)}{2\,(-1+\mathrm{Ms})},\ -\frac{(-4+d)\,\mathrm{slashedA}\,(-7+7\,\mathrm{Ms}-9x)}{4\,(-3+d)\,(-1+\mathrm{Ms})}, \\
&\frac{1}{16\,(-3+d)\,(-1+\mathrm{Ms})}\,\mathrm{slashedB1}\,\big(-92+26d+178\,\mathrm{Ms}-51d\,\mathrm{Ms}- \\
&\quad 28\,\mathrm{Ms}^2+8d\,\mathrm{Ms}^2-134x+37dx+54\,\mathrm{Ms}\,x-15d\,\mathrm{Ms}\,x-26x^2+7dx^2\big), \\
&\frac{(-10+3d)\,\mathrm{slashedB2}\,(-6+\mathrm{Ms}-x)\,(1+x)^2}{8\,(-3+d)\,(-1+\mathrm{Ms})\,x},\ -\frac{(-4+d)\,\mathrm{tribox}\,(-1+\mathrm{Ms}-x)\,(\mathrm{Ms}-x)}{8\,(-3+d)\,(-1+\mathrm{Ms})}, \\
&-\frac{(-4+d)\,\mathrm{dbox1}\,x}{8\,(-3+d)\,(-1+\mathrm{Ms})},\ -\frac{\mathrm{dbox2}\,(12-3d-12\,\mathrm{Ms}+3d\,\mathrm{Ms}+2x)}{4\,(-3+d)\,(-1+\mathrm{Ms})}\Bigg\}
\end{aligned}$$

Integration-by-parts (IBP) reduction





$$\Bigg\{\frac{(-10+3d)\ (-8+3d)\ \text{sunset1}\ \left(78-24d-10\,\text{Ms}+3d\,\text{Ms}+34x-9dx-4\,\text{Ms}\,x+d\,\text{Ms}\,x+4x^2-dx^2\right)}{16\ (-4+d)^2\ (-3+d)\ (-1+\text{Ms})\ x},$$

$$\Big((-10+3d)\ (-8+3d)\ \text{sunset2}\ \big(-24+6d+28\,\text{Ms}-7d\,\text{Ms}-4\,\text{Ms}^2+d\,\text{Ms}^2-30x+$$

$$9dx+34\,\text{Ms}\,x-10d\,\text{Ms}\,x-4\,\text{Ms}^2x+d\,\text{Ms}^2x+4x^2+dx^2+4\,\text{Ms}\,x^2-d\,\text{Ms}\,x^2\big)\Big)\Big/$$

$$\left(16\ (-4+d)^2\ (-3+d)\ (-1+\text{Ms})^2\ x\right),\ -\frac{(-10+3d)\ (-8+3d)\ \text{sunset3}\ (-1+\text{Ms}-2x)}{4\ (-4+d)^2\ (-3+d)\ (-1+\text{Ms})^2},$$

$$\frac{(-10+3d)\ (-8+3d)\ \text{sunset4}\ \left(4-5\,\text{Ms}+\text{Ms}^2+3x-4\,\text{Ms}\,x\right)}{8\ (-4+d)^2\ (-1+\text{Ms})^2\ \text{Ms}},$$

$$\frac{\text{bubtri}\ (-10+3d)\ \left(9-11\,\text{Ms}+2\,\text{Ms}^2+2x-5\,\text{Ms}\,x\right)}{8\ (-4+d)\ (-1+\text{Ms})^2},\ -\frac{9\ (-10+3d)\ \text{tribub}}{8\ (-4+d)\ (-1+\text{Ms})},$$

$$-\frac{\text{dbub1}\ (7-2d-7\,\text{Ms}+2d\,\text{Ms}+8x-2dx)}{2\ (-4+d)\ (-1+\text{Ms})^2},\ -\frac{(-3+d)\ \text{dbub2}\ (-2+\text{Ms}-x)}{2\ (-4+d)\ (-1+\text{Ms})},$$

$$-\frac{(-4+d)\ \text{tritri}\ \left(2-5\,\text{Ms}+2\,\text{Ms}^2+\text{Ms}^3+x-4\,\text{Ms}\,x-\text{Ms}^2x\right)}{4\ (-3+d)\ (-1+\text{Ms})^2},$$

$$-\frac{(-10+3d)\ \text{tribubA}\ (-6+\text{Ms}-x)\ (\text{Ms}-x)\ (-1+x)}{8\ (-4+d)\ (-1+\text{Ms})\ x},$$

$$\frac{1}{8\ (-4+d)\ (-1+\text{Ms})^2\ x}\ (-10+3d)\ \text{tribubB}$$

$$\left(-6\,\text{Ms}+7\,\text{Ms}^2-\text{Ms}^3-6x+10\,\text{Ms}\,x-5\,\text{Ms}^2x+\text{Ms}^3x-9x^2+7\,\text{Ms}\,x^2-2\,\text{Ms}^2x^2-x^3+\text{Ms}\,x^3\right),$$

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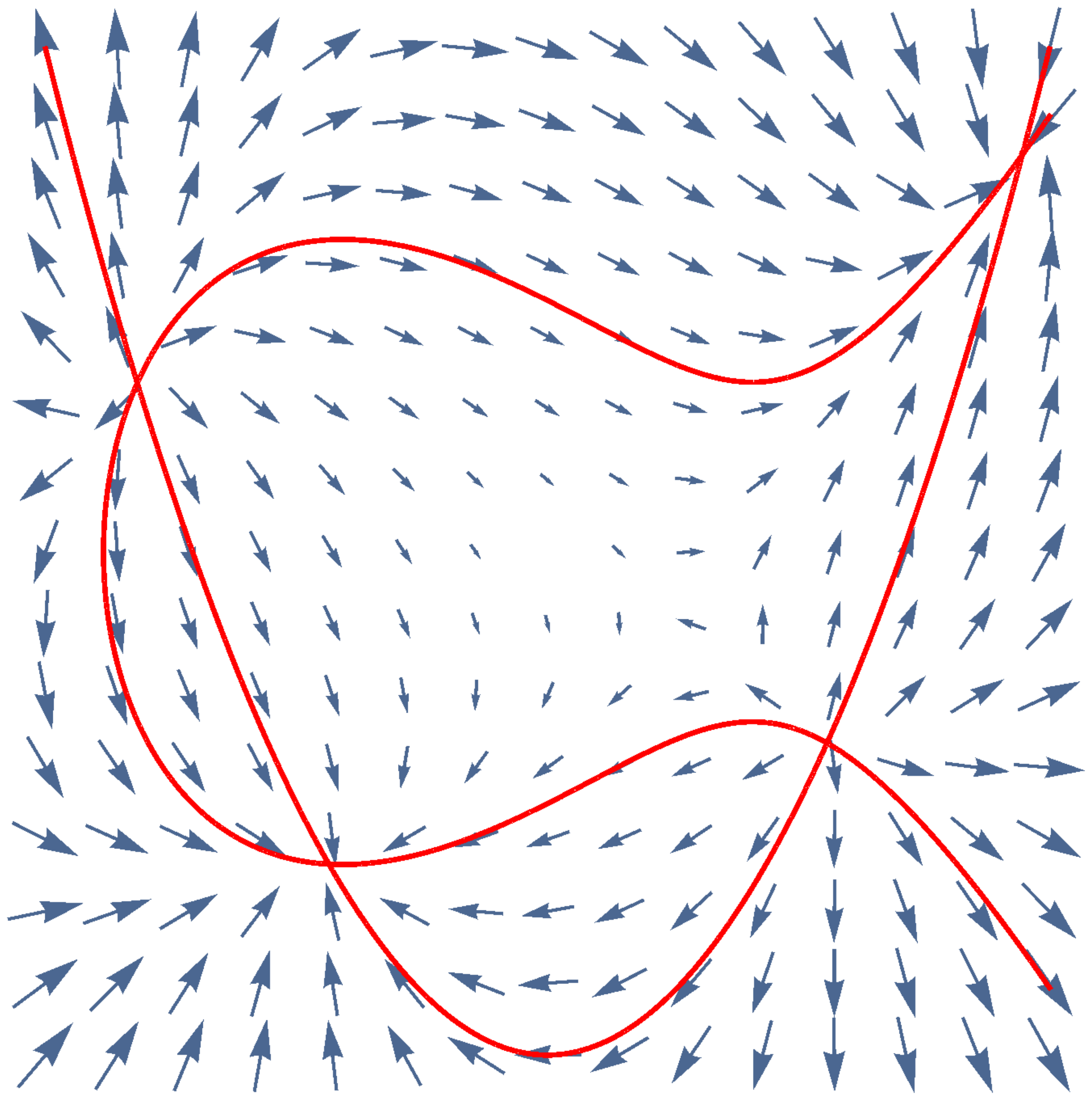
$$28\,\text{Ms}^2+8d\,\text{Ms}^2-134x+37dx+54\,\text{Ms}\,x-15d\,\text{Ms}\,x-26x^2+7dx^2\big),$$

$$\frac{(-10+3d)\ \text{slashedB2}\ (-6+\text{Ms}-x)\ (1+x)^2}{8\ (-3+d)\ (-1+\text{Ms})\ x},\ -\frac{(-4+d)\ \text{tribox}\ (-1+\text{Ms}-x)\ (\text{Ms}-x)}{8\ (-3+d)\ (-1+\text{Ms})},$$

$$-\frac{(-4+d)\ \text{dbox1}\ x}{8\ (-3+d)\ (-1+\text{Ms})},\ -\frac{\text{dbox2}\ (12-3d-12\,\text{Ms}+3d\,\text{Ms}+2x)}{4\ (-3+d)\ (-1+\text{Ms})}\Bigg\}$$



Tangent Algebra (blue arrows)
of
Affine Varieties (red curves)

Multi-loop amplitudes

- Unitarity

Unitarity: Bern, Dixon, Kosower 1994 …
General unitarity: Britto, Cachazo, Feng 2004 …
2-loop maximal unitarity: Kosower, Larsen 2011, Caron-Huot, Larsen 2012, Johansson, Kosower, Larsen 2011, 2013

- Integrand reduction

OPP method: Ossola, Papadopoulos, Pittau 2006, …
Integrand reduction via Groebner basis: YZ 2012, Mastrolia, Mirabella, Ossola, Peraro 2012, …

- Integral reduction and IBP identities

IBP: Chetyrkin, Tkachov 1981, Laporta 2001, …
IBP codes: FIRE (Smirnov), Reduze (von Manteuffel, Studerus), LiteRed (Lee) …
Syzygy approach: Gluza, Kjada, Kosower 2010

- Integral evaluation

Numeric: SecDec (Borowka, Heinrich, Jahn, Jones, Kerner, Schlenk, Zirke) Fiesta (Smirnov)
Differential equations: Kotikov 1991, Bern, Dixon Kosower 1994, Henn 2013, …
Symbol: Goncharov, Spradlin, Vergu, Volovich 2010 …

Multi-loop amplitudes

- Unitarity

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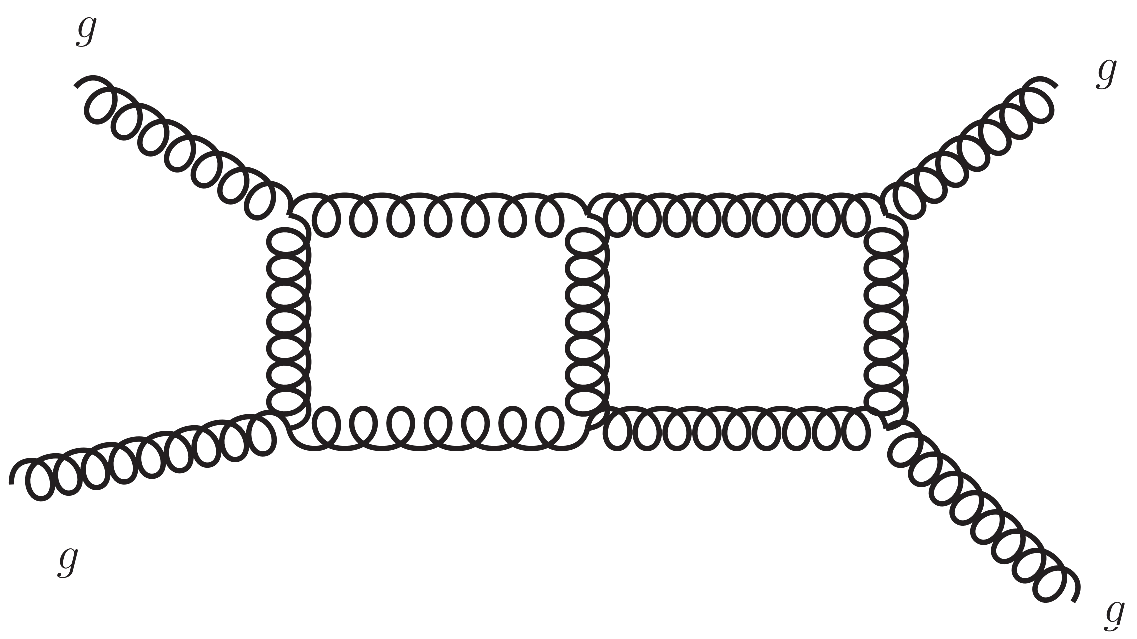
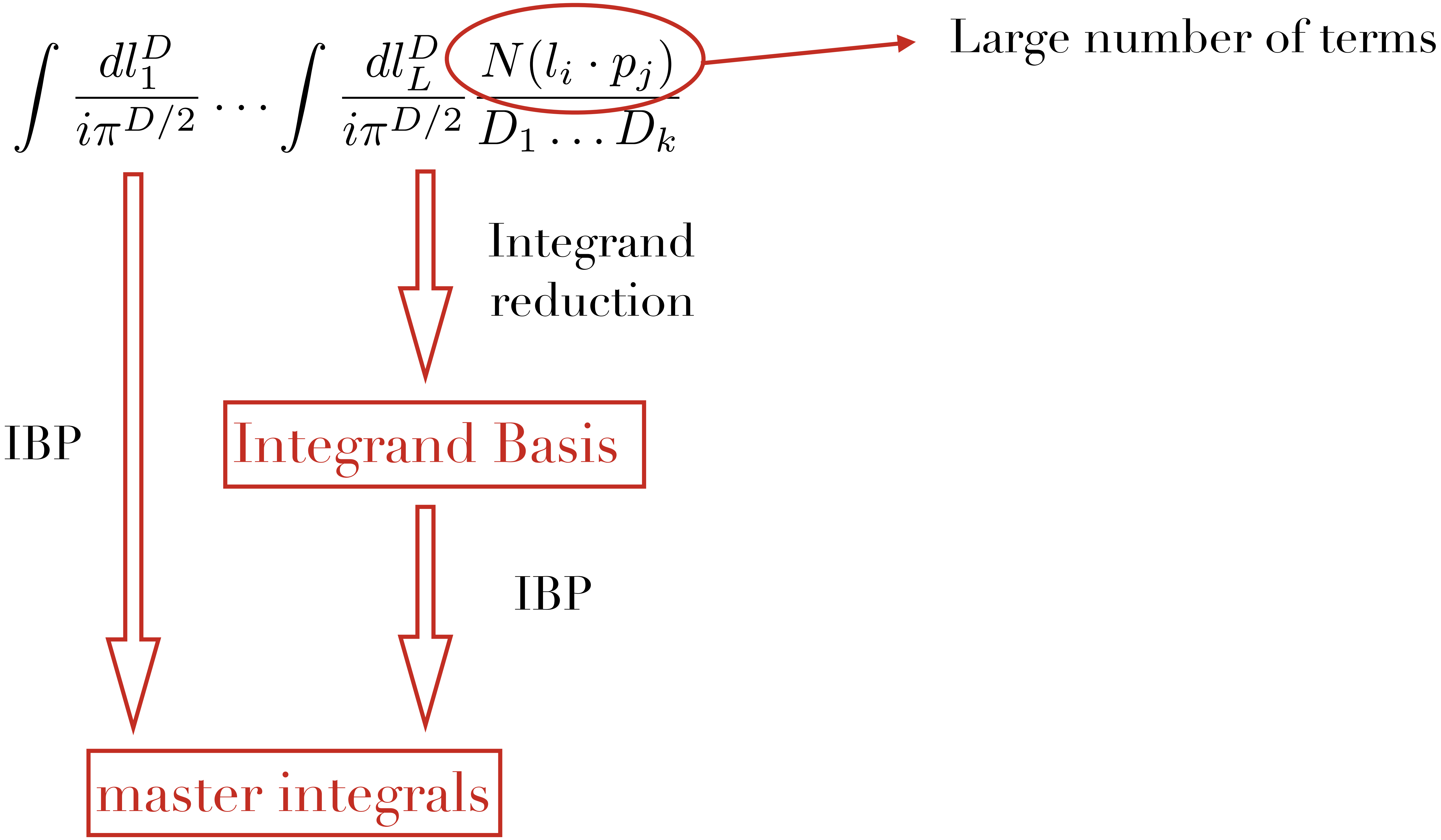
Multi-loop Integration-by-parts reduction



massless/massive, supersymmetric/non-supersymmetric

crucial for the next-to-next-to-leading (NNLO) order of LHC processes



$$\int \frac{dl_1^D}{i\pi^{D/2}} \cdots \int \frac{dl_L^D}{i\pi^{D/2}} \frac{N(l_i \cdot p_j)}{D_1 \ldots D_k}$$

