A numerical approach to NNLO calculations in QCD

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Outline

- Introduction
- The method
  - simple example
  - multi-loop integrals
  - phase space integrals
  - mixture one loop + single unresolved
- Summary and outlook

Introduction

Do we need NNLO ?

number of processes/observables where (NLO) theory error dominates (or will dominate at LC)

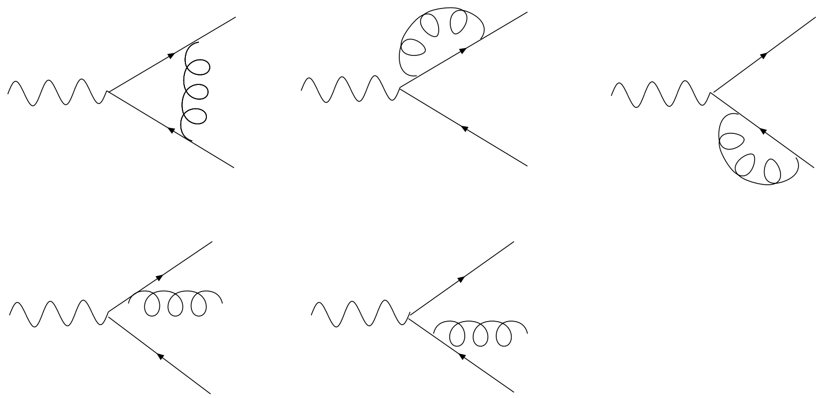
$\longrightarrow$ see S. Dittmaier's talk

some pro–NNLO arguments:

- reduction of scale dependence
- better modeling of jets
- better description of "intrinsic k_T" in hadronic initial state
- improved PDF's
- ...
- better understanding of infrared structure of QCD

"conventional" procedure to calculate partonic cross sections:

example NLO: $$\sigma^{NLO} = \underbrace{\sigma^{V}}_{\text{infinite}} + \underbrace{\sigma^{R}}_{\text{infinite}}$$

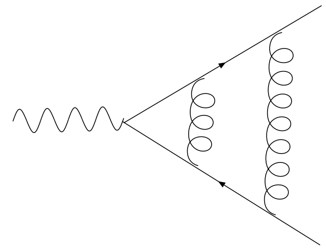


virtual: $\sum_{i=0}^{2} P_i/\epsilon^i$ real: subtraction of $d\sigma^S$

$$\sigma^{NLO} = \underbrace{\int_{m+1} \left[d\sigma^R - d\sigma^S\right]_{\epsilon=0}}_{\text{numerically}} + \underbrace{\int_m \left[\underbrace{d\sigma^V}_{\text{analytically}} + \underbrace{\int_1 d\sigma^S}_{\text{analytically}}\right]_{\epsilon=0}}_{\text{numerically}}$$

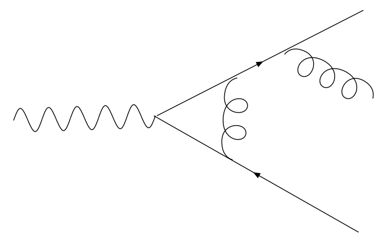
NNLO: example $e^+e^- \to 2\,\text{jets}$

- 2-loop virtual



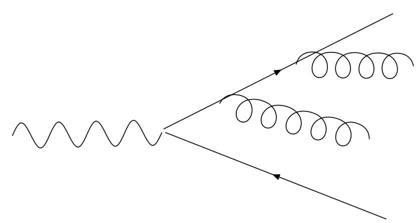
$\longrightarrow$ 2–particle phase space

- one-loop single real emission



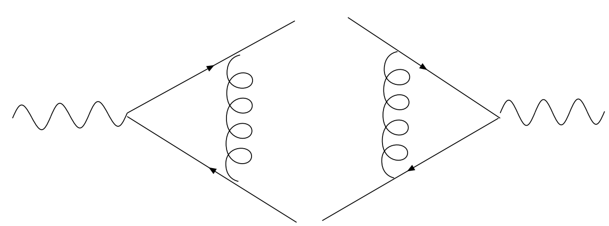
$\longrightarrow$ phase space: 1 unresolved particle

- double real emission



$\longrightarrow$ difficult phase space integrals:
up to two unresolved particles