Large number of terms

Integrand reduction

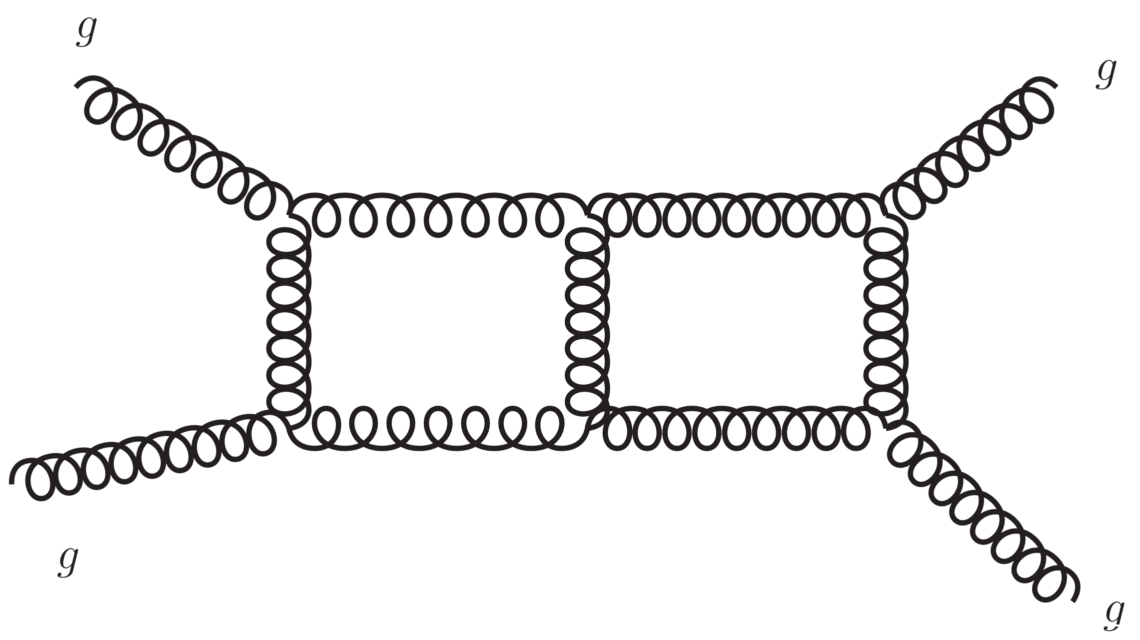
IBP

Integrand Basis

IBP

master integrals

Multi-loop Integration-by-parts reduction



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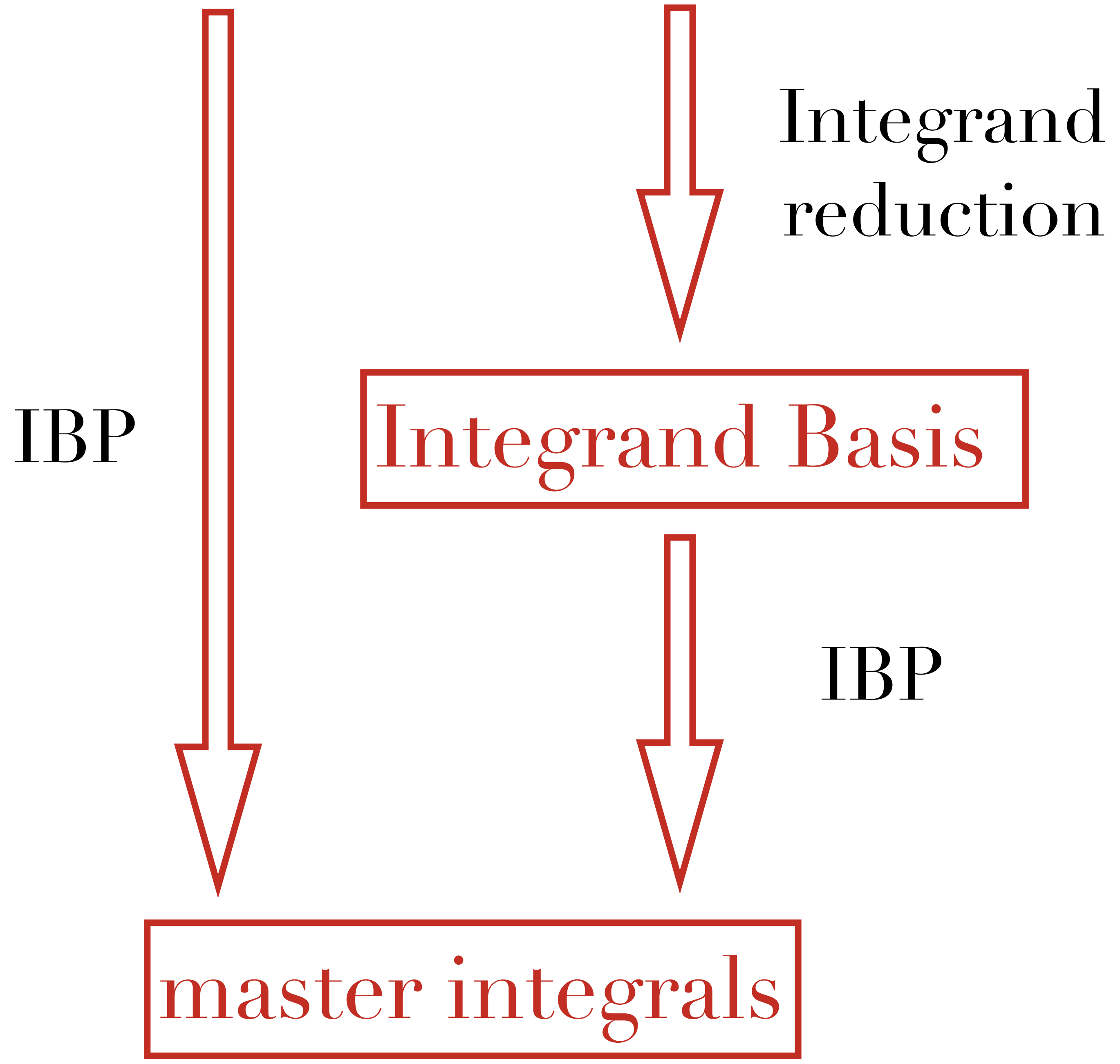
$$\int \frac{dl_1^D}{i\pi^{D/2}} \cdots \int \frac{dl_L^D}{i\pi^{D/2}} \frac{N(l_i \cdot p_j)}{D_1 \ldots D_k}$$

Large number of terms

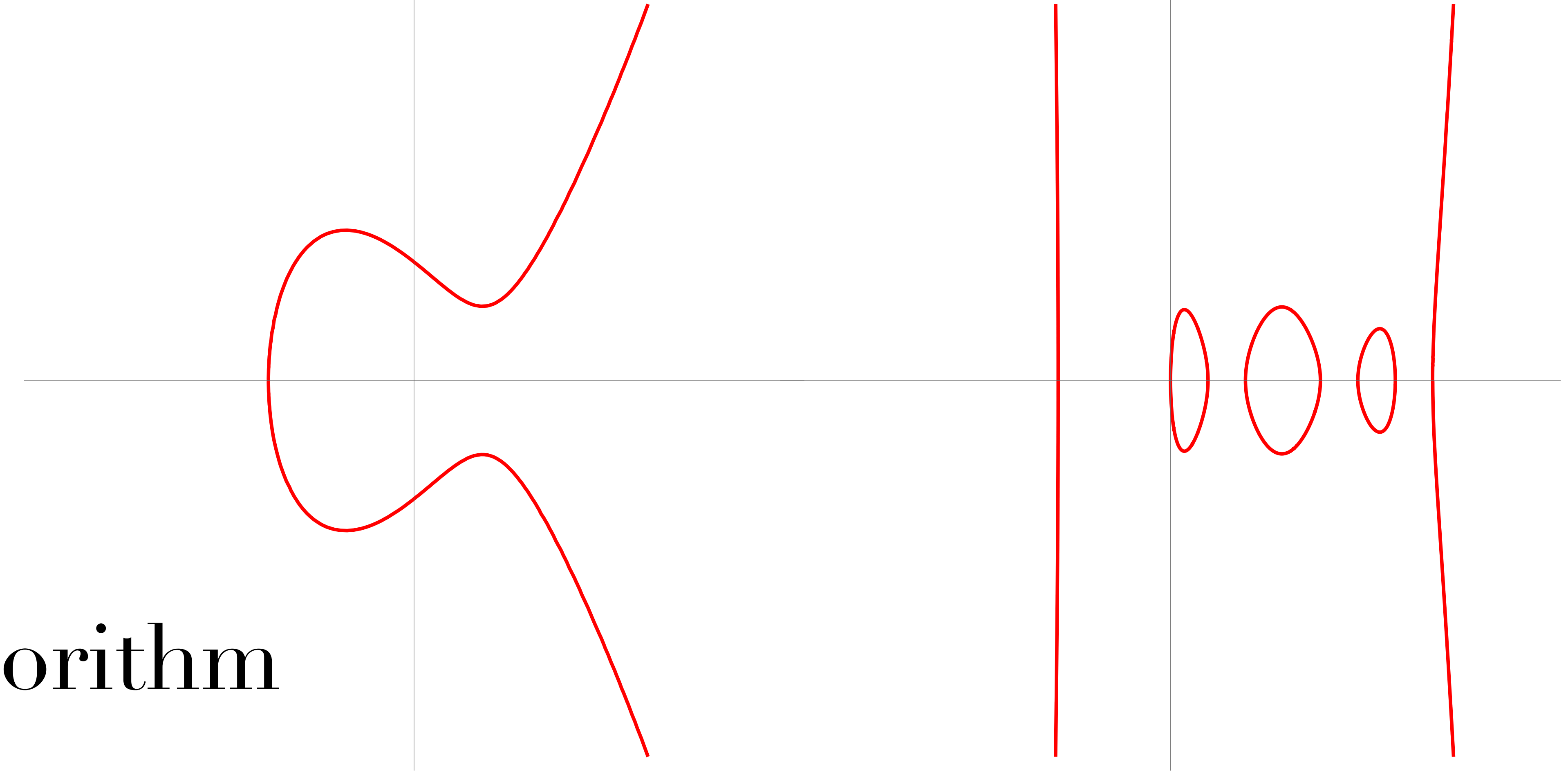


Integral reduction: Integration-by-parts (IBP)

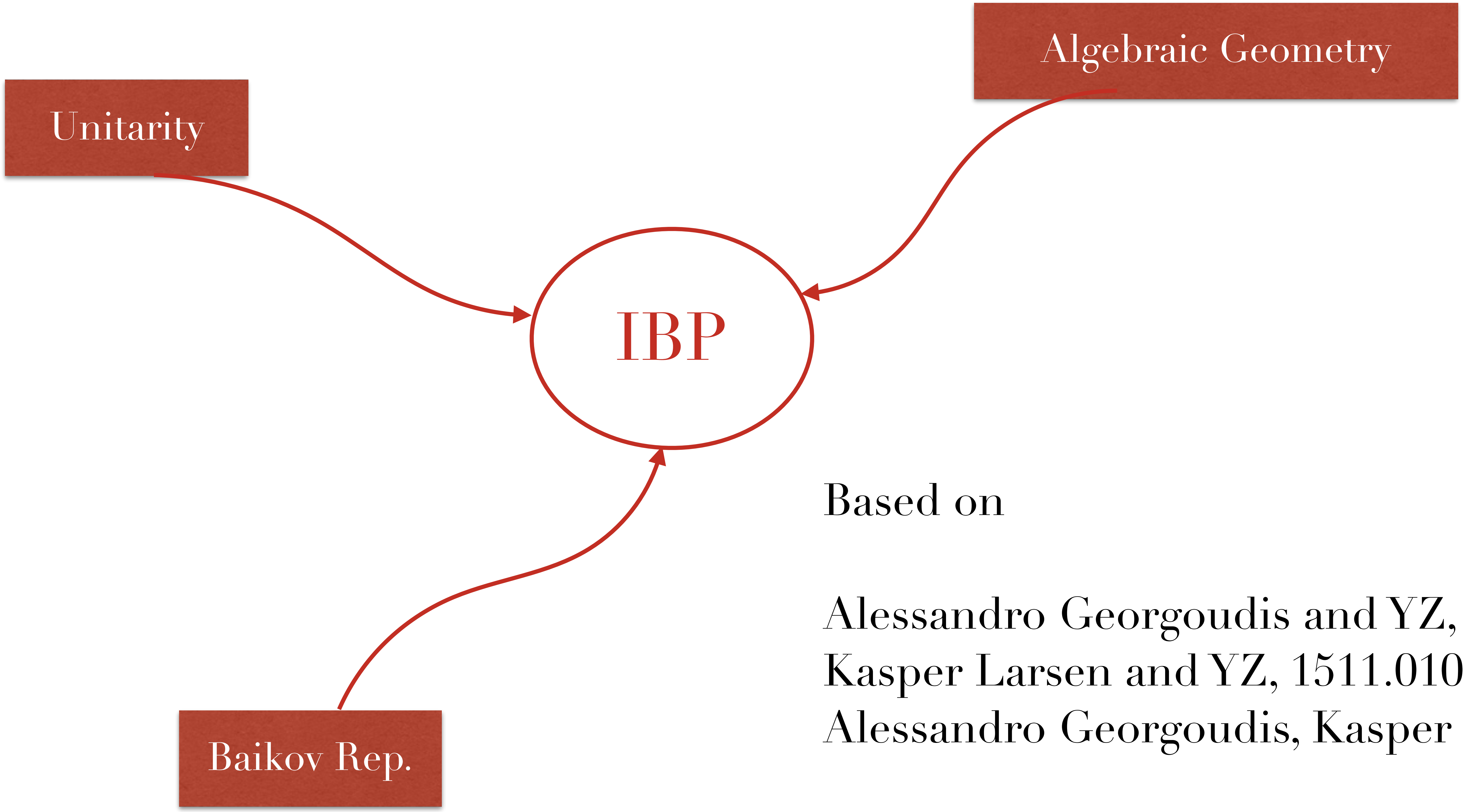
$$\int \frac{dl_1^D}{i\pi^{D/2}} \cdots \int \frac{dl_L^D}{i\pi^{D/2}} \frac{\partial}{\partial l_i^\mu} \left(\frac{v_i^\mu}{D_1 \ldots D_k} \right) = 0$$

difficult to find and sort IBPs for massive/multi-leg high-loop diagrams

Outline



- IBP: Unitarity + Tangent algebra (syzygy) algorithm



Based on

Alessandro Georgoudis and YZ, 1507.06310
Kasper Larsen and YZ, 1511.01071.
Alessandro Georgoudis, Kasper Larsen and YZ, to appear

Set up

Unitarity

Algebraic geometry

inspired by
Gluza, Kjada, Kosower 2010

Dimensional Regularization $D = 4 - 2\epsilon$

Important for studying IR/UV divergence

K. Larsen and YZ, 1511.01071

See also: H.Ita 1510.05626

2-loop

$$l_1 = l_1^{[4]} + l_1^{\perp}, \quad l_2 = l_2^{[4]} + l_2^{\perp}$$

$$\mu_{11} = -(l_1^{\perp})^2, \quad \mu_{22} = -(l_2^{\perp})^2, \quad \mu_{12} = -l_1^{\perp} \cdot l_2^{\perp}$$

Dimensional decomposition

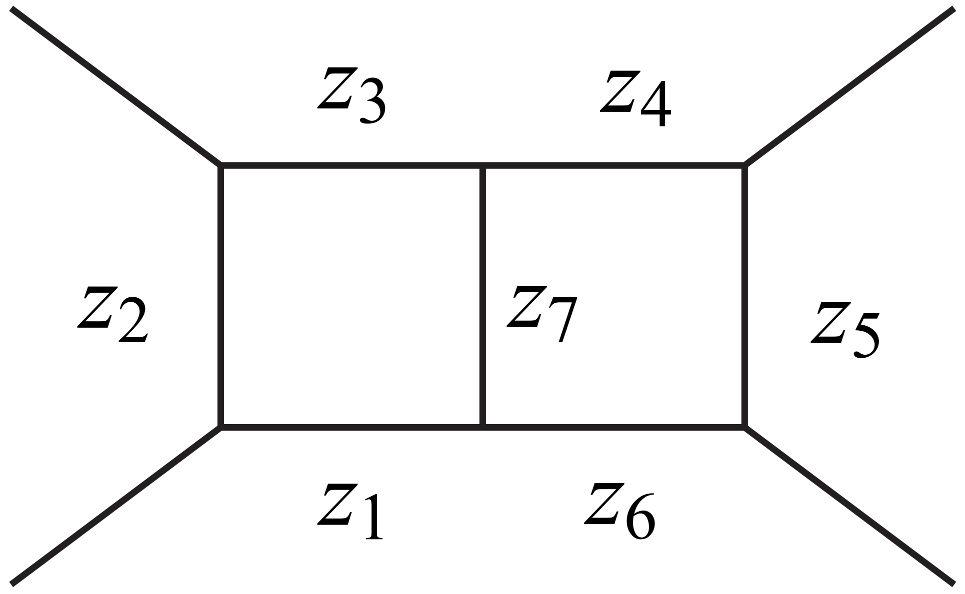
$$\int \frac{d^D l_1}{i\pi^{D/2}} \int \frac{d^D l_2}{i\pi^{D/2}} \frac{N}{D_1 \ldots D_k} \propto \int_0^{\infty} d\mu_{11} \int_0^{\infty} d\mu_{22} \int_{-\sqrt{\mu_{11}\mu_{22}}}^{\sqrt{\mu_{11}\mu_{22}}} d\mu_{12} \left(\mu_{11}\mu_{22} - \mu_{12}^2\right)^{\frac{D-7}{2}} \int d^4 l_1 d^4 l_2 \frac{N}{D_1 \ldots D_k}$$

L-loop

$$\int \frac{d^D l_1}{i\pi^{D/2}} \ldots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{N}{D_1 \ldots D_k} \propto \int \prod_{1 \leq i \leq j \leq L} d\mu_{ij} \det(\mu_{ij})^{\frac{D-5-L}{2}} \int d^4 l_1 \ldots d^4 l_L \frac{N}{D_1 \ldots D_k}$$

Baikov parametrization

Baikov 1996



$$
\begin{aligned}
z_i &\equiv D_i, \quad i = 1, \ldots 7 \\
z_8 &\equiv (l_1 + p_4)^2/2, \quad z_9 \equiv (l_2 + p_1)^2/2
\end{aligned}
$$

4-point double box: integrate out the spurious directions,
11-2=9 variables

$$
\Big\{x1 \to \frac{z1 - z2}{2},\ x2 \to \frac{1}{2}(s + z2 - z3),\ y3 \to \frac{1}{2}(-z5 + z6),\ y1 \to \frac{1}{2}(-z6 + 2\,z9),\ y2 \to \frac{1}{2}(-s + z4 - 2\,z9),\ x3 \to \frac{1}{2}(-z1 + 2\,z8),
$$

$$
\mu 11 \to \frac{s^2 t^2 - 2\,s\,t^2\,z1 + t^2\,z1^2 - 2\,s^2\,t\,z2 + 2\,s\,t\,z1\,z2 + s^2\,z2^2 - 2\,s\,t^2\,z3 - 4\,s\,t\,z1\,z3 - 2\,t^2\,z1\,z3 + 2\,s\,t\,z2\,z3 + t^2\,z3^2 - 4\,s^2\,t\,z8 + 4\,s\,t\,z1\,z8 - 4\,s^2\,z2\,z8 - 8\,s\,t\,z2\,z8 + 4\,s\,t\,z3\,z8 + 4\,s^2\,z8^2}{4\,s\,t\,(s + t)},
$$

$$
\begin{aligned}
\mu 12 \to -\frac{1}{4\,s\,t\,(s + t)}\Big(& s^2\,t^2 - s\,t^2\,z1 - s^2\,t\,z2 - s\,t^2\,z3 - s\,t^2\,z4 - 2\,s\,t\,z1\,z4 - t^2\,z1\,z4 + s\,t\,z2\,z4 + t^2\,z3\,z4 - s^2\,t\,z5 + s\,t\,z1\,z5 - s^2\,z2\,z5 - 2\,s\,t\,z2\,z5 + s\,t\,z3\,z5 - s\,t^2\,z6 + t^2\,z1\,z6 + s\,t\,z2\,z6 - \\
& 2\,s\,t\,z3\,z6 - t^2\,z3\,z6 + 2\,s^2\,t\,z7 + 2\,s\,t^2\,z7 - 2\,s^2\,t\,z8 + 2\,s\,t\,z4\,z8 + 2\,s^2\,z5\,z8 + 2\,s\,t\,z6\,z8 - 2\,s^2\,t\,z9 + 2\,s\,t\,z1\,z9 + 2\,s^2\,z2\,z9 + 2\,s\,t\,z3\,z9 - 4\,s^2\,z8\,z9 - 8\,s\,t\,z8\,z9\Big),
\end{aligned}
$$

$$
\mu 22 \to \frac{s^2 t^2 - 2\,s\,t^2\,z4 + t^2\,z4^2 - 2\,s^2\,t\,z5 + 2\,s\,t\,z4\,z5 + s^2\,z5^2 - 2\,s\,t^2\,z6 - 4\,s\,t\,z4\,z6 - 2\,t^2\,z4\,z6 + 2\,s\,t\,z5\,z6 + t^2\,z6^2 - 4\,s^2\,t\,z9 + 4\,s\,t\,z4\,z9 - 4\,s^2\,z5\,z9 - 8\,s\,t\,z5\,z9 + 4\,s\,t\,z6\,z9 + 4\,s^2\,z9^2}{4\,s\,t\,(s + t)}\Big\}
$$

$$
\int \frac{d^D l_1}{\pi^{D/2}} \frac{d^D l_2}{\pi^{D/2}} \frac{N}{D_1 D_2 \ldots D_7} \propto \int \left(\prod_{i=1}^{9} dz_i \right) F(z)^{\frac{D-6}{2}} \frac{N(z)}{\underline{z_1 \ldots z_7}}
$$

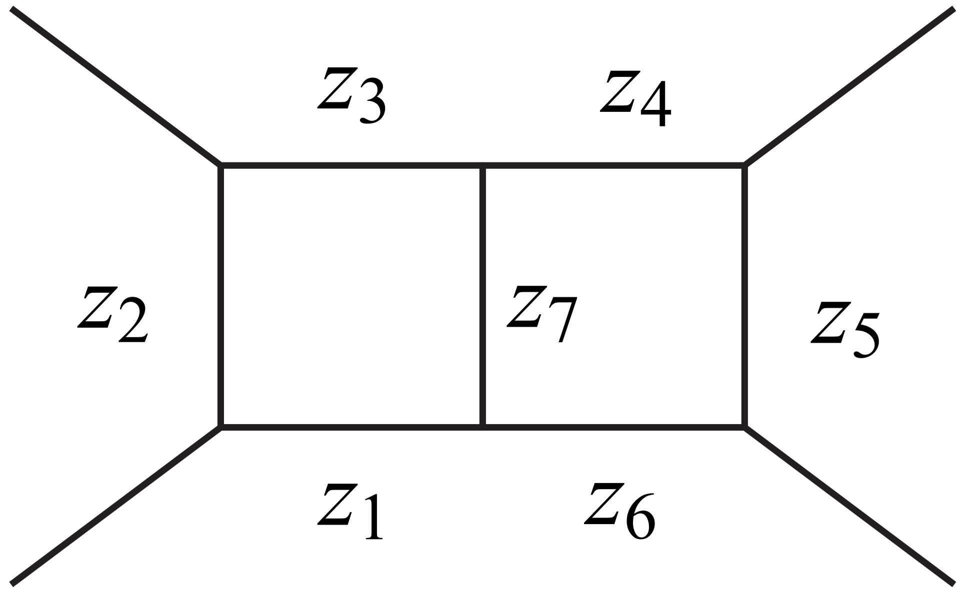
Linear

polynomial

- Easy to control the unitarity cut
- Dimension dependence is explicit
- Works for higher-loops

Baikov parametrization

Baikov 1996



$$z_i \equiv D_i, \quad i = 1, \ldots 7$$
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4-point double box: integrate out the spurious directions,
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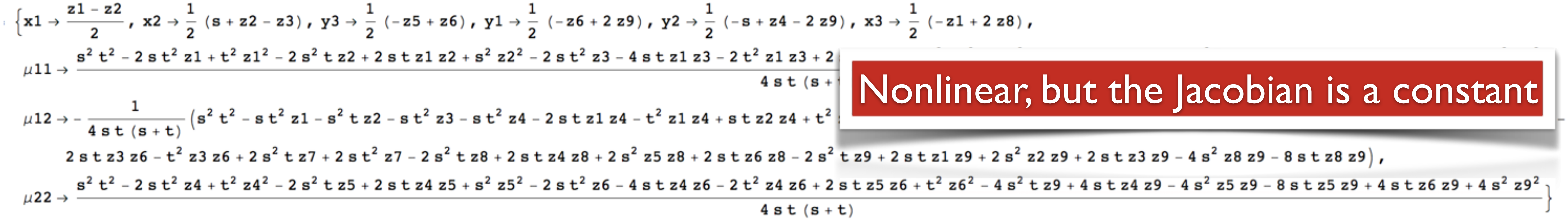

```
{x1 → (z1 - z2)/2, x2 → 1/2 (s + z2 - z3), y3 → 1/2 (-z5 + z6), y1 → 1/2 (-z6 + 2 z9), y2 → 1/2 (-s + z4 - 2 z9), x3 → 1/2 (-z1 + 2 z8),
μ11 → (s² t² - 2 s t² z1 + t² z1² - 2 s² t z2 + 2 s t z1 z2 + s² z2² - 2 s t² z3 - 4 s t z1 z3 - 2 t² z1 z3 + 2
      / 4 s t (s +
μ12 → - 1/(4 s t (s + t)) (s² t² - s t² z1 - s² t z2 - s t² z3 - s t² z4 - 2 s t z1 z4 - t² z1 z4 + s t z2 z4 + t²
    2 s t z3 z6 - t² z3 z6 + 2 s² t z7 + 2 s t² z7 - 2 s² t z8 + 2 s t z4 z8 + 2 s² z5 z8 + 2 s t z6 z8 - 2 s² t z9 + 2 s t z1 z9 + 2 s² z2 z9 + 2 s t z3 z9 - 4 s² z8 z9 - 8 s t z8 z9),
μ22 → (s² t² - 2 s t² z4 + t² z4² - 2 s² t z5 + 2 s t z4 z5 + s² z5² - 2 s t² z6 - 4 s t z4 z6 - 2 t² z4 z6 + 2 s t z5 z6 + t² z6² - 4 s² t z9 + 4 s t z4 z9 - 4 s² z5 z9 - 8 s t z5 z9 + 4 s t z6 z9 + 4 s² z9²) / (4 s t (s + t))}
```

Nonlinear, but the Jacobian is a constant

$$\int \frac{d^D l_1}{\pi^{D/2}} \frac{d^D l_2}{\pi^{D/2}} \frac{N}{D_1 D_2 \ldots D_7} \propto \int \left(\prod_{i=1}^{9} dz_i \right) F(z)^{\frac{D-6}{2}} \frac{N(z)}{\underline{z_1 \ldots z_7}}$$

Linear

polynomial

- Easy to control the unitarity cut
- Dimension dependence is explicit
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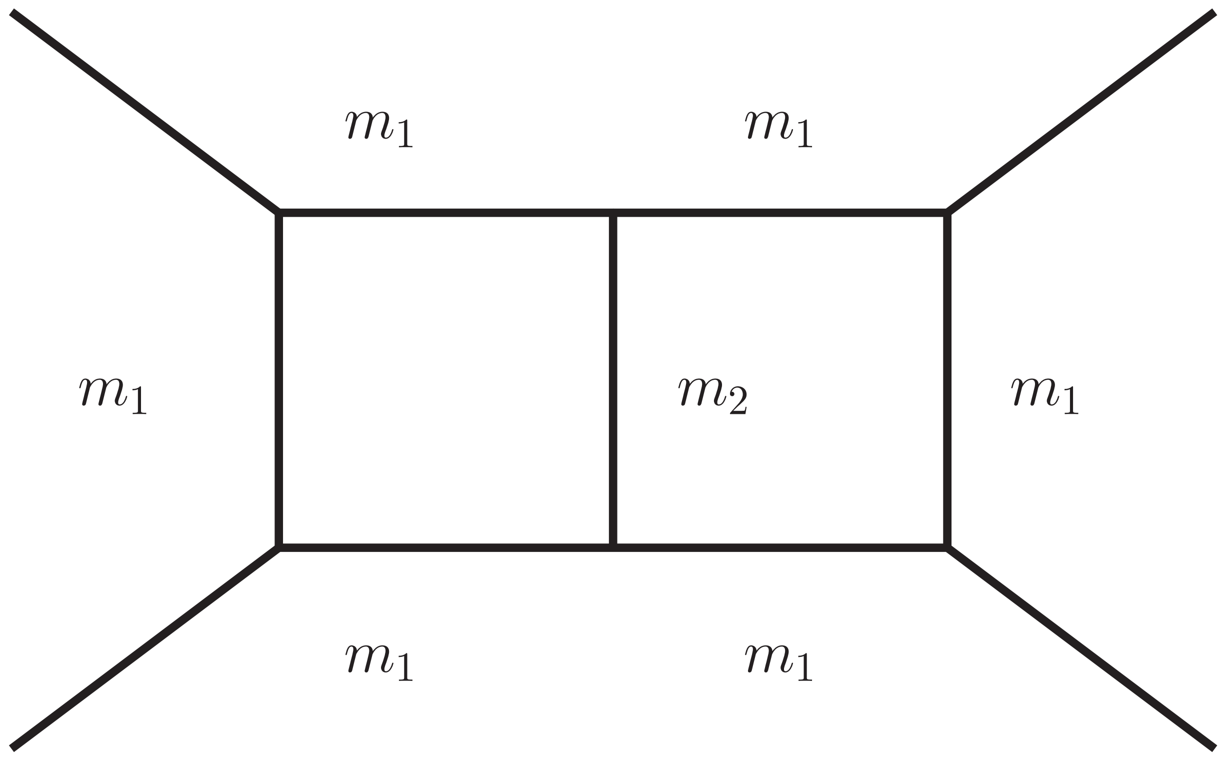
Maximal cut (toy example)



Unitarity cut $\displaystyle \left.\frac{1}{D_1 \ldots D_k}\right|_{\text{cut}} \propto \delta(D_1)\ldots\delta(D_k)$

$$I^D_{\text{dbox}}|_{\text{cut}} \propto \int\int dz_8 dz_9 \boxed{F(z_8,z_9)}^{\frac{D-6}{2}} N(z_8,z_9)$$

measure
on the cut

		$F(x,y) = 0$
Case I	$m_1 = m_2 = 0$	reducible curve: two lines plus one conic
Case II	$m_1 \neq 0, m_2 = 0$	deformed elliptic curve
Case III	$m_1 \neq 0, m_2 \neq 0$	elliptic curve

dimension
shifted

Integral reduction
$$0 = \int d\left[(-\alpha_9 dz_8 + \alpha_8 dz_9) F^{\frac{D-6}{2}}\right]$$
$$= \int \left[\left(\frac{\partial\alpha_8}{\partial z_8} + \frac{\partial\alpha_9}{\partial z_9}\right) F^{\frac{D-6}{2}} + (\alpha_8 F_{z_8} + \alpha_9 F_{z_9})\left(\frac{D-6}{2}\right) F^{\boxed{\frac{D-8}{2}}}\right] dx \wedge dy$$

Require

$$\boxed{\alpha_8 F_{z_8} + \alpha_9 F_{z_9} + \alpha F = 0}$$

Syzygy (συζυγία) equation Gluza, Kjada, Kosower 2010

Tangent algebra

Larsen, YZ 1511.01071

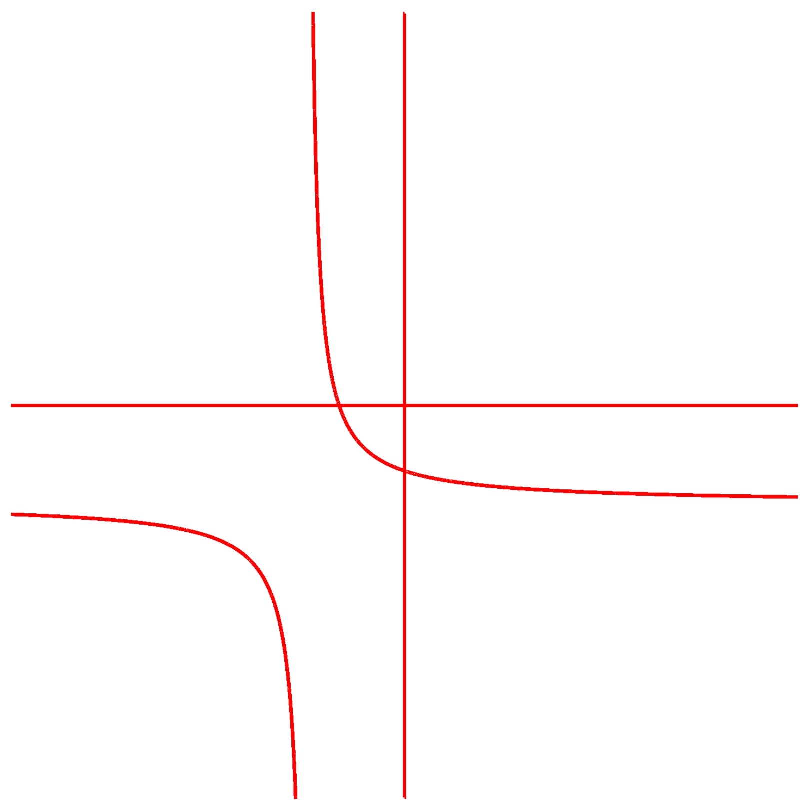
$F=0$ defines an affine variety V. The solution set of $\alpha_8 F_{z_8}+\alpha_9 F_{z_9}+\alpha F=0$ is the tangent algebra of V, i.e., polynomial vector fields such that

$$\left(\alpha_8 \frac{\partial}{\partial z_8}+\alpha_9 \frac{\partial}{\partial z_9}\right) F \in \langle F\rangle .$$

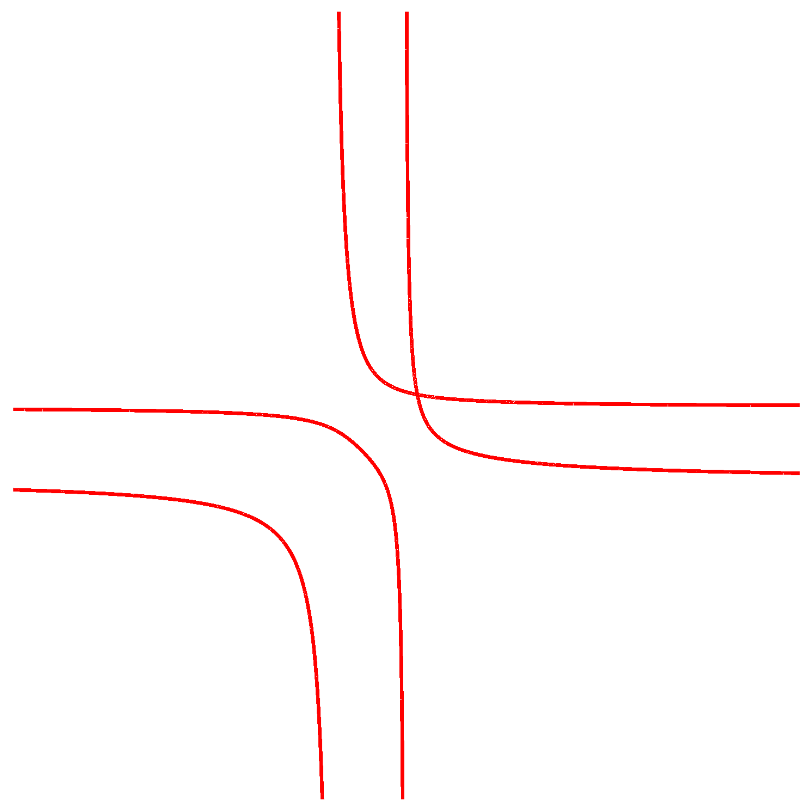
- (infinite-dimensional) Lie algebra
- Module over polynomial ring

$\alpha_8 F_{z_8}+\alpha_9 F_{z_9}+\alpha F=0$ defines syzygy for generator of the Jacobian ideal $J=\langle F_{z_8}, F_{z_9}, F\rangle$.