- squares of one-loop amplitudes



problems at NNLO (in "conventional" approach):

- find appropriate subtraction terms
- analytical integration of subtraction terms in $D = 4 - 2\epsilon$ dimensions

[see work of A. Gehrmann-De Ridder, T. Gehrmann, N. Glover, W. Kilgore, D. A. Kosower, S. Weinzierl]

problem in general:

- construction of a stable and fast Monte Carlo program main problem in perturbative QCD:

infrared (soft and collinear) divergences:

overlapping structure

example (in parameter space, using dimensional regularization):

$$\int_0^1 dx_1\, dx_2\, x_1^{-1-\epsilon}\, [x_1 + x_2]^{-1}$$

occurrence:

multi-loop integrals, phase space integrals, combinations

possible solution:

sector decomposition

History

- K. Hepp [Commun. Math. Phys. 2 (1966)]: for overlapping UV divergences
- M. Roth and A. Denner [Nucl. Phys. B 479 (1996)]: high-energy approximation

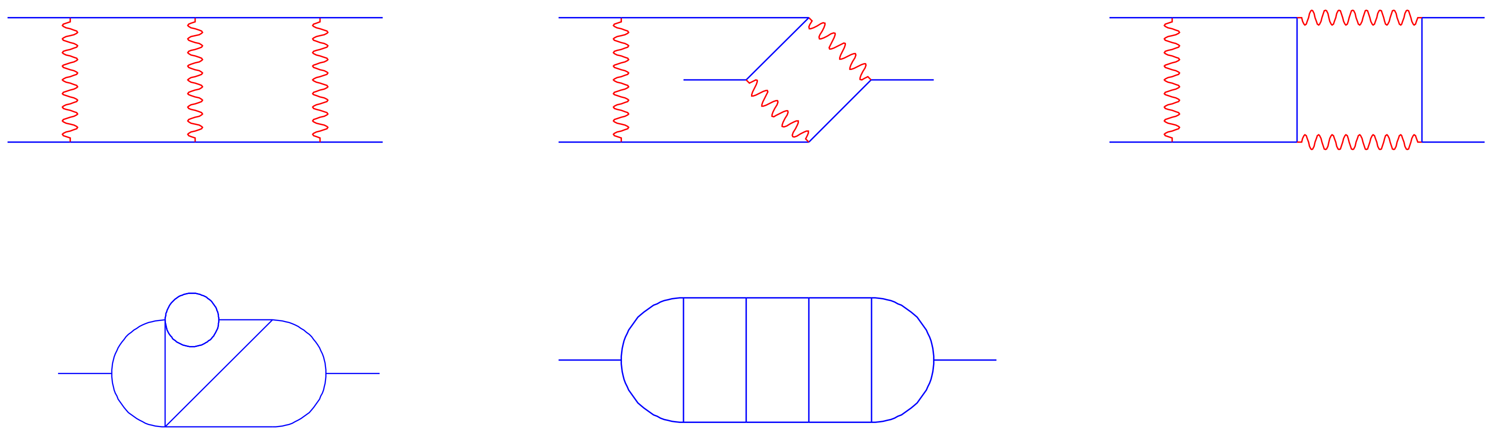
of one-loop Feynman integrals

- iterative, automated algorithm for multi-loop integrals:

T. Binoth, G.H. [Nucl. Phys. B 585 (2000)]

- applied to two-loop box diagrams, massless 3-loop box,
  2-point functions up to 5 loops

T. Binoth, G.H. [Nucl. Phys. B 680 (2004)]



application to phase space integrals:

- G.H., Nucl. Phys. Proc. Suppl. 116 (2003)
(loops & legs 2002)

- A. Gehrmann-De Ridder, T. Gehrmann and G.H., hep-ph/0311276
all master phase space integrals for $1 \to 4$ processes

- C. Anastasiou, K. Melnikov and F. Petriello, hep-ph/0311311
inclusion of jet function, N_f-part of $e^+e^- \to 2, 3, 4\,\text{jets}$ at $\mathcal{O}(\alpha_s^2)$

- T. Binoth and G.H., hep-ph/0402265
partial results for $e^+e^- \to 2\,\text{jets}$