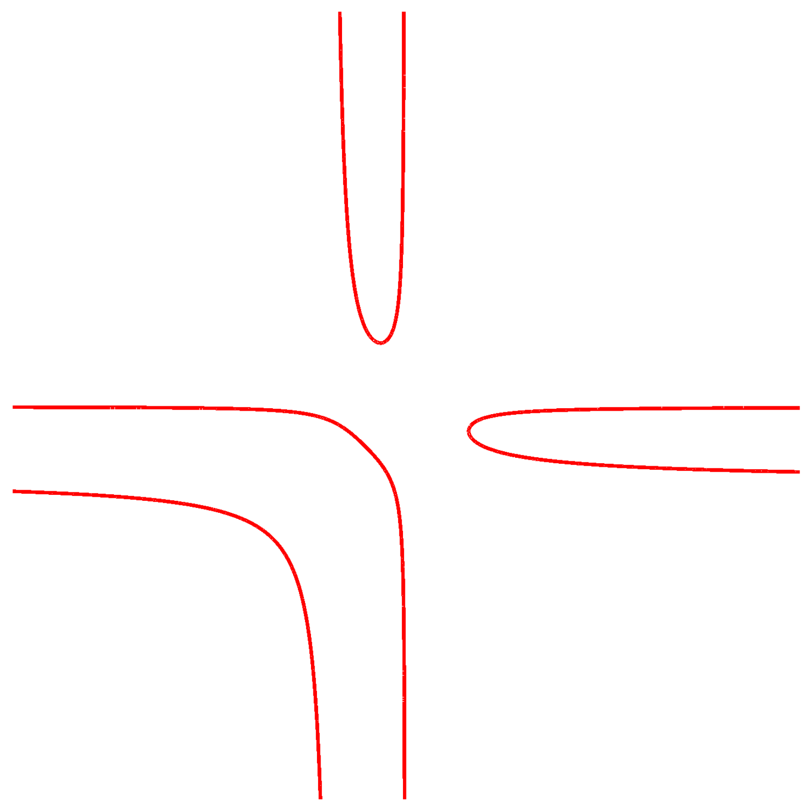
characterizes singular points of V



Case I, 3 singular points



Case II, 1 singular point



Case III, no singular point

Tangent algebra

Larsen, YZ 1511.01071

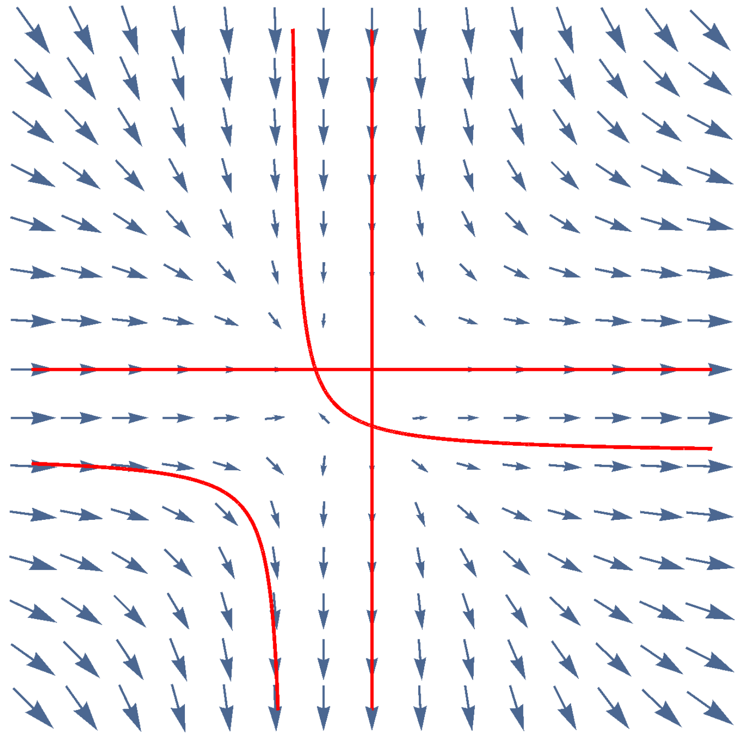
$F=0$ defines an affine variety V. The solution set of $\alpha_8 F_{z_8}+\alpha_9 F_{z_9}+\alpha F=0$ is the tangent algebra of V, i.e., polynomial vector fields such that

$$\left(\alpha_8 \frac{\partial}{\partial z_8}+\alpha_9 \frac{\partial}{\partial z_9}\right) F \in \langle F\rangle .$$

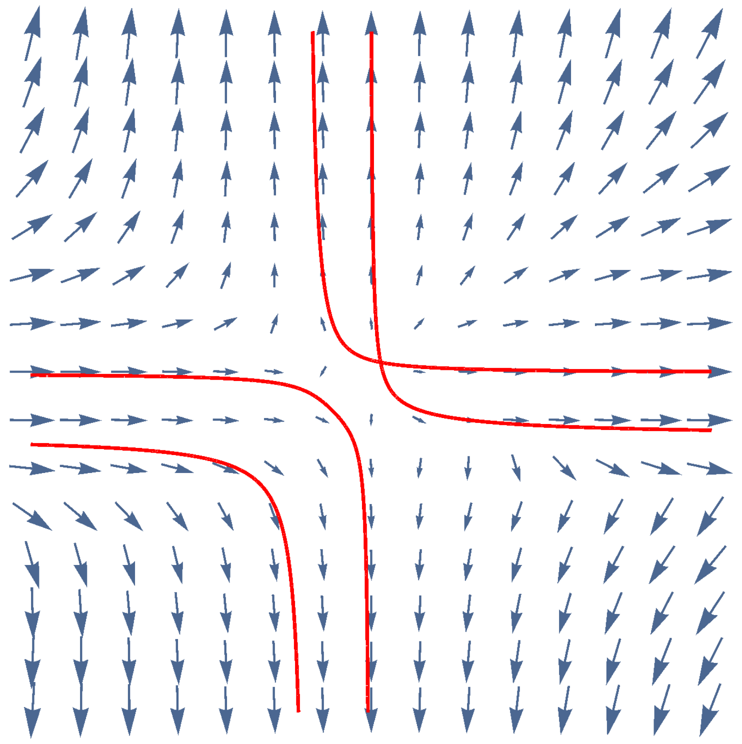
- (infinite-dimensional) Lie algebra
- Module over polynomial ring

$\alpha_8 F_{z_8}+\alpha_9 F_{z_9}+\alpha F=0$ defines syzygy for generator of the Jacobian ideal $J=\langle F_{z_8}, F_{z_9}, F\rangle$.

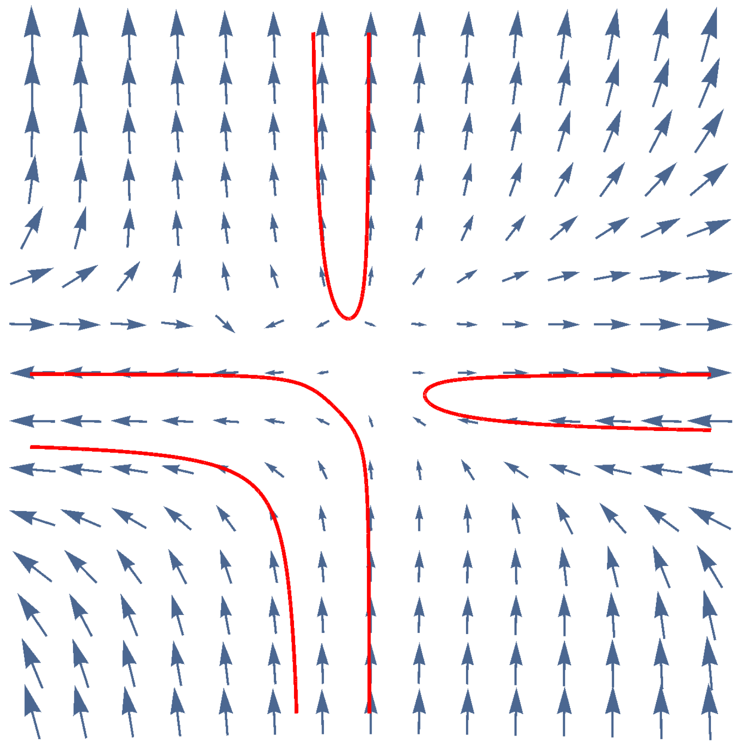
characterizes singular points of V



Case I, 3 singular points



Case II, 1 singular point



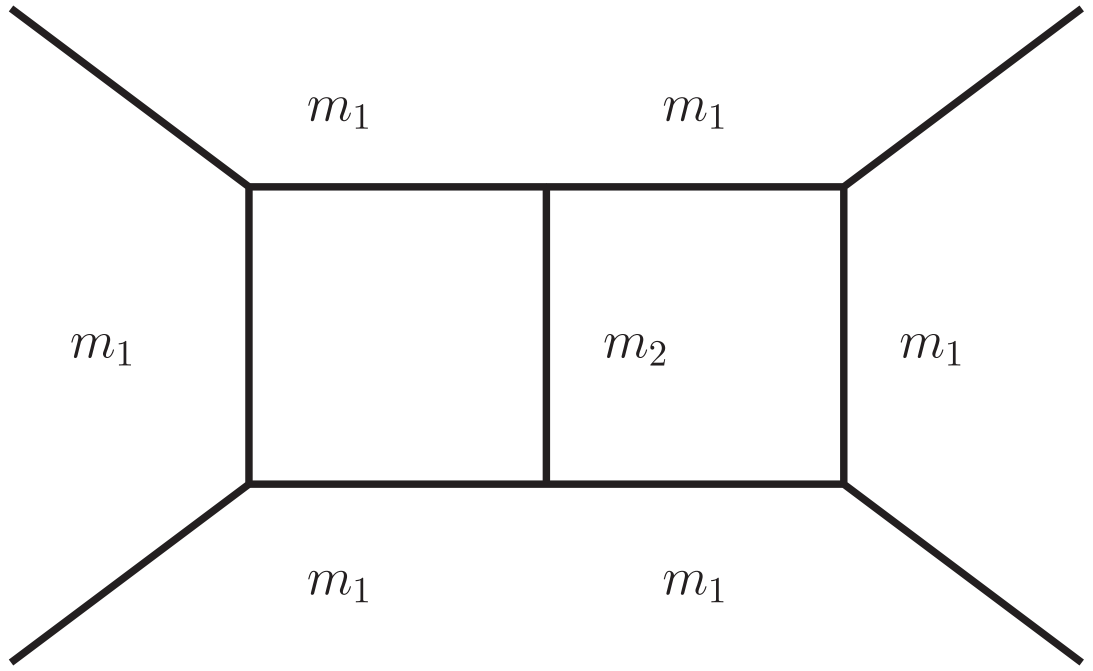
Case III, no singular point

We can find tangent algebras in all these case, but before the calculation…

Tangent algebra and singular points

Quillen–Suslin theorem: Syzygy for polynomials without common root is a free module.

$F=0$ is smooth → F_{z_8}, F_{z_9}, F has no common root → tangent algebra is a free module



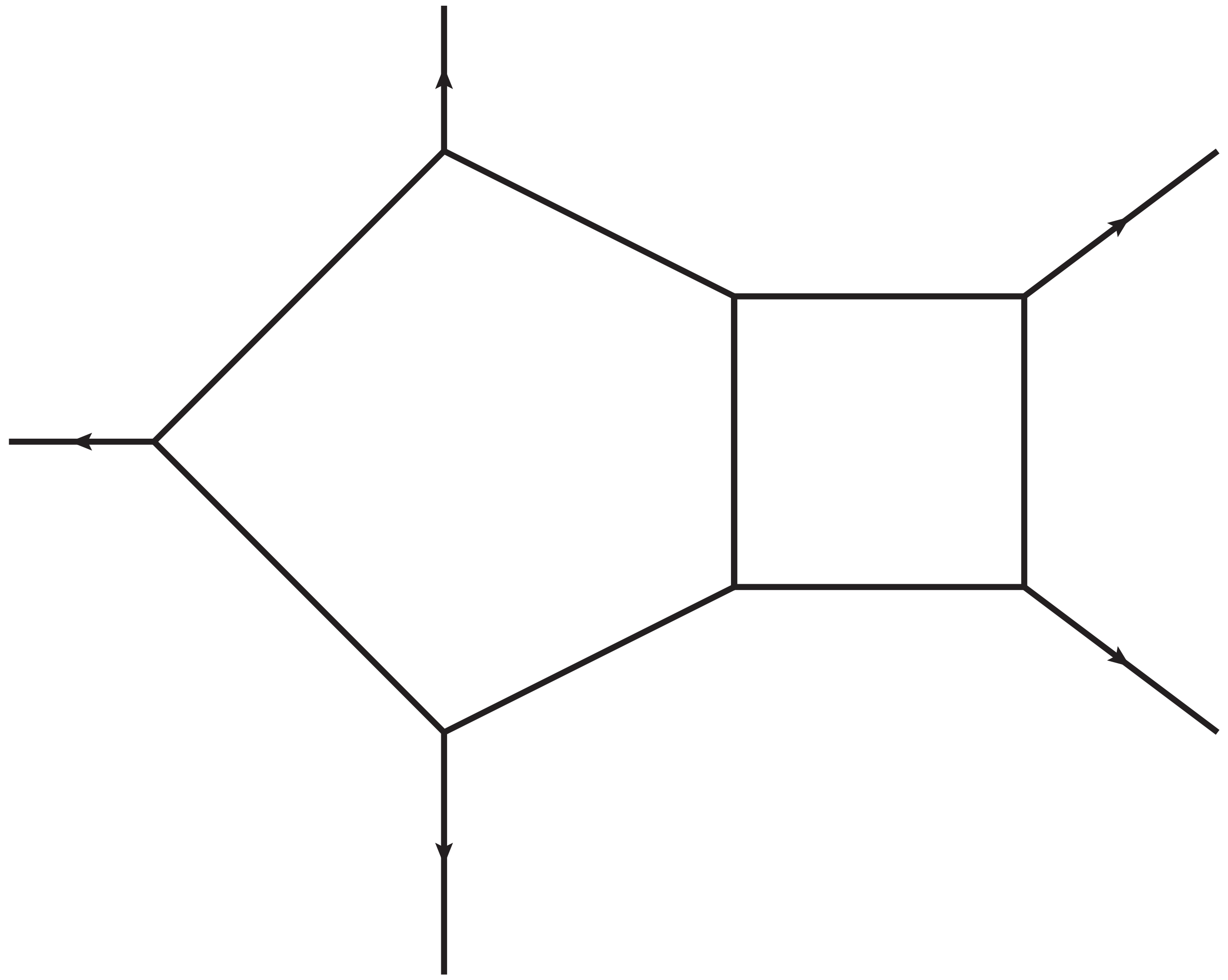
Case III, $m_1 \neq 0, m_2 \neq 0$ has the simplest tangent algebra (generated by principle syzygies). For case I, II, the tangent algebras are generated by principle syzygy + weighted Euler vectors around the singular points.

All cases' algebra can be automatically found by algebraic geometry softwares Macaulay2/Singular (based on Gröbner basis computation)

$$\int \frac{dl_1^D}{i\pi^{D/2}} \int \frac{dl_2^D}{i\pi^{D/2}} \frac{-\alpha(D-6)/2 + \partial\alpha_8/\partial z_8 + \partial\alpha_9/\partial z_9}{D_1 \ldots D_7} = 0 + \ldots$$

get all on-shell part of D-dim IBPs

Maximal cut, 5-point, (toy example)



measure form

$$I^D_{\text{pentabox}}|_{\text{cut}} \quad \equiv \quad \int\int\int dx dy_1 dy_2 N(x,y_1,y_2) F(x,y_1,y_2)^{\frac{D-7}{2}}$$

$$F(x,y_1,y_2) = 0 \quad \text{surface}$$

$$\boxed{\alpha F_x + \beta F_{y_1} + \gamma F_{y_2} + \delta F = 0}$$

Syzygy equation

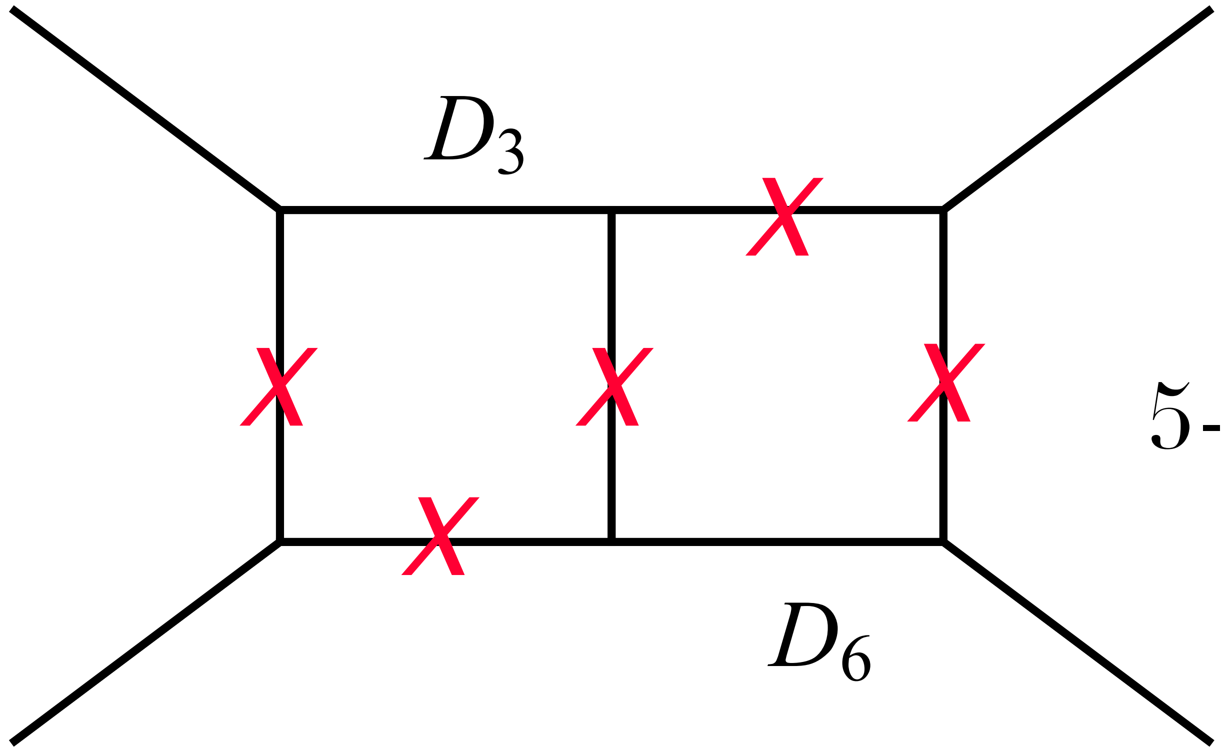
$$\begin{aligned} 0 &= \int d[(\alpha dy_1 \wedge dy_1 + \beta dy_2 \wedge dx + \gamma dx \wedge dy_1) F^{\frac{D-7}{2}}] \\ &= \int \left[\left(\frac{\partial\alpha}{\partial x} + \frac{\partial\beta}{\partial y_1} + \frac{\partial\gamma}{\partial y_2}\right) - \delta\left(\frac{D-7}{2}\right)\right] F^{\frac{D-7}{2}} dx \wedge dy \end{aligned}$$

get all on-shell part of D-dim IBPs

Non-maximal cut

$$2\times 4+3-2-5=4 \text{ variables left, } z_3, z_6, z_8, z_9$$

4D mu's spurious 5-cut



5-cut

$$I^D_{\text{dbox}}|_{\text{5-cut}} \propto \int dz_3 dz_6 dz_8 dz_9\; N\, F(z)^{\frac{D-6}{2}} z_3^{-1} z_6^{-1}$$

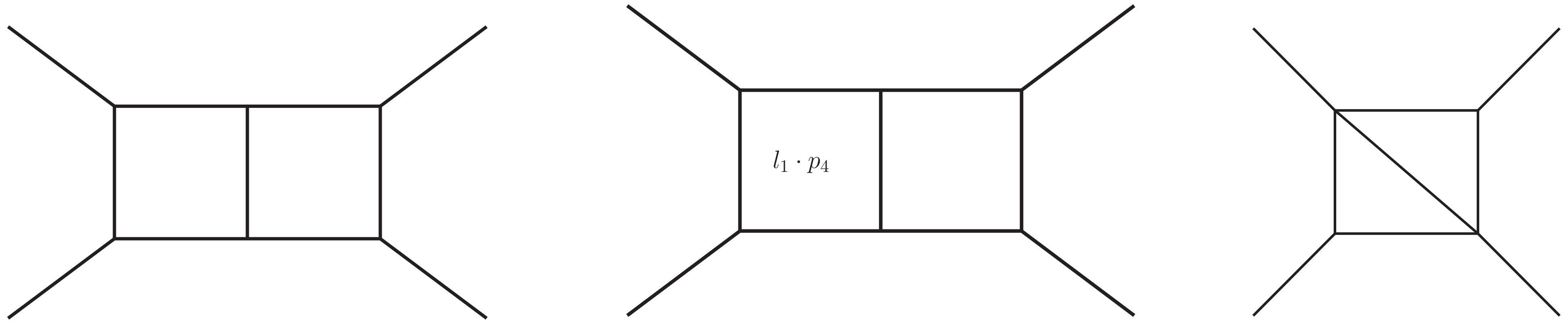
$$0=\int d\Big((\alpha_3 dz_6\wedge dz_8\wedge dz_9-\alpha_6 dz_8\wedge dz_9\wedge dz_3$$
$$+\alpha_8 dz_9\wedge dz_3\wedge dz_6-\alpha_9 dz_3\wedge dz_6\wedge dz_8)NF(z)^{\frac{D-6}{2}}z_3^{-1}z_6^{-1}\Big)$$

$$\boxed{\begin{aligned}\alpha_i\frac{\partial F}{\partial z_i}+\beta F&=0\\ \alpha_3+\beta_3 z_3&=0\\ \alpha_6+\beta_6 z_6&=0\end{aligned}}$$

Two simple linear equations →

$$\left.\begin{array}{c}\text{Syzygy for polynomials}\\ \{z_3F_{z_3}, z_6F_{z_6}, F_{z_8}, F_{z_9}, F\}\end{array}\right\}\ \text{Tangent algebra of } z_3z_6F=0$$

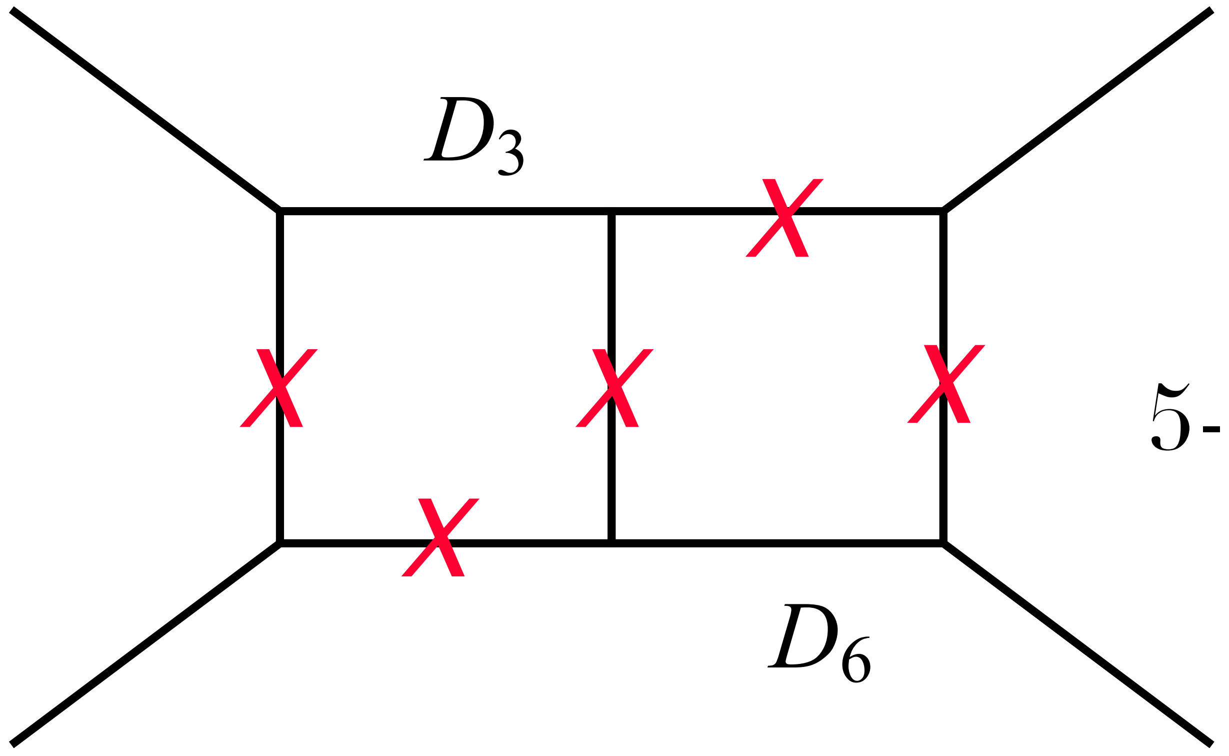
Reduce to 3 MIs



Non-maximal cut

$$2\times 4+3-2-5=4 \text{ variables left, } z_3, z_6, z_8, z_9$$

4D mu's spurious 5-cut



5-cut

$$I^D_{\text{dbox}}|_{\text{5-cut}} \propto \int dz_3 dz_6 dz_8 dz_9\, N\, F(z)^{\frac{D-6}{2}} z_3^{-1} z_6^{-1}$$

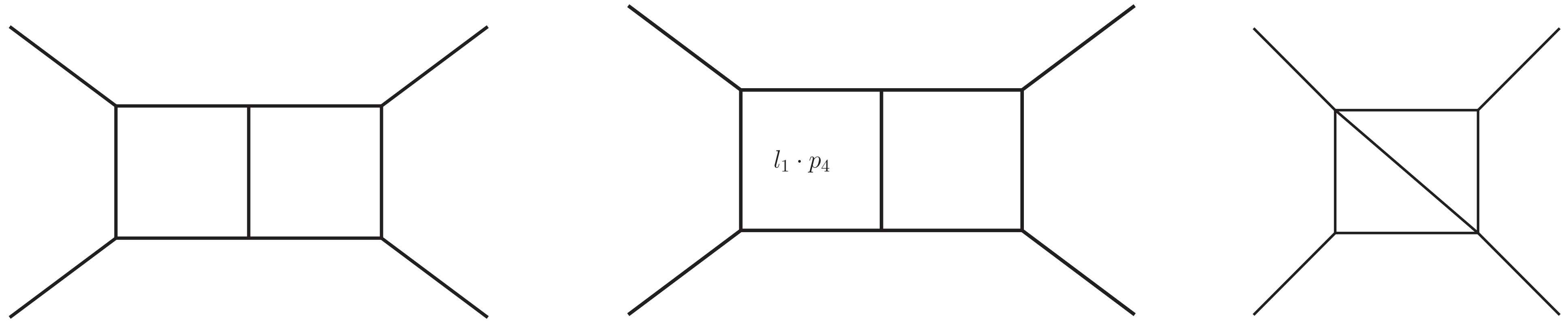
$$0=\int d\Big((\alpha_3 dz_6\wedge dz_8\wedge dz_9-\alpha_6 dz_8\wedge dz_9\wedge dz_3 + \alpha_8 dz_9\wedge dz_3\wedge dz_6-\alpha_9 dz_3\wedge dz_6\wedge dz_8) N F(z)^{\frac{D-6}{2}} z_3^{-1} z_6^{-1}\Big)$$

$$\alpha_i\frac{\partial F}{\partial z_i}+\beta F=0$$
$$\alpha_3+\beta_3 z_3=0$$
$$\alpha_6+\beta_6 z_6=0$$

Two simple linear equations

$$\left.\begin{array}{c}\text{Syzygy for polynomials}\\ \{z_3F_{z_3}, z_6F_{z_6}, F_{z_8}, F_{z_9}, F\}\end{array}\right\} \text{Tangent algebra of } z_3z_6F=0$$

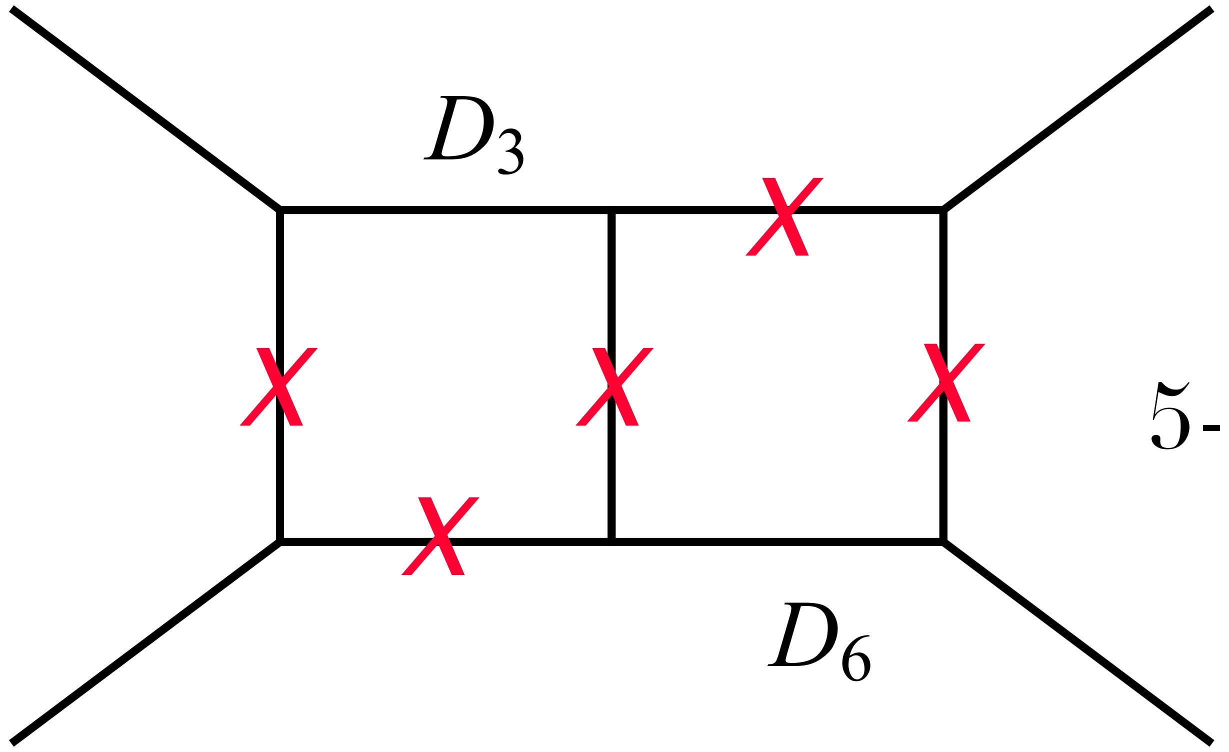
Reduce to 3 MIs



Non-maximal cut

$$2\times 4+3-2-5=4 \text{ variables left, } z_3, z_6, z_8, z_9$$

4D mu's spurious 5-cut



5-cut

$$I^D_{\text{dbox}}|_{\text{5-cut}} \propto \int dz_3 dz_6 dz_8 dz_9\, N \boxed{F(z)}^{\frac{D-6}{2}} \boxed{z_3^{-1}} \boxed{z_6^{-1}}$$

$$0=\int d\Big((\alpha_3 dz_6\wedge dz_8\wedge dz_9-\alpha_6 dz_8\wedge dz_9\wedge dz_3$$
$$+\alpha_8 dz_9\wedge dz_3\wedge dz_6-\alpha_9 dz_3\wedge dz_6\wedge dz_8)NF(z)^{\frac{D-6}{2}}z_3^{-1}z_6^{-1}\Big)$$

$$\boxed{\begin{gathered}\alpha_i\frac{\partial F}{\partial z_i}+\beta F=0\\ \boxed{\begin{gathered}\alpha_3+\beta_3 z_3=0\\ \alpha_6+\beta_6 z_6=0\end{gathered}}\end{gathered}}$$

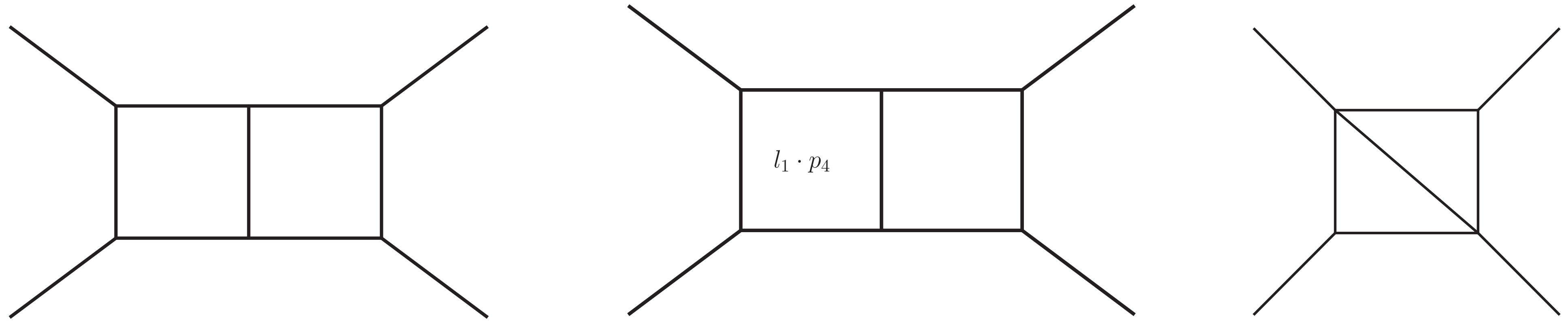
Remove doubled propagator, reduce # IBPs