- C. Anastasiou, K. Melnikov and F. Petriello, hep-ph/0402280
$e^+e^- \to 2\,\text{jets}$

The Method

simple idea:

$$
\begin{aligned}
I &= \int_0 dx_1\, dx_2\, x_1^{-1-\epsilon}\, (x_1+x_2)^{-1} \\
&= \int_0 dx_1\, dx_2\, x_1^{-1-\epsilon}\, (x_1+x_2)^{-1}\, [\underbrace{\Theta(x_1-x_2)}_{(1)} + \underbrace{\Theta(x_2-x_1)}_{(2)}]
\end{aligned}
$$

subst. (1) $x_2 = x_1\, t_2$ (2) $x_1 = x_2\, t_1$

$$
\begin{aligned}
I &= \int_0 dx_1\, x_1^{-1-\epsilon} \int_0^1 dt_2\, (1+t_2)^{-1} \\
&\quad + \int_0 dx_2\, x_2^{-1-\epsilon} \int_0^1 dt_1\, t_1^{-1-\epsilon}\, (1+t_1)^{-1}
\end{aligned}
$$

for more complicated functions one decomposition is not enough $\Rightarrow$ iterate

Multi-loop integrals:

Input: L-loop integral with N propagators after integration over loop momenta

$$G = (-1)^N \Gamma(N - LD/2) \int_0^\infty \prod_{l=1}^{N} dx_l \, \delta(1 - \sum_{i=1}^{N} x_i) \, \frac{\mathcal{U}(\vec{x})^{N-(L+1)D/2}}{\mathcal{F}(\vec{x}, \{s, m^2\})^{N-LD/2}}$$

example: on-shell massless 2-loop box

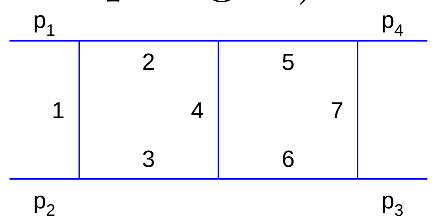
$$G = -\Gamma(3+2\epsilon) \int_0^\infty \prod_{l=1}^{7} dx_l \, \delta(1 - \sum_{i=1}^{7} x_i) \frac{\mathcal{U}^{1+3\epsilon}}{\mathcal{F}^{3+2\epsilon}}$$

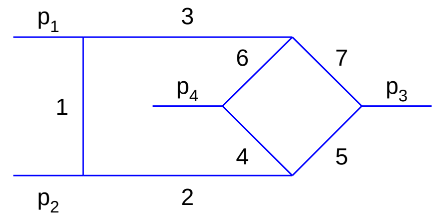
$$\begin{aligned}\mathcal{U} &= (x_1 + x_2 + x_3)(x_5 + x_6 + x_7) \\ &\quad + x_4(x_1 + x_2 + x_3 + x_5 + x_6 + x_7)\end{aligned}$$

$$\begin{aligned}\mathcal{F} &= (-\mathtt{s})\Big\{ x_2 x_3 (x_4 + x_5 + x_6 + x_7) \\ &\quad + x_5 x_6 (x_1 + x_2 + x_3 + x_4) \\ &\quad + x_2 x_4 x_6 + x_3 x_4 x_5 \Big\} \\ &\quad + (-\mathtt{t})\, x_1 x_4 x_7\end{aligned}$$

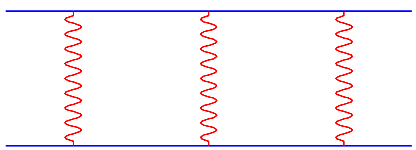
some examples (∗: no analytical result exists)

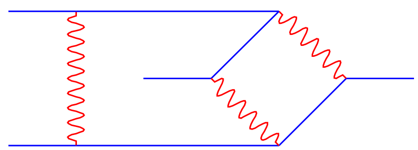
∗ massless 2-loop 4-point functions with 2 off-shell legs
(planar and non-planar topologies)

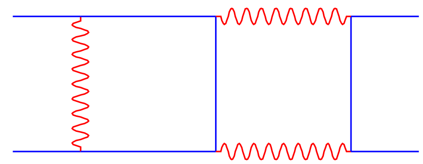




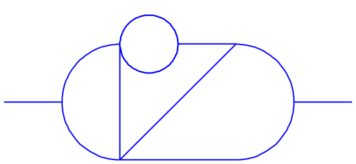
(∗) all 4-point master integrals needed for the calculation of 2-loop Bhabha scattering (massive fermions)