→

Two simple linear equations

$$\left.\begin{array}{c}\text{Syzygy for polynomials}\\ \{z_3F_{z_3}, z_6F_{z_6}, F_{z_8}, F_{z_9}, F\}\end{array}\right\} \text{Tangent algebra of } z_3z_6F=0$$

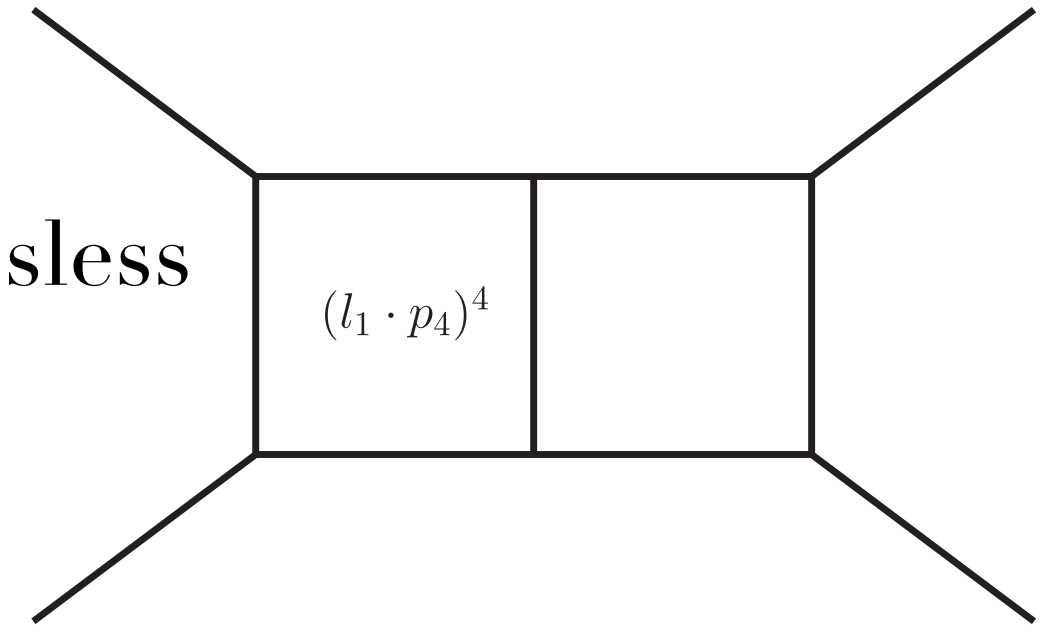
Reduce to 3 MIs



Complete reduction

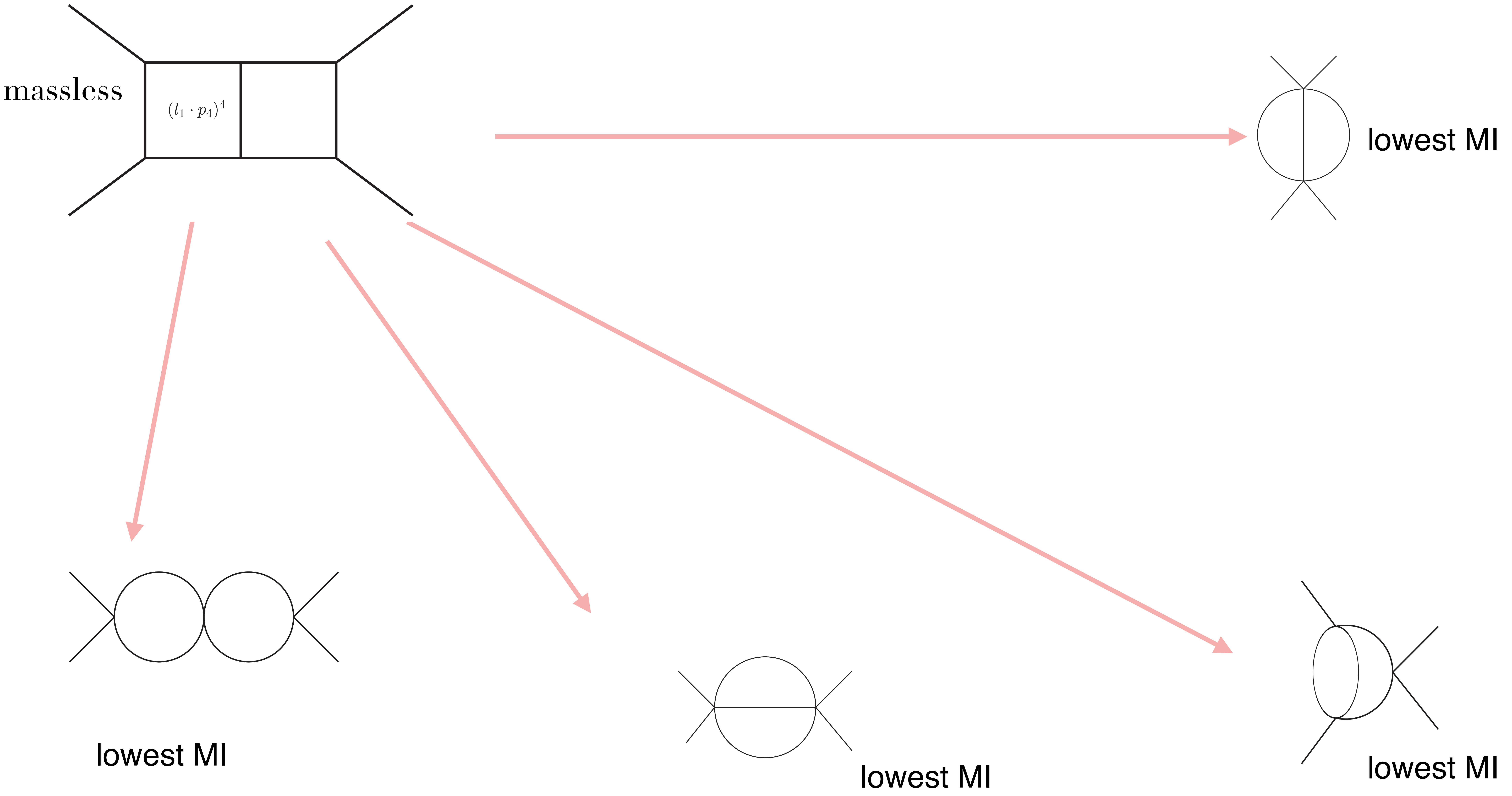
code powered by
Mathematica/Macaulay2/Singular

massless



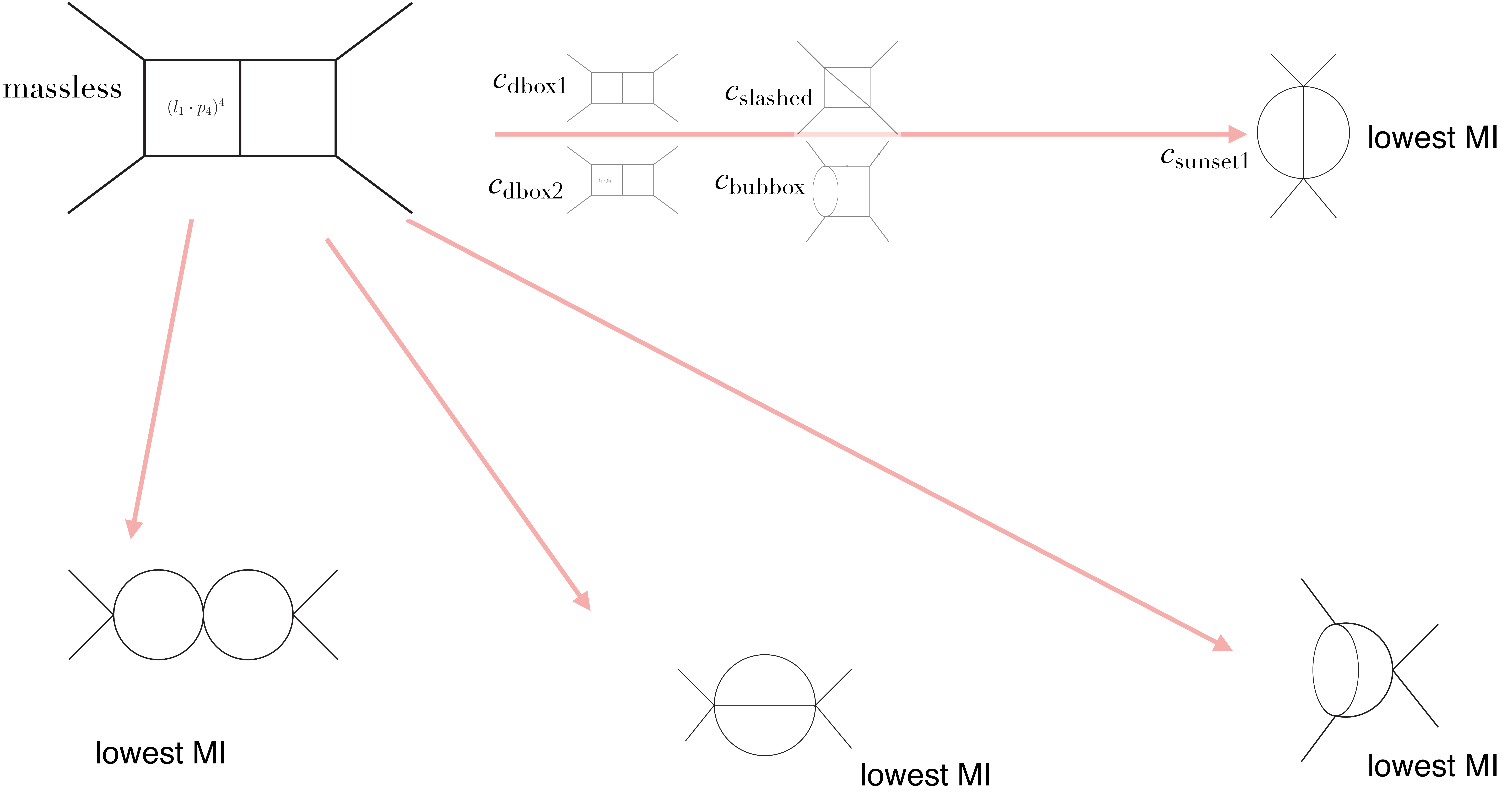
Complete reduction

code powered by
Mathematica/Macaulay2/Singular



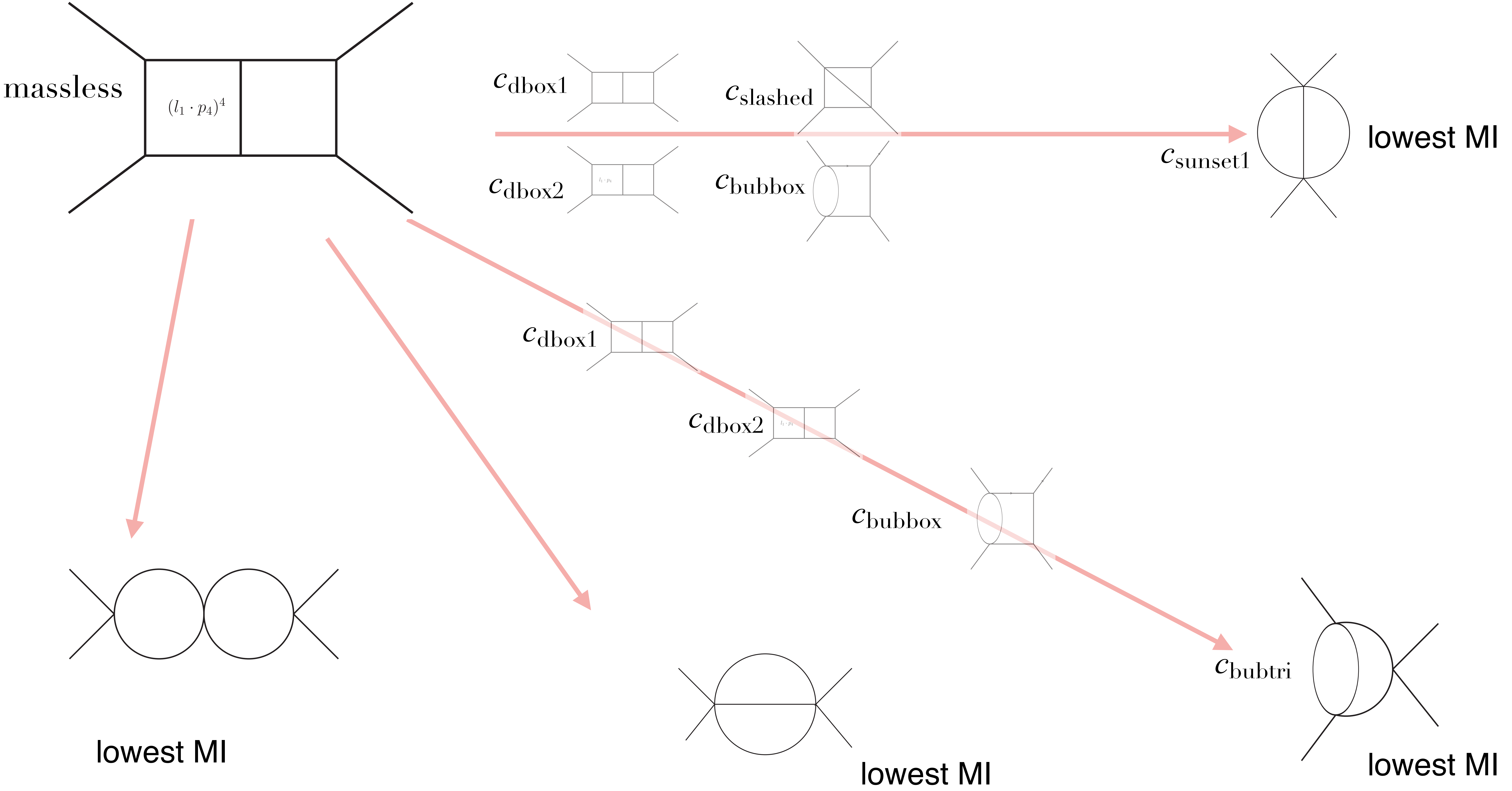
Complete reduction

code powered by
Mathematica/Macaulay2/Singular



Complete reduction

code powered by
Mathematica/Macaulay2/Singular



Complete reduction

code powered by
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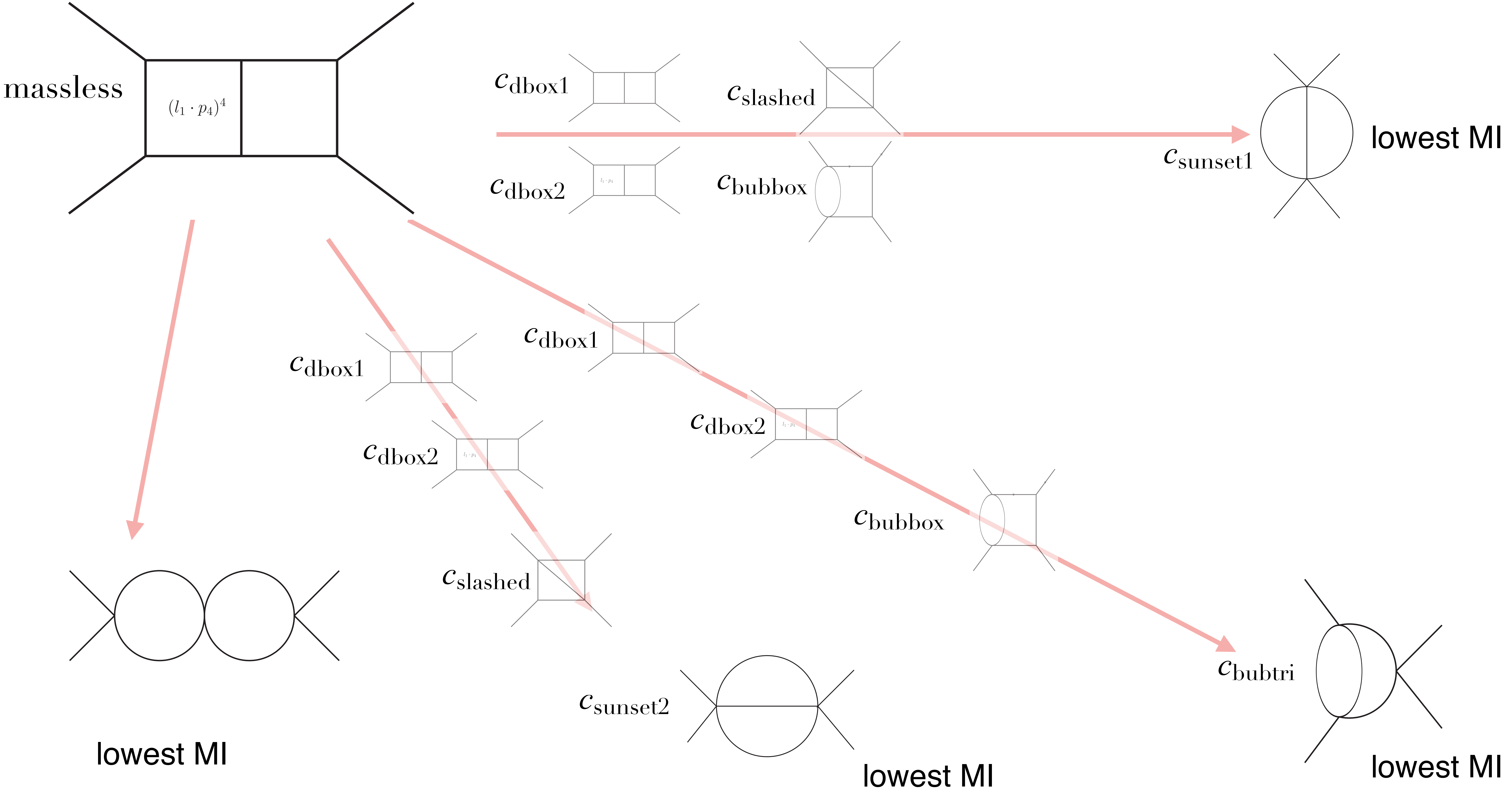


massless

$(l_1 \cdot p_4)^4$

c_{dbox1}

c_{slashed}

c_{dbox2}

c_{bubbox}

c_{sunset1}

lowest MI

c_{dbox1}

c_{dbox1}

c_{dbox2}

c_{dbox2}

c_{bubbox}

c_{slashed}

c_{bubtri}

c_{sunset2}

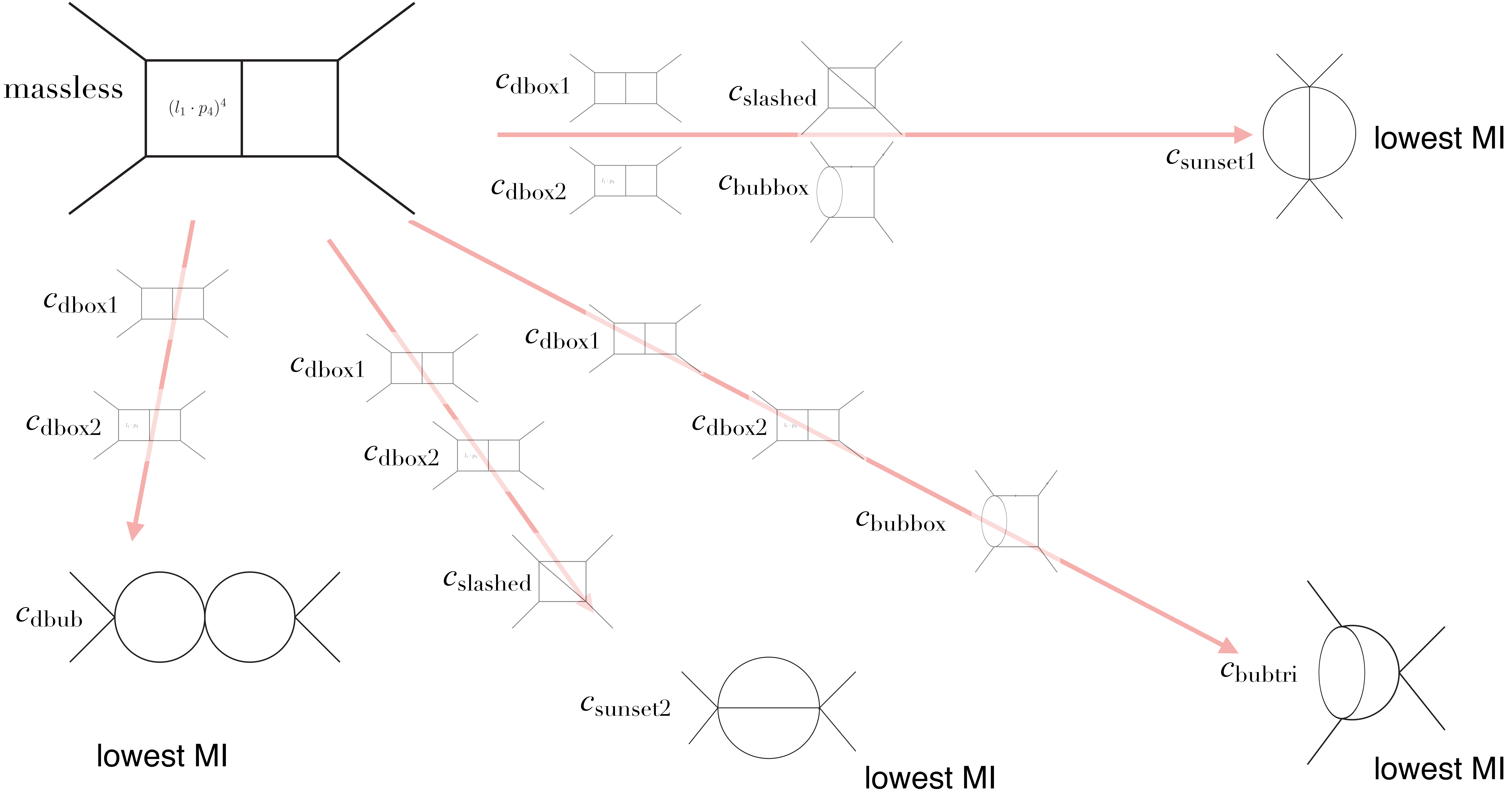
lowest MI

lowest MI

lowest MI

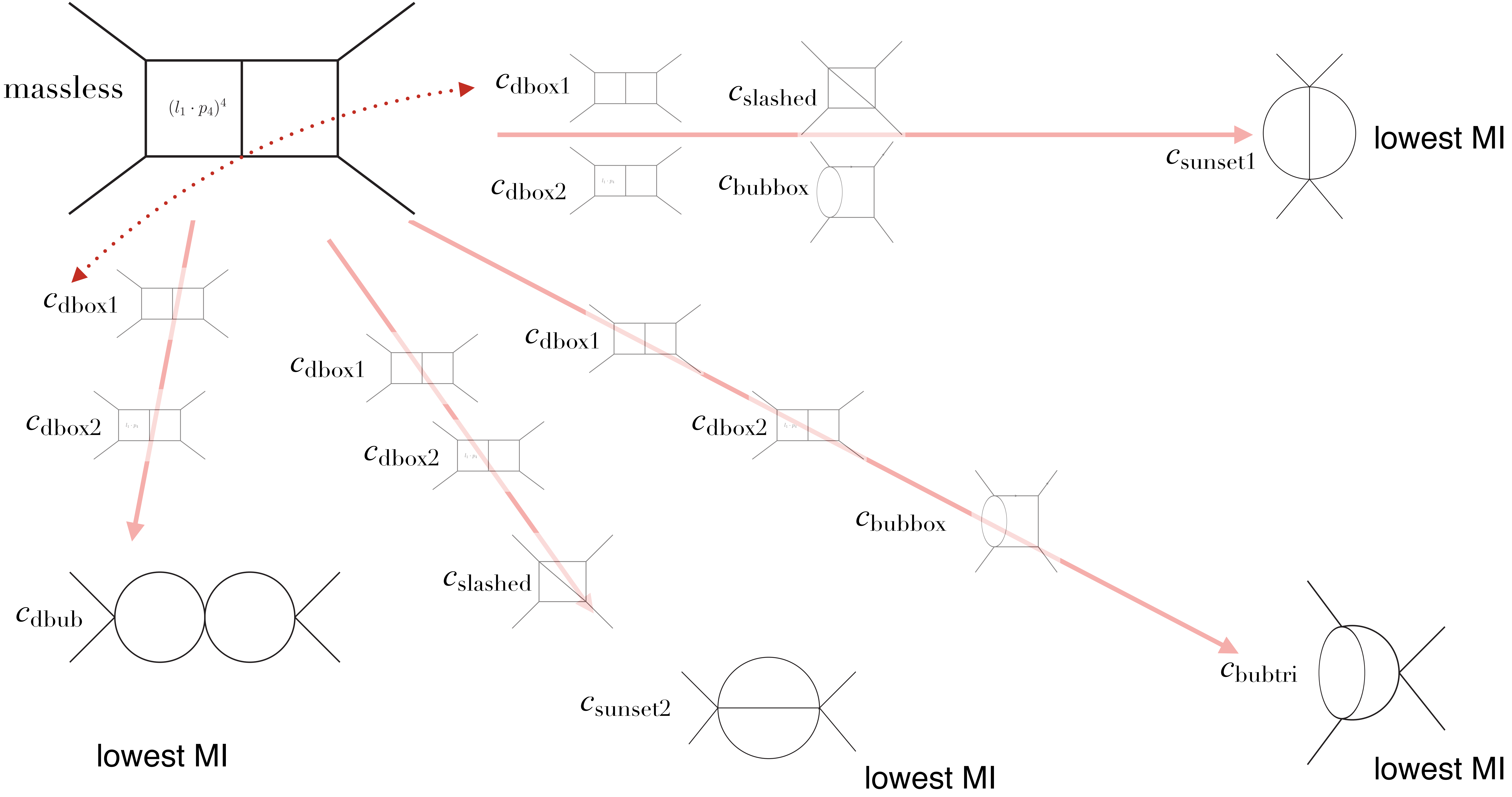
Complete reduction

code powered by
Mathematica/Macaulay2/Singular



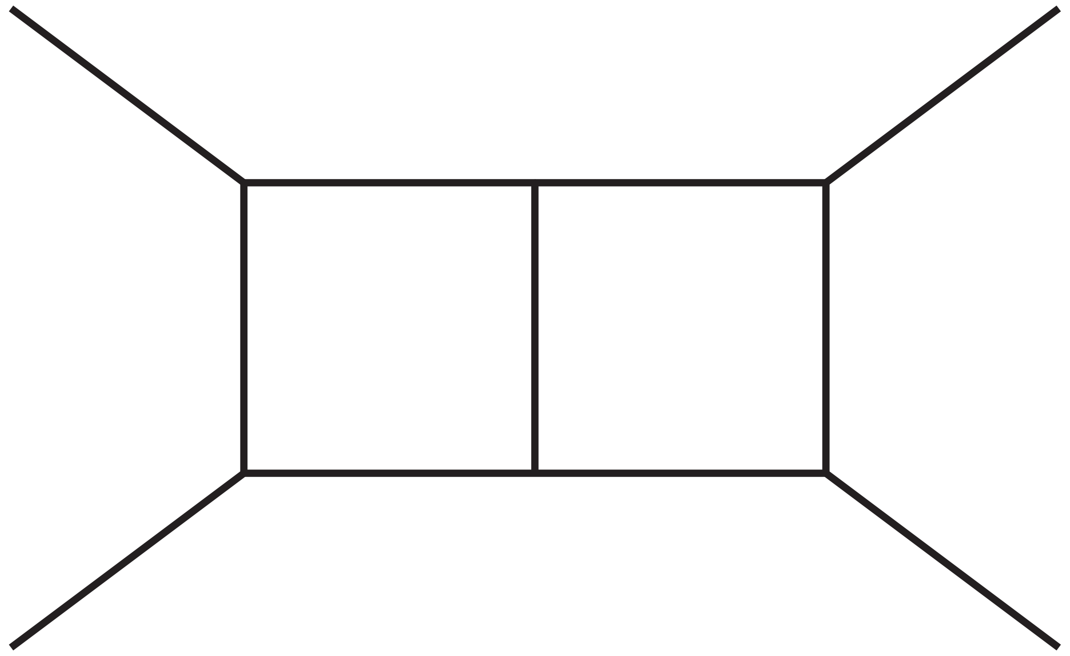
Complete reduction

code powered by
Mathematica/Macaulay2/Singular

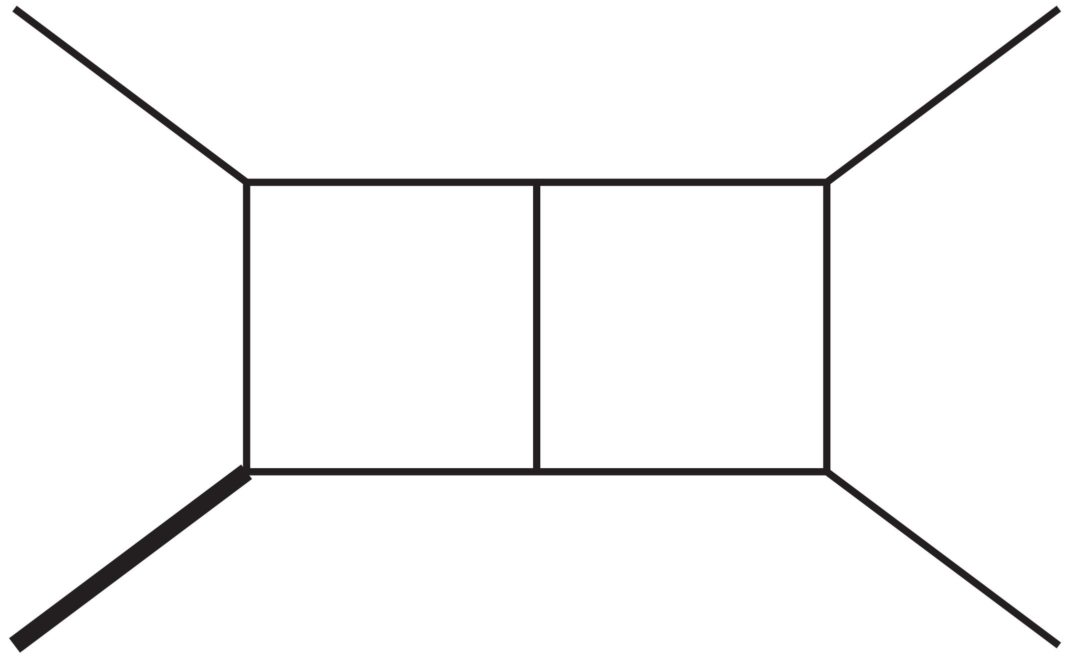


Complete IBP reduction, examples

primitive implementation powered by
Mathematica/Macaulay2/Singular

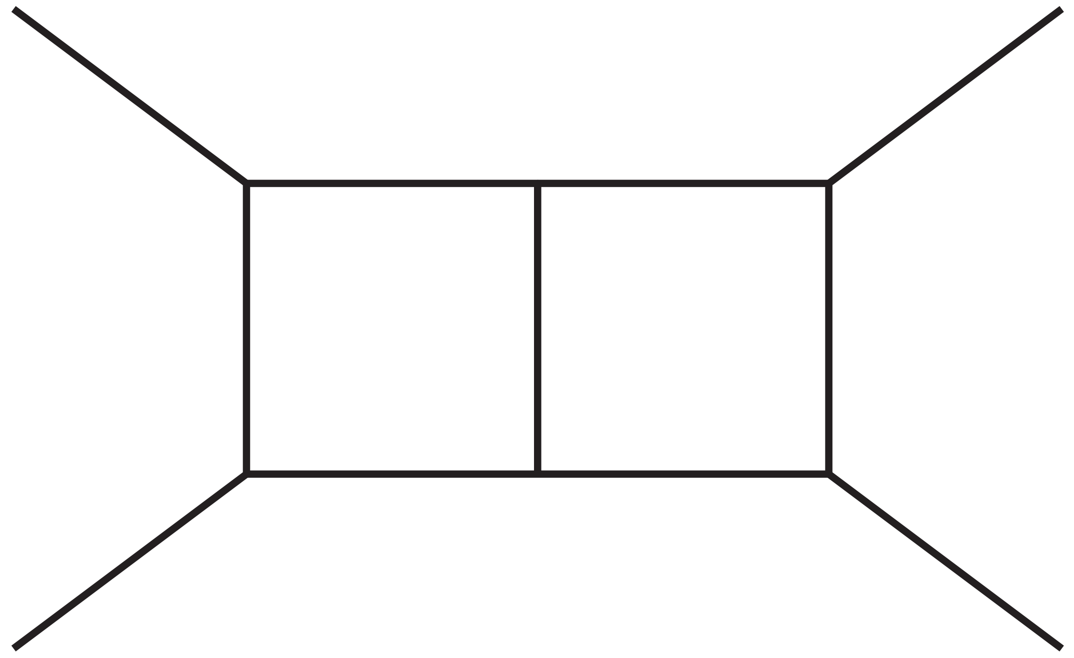


Massless
complete reduction
of all integrals with rank<=4
to 8 MIs in 39 seconds



Complete IBP reduction, examples

primitive implementation powered by
Mathematica/Macaulay2/Singular



Massless
complete reduction
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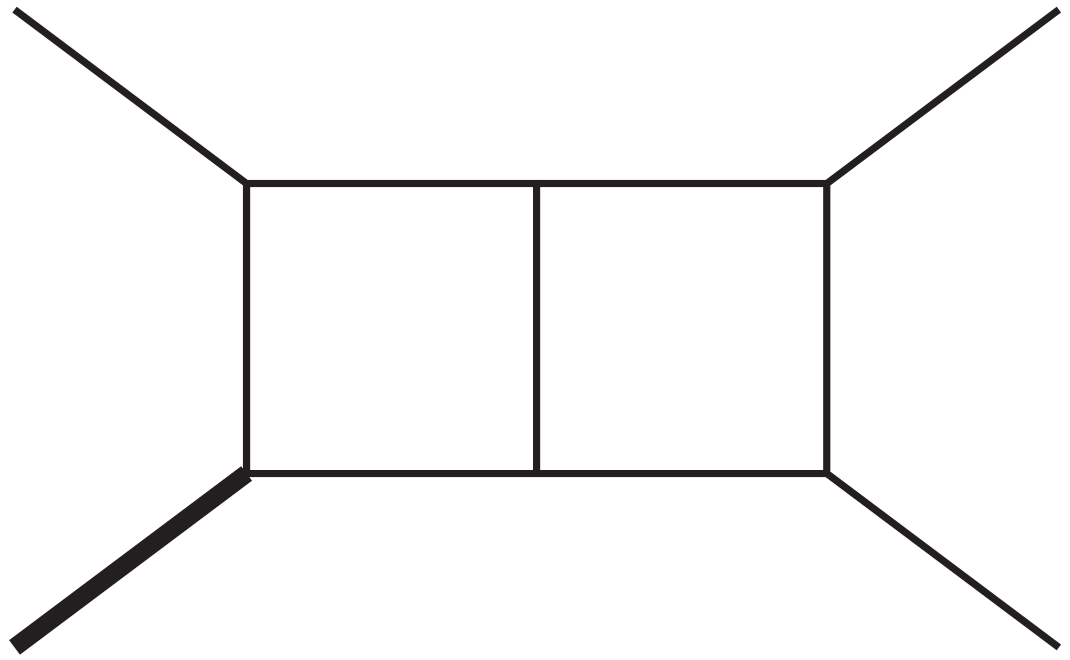
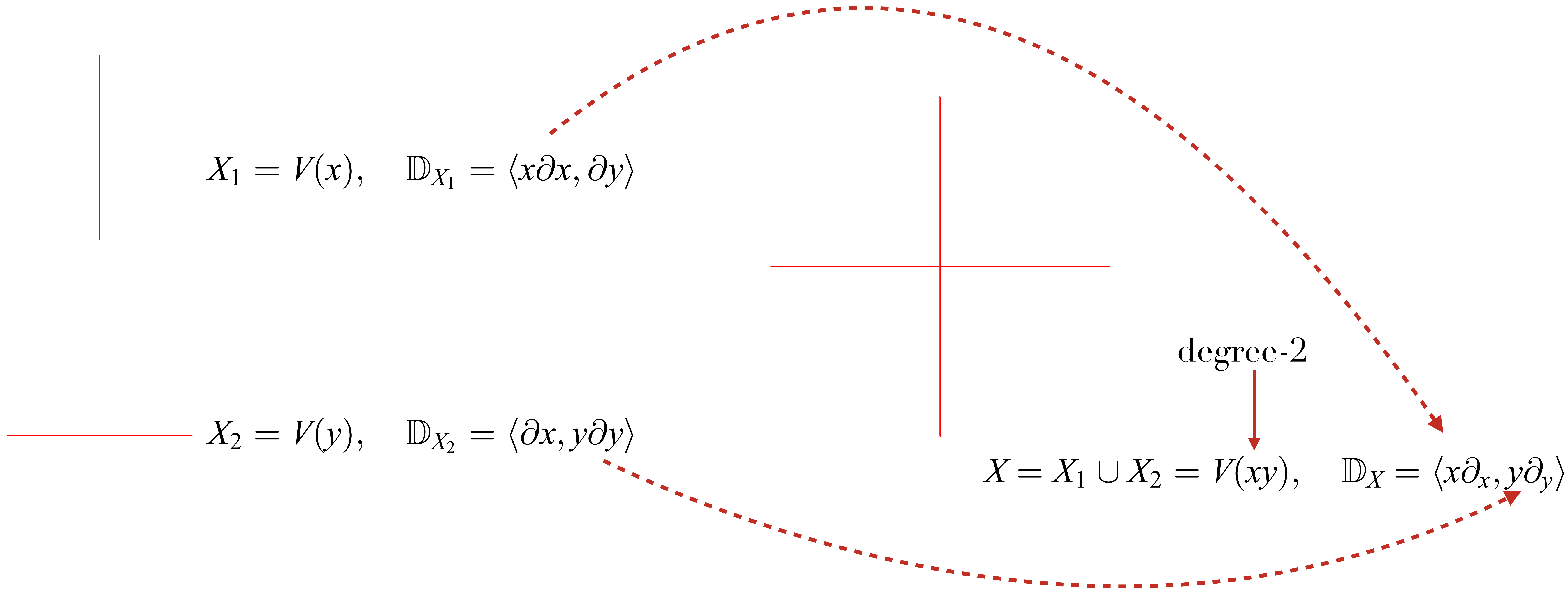


One-Mass
complete reduction
of all integrals with rank<=4
to 19 MIs in 211 seconds

more about tangent algebra

Let X be an affine variety, $X = X_1 \cup X_2 \ldots \cup X_k$ (irreducible components).
The tangent algebra of X, $\mathbb{D}_X$ is,

$$\mathbb{D}_X = \mathbb{D}_{X_1} \cap \mathbb{D}_{X_2} \ldots \cap \mathbb{D}_{X_k}.$$



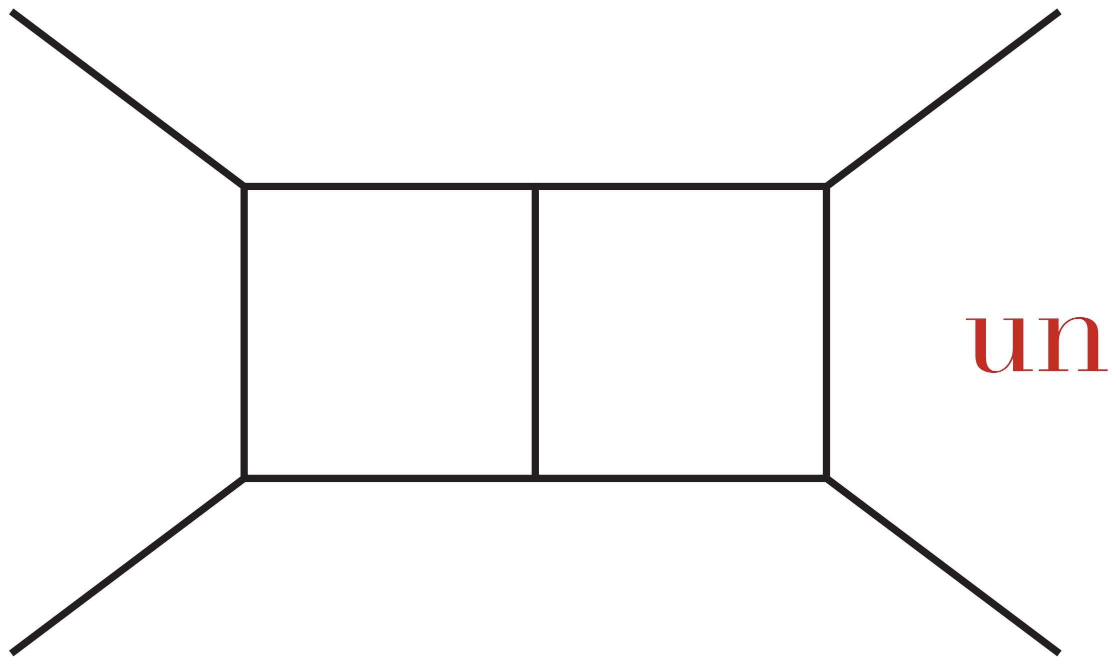
$X_1 = V(x), \quad \mathbb{D}_{X_1} = \langle x\partial x, \partial y\rangle$

$X_2 = V(y), \quad \mathbb{D}_{X_2} = \langle \partial x, y\partial y\rangle$

degree-2

$X = X_1 \cup X_2 = V(xy), \quad \mathbb{D}_X = \langle x\partial_x, y\partial_y\rangle$

Intersection of tangent algebras



uncut

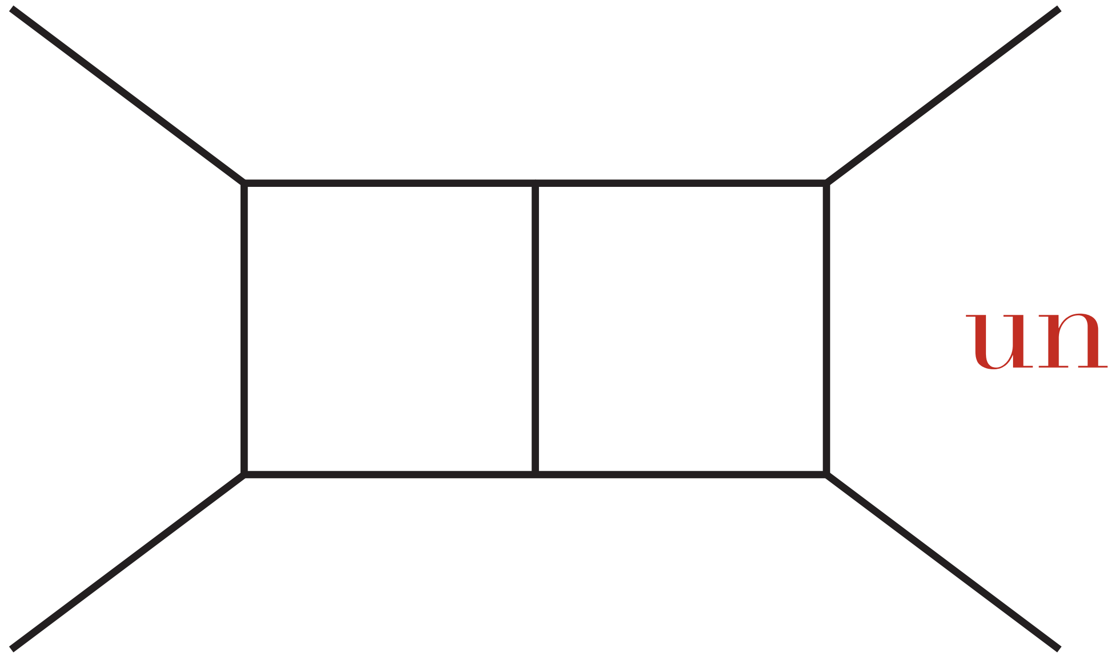
$$0=\int d\Big(\sum_{i=1}^{9}\frac{(-1)^{i+1}a_iF(z)^{\frac{D-6}{2}}}{z_1\cdots z_7}dz_1\wedge\cdots\widehat{dz_i}\cdots\wedge dz_9\Big)$$

$$\boxed{\sum_{i=1}^{9}a_i\frac{\partial}{\partial z_i}F \quad\propto\quad F} \longrightarrow$$

$$M_1=\Big\{-\mu_{11}\frac{\partial}{\partial\mu_{11}}+\mu_{22}\frac{\partial}{\partial\mu_{22}},2\mu_{12}\frac{\partial}{\partial\mu_{22}}+\mu_{11}\frac{\partial}{\partial\mu_{12}},$$
$$2\mu_{12}\frac{\partial}{\partial\mu_{11}}+\mu_{22}\frac{\partial}{\partial\mu_{12}},\mu_{11}\frac{\partial}{\partial\mu_{11}}+\mu_{22}\frac{\partial}{\partial\mu_{22}}+\mu_{12}\frac{\partial}{\partial\mu_{12},},\frac{\partial}{\partial l_1^{[4]}},\frac{\partial}{\partial l_2^{[4]}}\Big\}$$