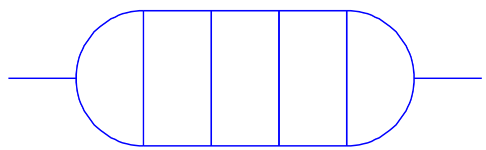




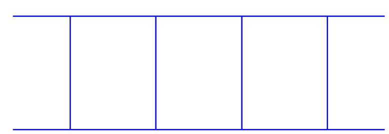


(∗) two-point-functions with 3, 4 and 5 loops, among them a 4-loop graph with an UV subdivergence, terms up to order ϵ for 3-loop and 4-loop graphs

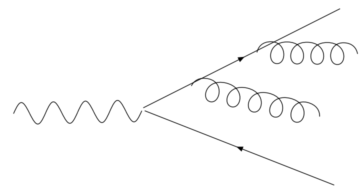




- the planar massless 3-loop 4-point-function with on-shell legs
  [V.A.Smirnov, May 03]



Phase space integrals



integration over scaled invariants

$x_1 = s_{12}/q^2, x_2 = s_{13}/q^2, x_3 = s_{23}/q^2, x_4 = s_{14}/q^2, x_5 = s_{24}/q^2, x_6 = s_{34}/q^2$

example $1 \to 4$ phase space:

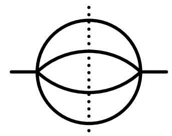
$$\int d\Phi^{(4)} \, |M_4|^2 \quad = \quad C_\Gamma^{(4)} \int \prod_{j=1}^{6} dx_j \, \delta(1 - \sum_{i=1}^{6} x_i) \, \left[\lambda(x_1 x_6, x_2 x_5, x_3 x_4)\right]^{\frac{D-5}{2}} \, |M_4|^2$$

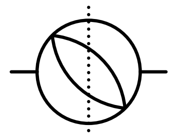
$$|M_4|^2 \quad \sim \quad \frac{\mathcal{P}_1(x_i,\epsilon)}{x_2^2 (x_2 + x_4 + x_6)^2} + \frac{\mathcal{P}_2(x_i,\epsilon)}{(x_2 + x_4 + x_6)(x_3 + x_5 + x_6) x_4 x_5} + \ldots$$

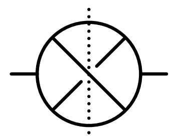
$$\lambda(x,y,z) \quad = \quad 2(xy + xz + yz) - (x^2 + y^2 + z^2)$$

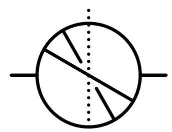
$\lambda \sim$ Gram determinant $\Delta_4 = \det(G)$, $G_{ij} = 2 p_i \cdot p_j \to$ nonlinear constraint application: master integrals for $e^+e^- \to 2\,$jets:

[A. Gehrmann-De Ridder, T. Gehrmann, G.H.]

$R_4 =$  $= \int d\Phi^{(4)} = P_4$

$R_6 =$  $= \int d\Phi^{(4)} \frac{1}{s_{134} s_{234}}$

$R_{8,a} =$  $= \int d\Phi^{(4)} \frac{1}{s_{13} s_{23} s_{14} s_{24}}$

$R_{8,b} =$  $= \int d\Phi^{(4)} \frac{1}{s_{13} s_{134} s_{23} s_{234}}$

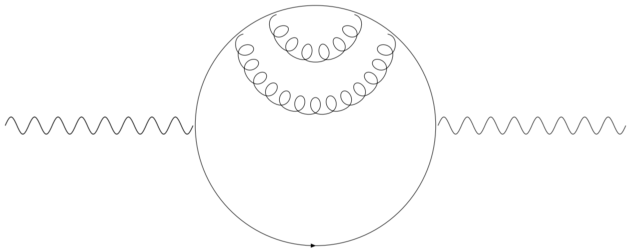
$$
\begin{aligned}
R_4 &= S_\Gamma (q^2)^{2-2\epsilon} \left[0.08335 + 0.8195\,\epsilon + 4.141\epsilon^2 + 14.40\,\epsilon^3 + 38.88\,\epsilon^4 + \mathcal{O}(\epsilon^5)\right] \\
R_6 &= S_\Gamma (q^2)^{-2\epsilon} \left[0.64498 + 7.0423\epsilon + 40.507\epsilon^2 + \mathcal{O}(\epsilon^3)\right] \\
R_{8,a} &= S_\Gamma (q^2)^{-2-2\epsilon} \left[\frac{5.0003}{\epsilon^4} - \frac{0.0013}{\epsilon^3} - \frac{65.832}{\epsilon^2} - \frac{151.53}{\epsilon} + 37.552 + \mathcal{O}(\epsilon)\right] \\
R_{8,b} &= S_\Gamma (q^2)^{-2-2\epsilon} \left[\frac{0.74986}{\epsilon^4} - \frac{0.00009}{\epsilon^3} - \frac{14.001}{\epsilon^2} - \frac{52.911}{\epsilon} - 99.031 + \mathcal{O}(\epsilon)\right]
\end{aligned}
$$

agreement of numerical and analytical results better than 1%

without reduction to master integrals: [T.Binoth, G.H., hep-ph/0402265]

split into topologies ($T_1 \ldots T_8$)

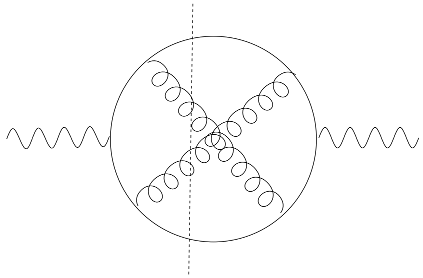
simple topology:



takes $\sim$ 1 hour for precision of 0.1% (Pentium IV 2.2 GHz)

most complicated topology:

$\sim$ 9 hours for precision of 0.1%



$1 \to 5$ phase space: 9 independent invariants

$$\int d\Phi^{(5)} \quad = \quad C_\Gamma^{(5)} \int \prod_{j=1}^{10} dx_j \, \delta(1 - \sum_{i=1}^{10} x_i) \, [-\Delta_5(\vec{x})]^{\frac{D-6}{2}} \, \Theta(-\Delta_5)$$

simplest example: 5-particle phase space volume

analytically: [A. Gehrmann-De Ridder, T. Gehrmann, G.H.]

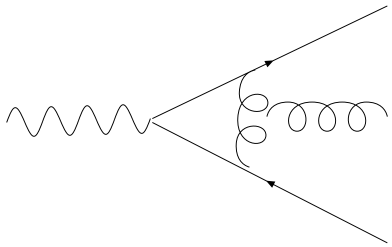
$$\begin{aligned} \int d\Phi^{(5)} &= (4\pi)^{4\epsilon-7} \frac{\Gamma(1-\epsilon)^5}{2\Gamma(4-4\epsilon)\Gamma(5-5\epsilon)} \\ &= \frac{(4\pi)^{4\epsilon-7}}{\Gamma(1-2\epsilon)\Gamma(2-2\epsilon)} \Big[0.0034722 + 0.0546875\epsilon \\ &\quad +0.443355\epsilon^2 + 2.47433\epsilon^3 + 10.7292\epsilon^4 + \mathcal{O}(\epsilon^5) \Big] \end{aligned}$$

sector decomposition:

$$\begin{aligned} \int d\Phi^{(5)} &= \frac{(4\pi)^{4\epsilon-7}}{\Gamma(1-2\epsilon)\Gamma(2-2\epsilon)} \Big[0.00347 + 0.05469\epsilon \\ &\quad +0.44336\epsilon^2 + 2.47424\epsilon^3 + 10.7283\epsilon^4 + \mathcal{O}(\epsilon^5) \Big] \end{aligned}$$

accuracy 0.5%, CPU time 10 minutes

One loop + single unresolved combined:



- most complicated one-loop structure occurring in $e^+e^- \to 2\,\text{jets}$:
  box graph with one off-shell leg
- can be expressed by Hypergeometric functions ${}_2F_1(1, -\epsilon, 1-\epsilon; x_i/x_j)$
  $(x_1 = s_{12}/q^2, x_2 = s_{13}/q^2, x_3 = s_{23}/q^2)$
- use parameter representation of ${}_2F_1$

$${}_2F_1(1, -\epsilon, 1-\epsilon; z) = \frac{\Gamma(1-\epsilon)}{\Gamma(-\epsilon)} \int_0^1 dt\, t^{-1-\epsilon}(1-tz)^{-1}$$

- combine with 3-particle phase space

$\int d\phi_3 = \int dx_1 dx_2 dx_3\, \delta(1 - x_1 - x_2 - x_3)\,(x_1\, x_2\, x_3)^{-\epsilon}$

$\to$ 4-dimensional parameter integral $\to$ sector decomposition

$e^+e^- \to 3\,\text{jets}$: pentagons with one off-shell leg

$$
\begin{aligned}
I &= -\Gamma(3+\epsilon)\int dz_i\,\delta(1-\sum_{i=1}^{5} z_i)\,\mathcal{F}^{3+\epsilon}\\
-\mathcal{F} &= s_{12}\,z_1 z_5 + s_{23}\,z_1(z_3+z_4+z_5) + s_{13}\,z_1(z_1+z_2)\\
&\quad + s_{14}\,z_5(z_1+z_2+z_3) + s_{24}\,z_1(z_4+z_5) + s_{34}\,(z_1+z_2)(z_4+z_5)
\end{aligned}
$$

sector decomposition $\Rightarrow$

$$
I = \sum_{\alpha=0}^{2} P_\alpha/\epsilon^\alpha = \sum_{\alpha=0}^{2} 1/\epsilon^\alpha \int_0^1 \prod_{i=1}^{4-\alpha} dt_i \mathcal{G}(t_i, s_{12}, \ldots, s_{34})\ ,\ \lim_{t_i\to 0} \mathcal{G} \neq 0
$$

insert into 4-particle phase space

$\Rightarrow$ proceed with decomposition in $s_{12}, \ldots, s_{34} \sim x_1, \ldots, x_6$

Structure of the program

Input: number N of propagators resp. invariants, integrand

$\Downarrow$

loop/phase space subroutine

$\Downarrow$

SECDEC $\rightarrow$ poles manifestly factorized

$\Downarrow$

subtractions

$\Downarrow$

Laurent series in ϵ

$\Downarrow$

Creation of FORTRAN files

$\Downarrow$

Numerical integration of finite functions (BASES, S. Kawabata)

Subtractions:

$$
\begin{aligned}
&\int_0^1 dx\, x^{-1+\kappa\epsilon} \mathcal{F}(x,y) \\
&= \frac{1}{\kappa\epsilon} \int_0^1 dx\, \mathcal{F}(x,y)\, \delta(x) + \int_0^1 dx\, x^{-1+\kappa\epsilon} \left[\mathcal{F}(x,y) - \mathcal{F}(0,y)\right] \\
&= \frac{1}{\kappa\epsilon} \int_0^1 dx\, \mathcal{F}(x,y)\, \delta(x) + \sum_{n=0}^{\infty} \frac{(\kappa\epsilon)^n}{n!} \int_0^1 dx\, \left[\frac{\ln^n(x)}{x}\right]_+ \mathcal{F}(x,y)
\end{aligned}
$$

$\longrightarrow$ same procedure for all x_i

$\longrightarrow$ Laurent series

$$
\mathcal{I} = \sum_{k=-a}^{b} \epsilon^k\, C_k \; + \; \mathcal{O}(\epsilon^{b+1})
$$

$\longrightarrow$ inclusion of measurement function straightforward !

Summary and outlook

- Automated sector decomposition algorithm: powerful method to isolate overlapping infrared poles and to calculate numerically
  - multi-loop integrals
  - one-loop + real emission combined
  - double real radiation at NNLO
- advantages of the method:
  - robust and multi-purpose
  - double real emission:
    - no need to establish a subtraction scheme and to integrate analytically over complicated subtraction terms
    - inclusion of measurement function no problem → differential
    - generalisation to other processes than e^+e^- annihilation feasible
- drawbacks:
  - loop integrals with more than one scale: only Euclidean points so far
  - generates large number of (finite) functions