$$\boxed{\sum_{i=1}^{9}a_i\frac{\partial}{\partial z_i}z_j \quad\propto\quad z_j,\quad j=1,\ldots,7} \longrightarrow M_2=\Big\{z_1\frac{\partial}{\partial z_1},\ldots,z_7\frac{\partial}{\partial z_7},\frac{\partial}{\partial z_8},\frac{\partial}{\partial z_9}\Big\}$$

Intersection of tangent algebras



$$0 = \int d\left(\sum_{i=1}^{9} \frac{(-1)^{i+1} a_i F(z)^{\frac{D-6}{2}}}{z_1\cdots z_7}\, dz_1\wedge\cdots\widehat{dz_i}\cdots\wedge dz_9\right)$$

$$\boxed{\sum_{i=1}^{9} a_i\frac{\partial}{\partial z_i}F \quad\propto\quad F}$$

$$M_1 = \Big\{ -\mu_{11}\frac{\partial}{\partial\mu_{11}} + \mu_{22}\frac{\partial}{\partial\mu_{22}},\, 2\mu_{12}\frac{\partial}{\partial\mu_{22}} + \mu_{11}\frac{\partial}{\partial\mu_{12}},\, 2\mu_{12}\frac{\partial}{\partial\mu_{11}} + \mu_{22}\frac{\partial}{\partial\mu_{12}},\, \mu_{11}\frac{\partial}{\partial\mu_{11}} + \mu_{22}\frac{\partial}{\partial\mu_{22}} + \mu_{12}\frac{\partial}{\partial\mu_{12},},\, \frac{\partial}{\partial l_1^{[4]}},\, \frac{\partial}{\partial l_2^{[4]}}\Big\}$$

$$\boxed{\sum_{i=1}^{9} a_i\frac{\partial}{\partial z_i}z_j \quad\propto\quad z_j, \quad j = 1,\ldots,7}$$

$$M_2 = \Big\{z_1\frac{\partial}{\partial z_1},\ldots,z_7\frac{\partial}{\partial z_7},\frac{\partial}{\partial z_8},\frac{\partial}{\partial z_9}\Big\}$$

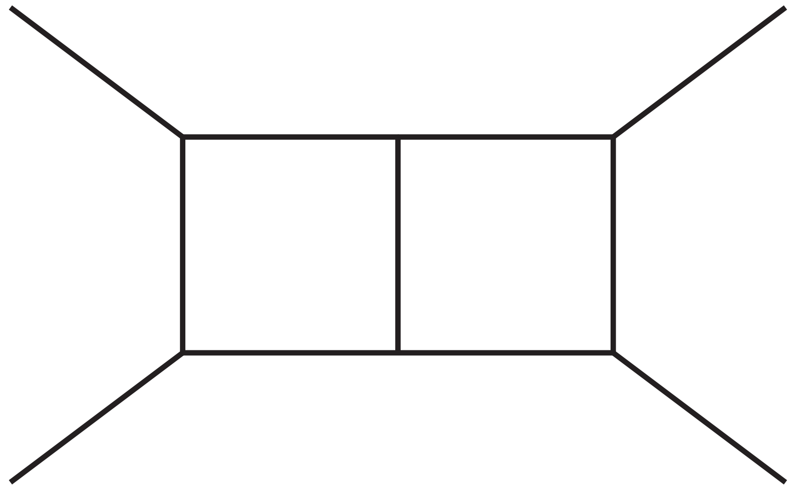
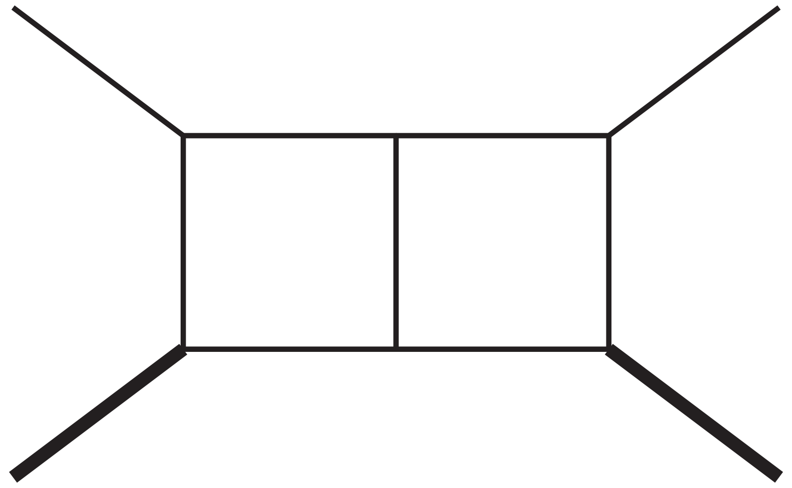
$M = M_1 \cap M_2$ is the module for generating IBPs without double propagator!

Intersection can be
obtained by Gröbner basis techniques

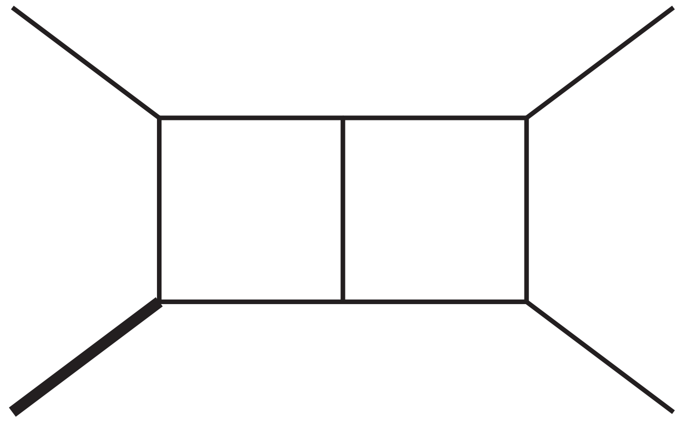
Intersection method, toy examples

Experimental implementation
K. Larsen and YZ

to find generators of tangent algebra

		Direct syzygy (Singular 'syz')	Intersection (code in Singular)
	massless uncut	$\sim$ 180 seconds	$\sim$ 5 seconds
	2 mass quadruple cut	?	$\sim$ 10 seconds

complete reduction



1 mass

211 seconds → 162 seconds (intersection method)

Summary

- Algebraic geometry approach for IBP reduction
- highly efficient for examples tested
- full control of arbitrary unitarity cut

Future directions

- Syzygy via ‘F5’ algorithm (Gröbner basis “without” reduction to zero)
- Infinite dimensional Lie-algebra structure (variety/tangent algebra dictionary)
- Large D limit
- A fully automatic program