Heavy Quark Perturbative Fragmentation Function at NNLO

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Heavy Quark Fragmentation

Let $\mathcal{Q}$ be a heavy quark. Consider the production of a $\mathcal{Q}$-flavored hadron $H_{\mathcal{Q}}$ in a hard scattering process.

Example: B-meson production in e^+e^-.

Such a process is characterized by two very different scales:

$$Q >> m$$

From the factorization theorem, the energy spectrum for $H_{\mathcal{Q}}$ reads:

$$\frac{d\sigma_H}{dz}(Q,m,z) = \frac{d\sigma_{\mathcal{Q}}}{dz}(Q,m,z) \otimes D^{\rm n.p.}_{\mathcal{Q}\to H}(z)$$

where:

- z - the energy of the observed hadron ($0 \leq z \leq 1$).
- $d\sigma_{\mathcal{Q}}(Q,m,z)$ - describes the perturbative production of massive quark $\mathcal{Q}$.
- $D^{\rm n.p.}_{\mathcal{Q}\to H}(z)$ - describes the transition $\mathcal{Q} \to H_{\mathcal{Q}}$ at scale $\sim m$.

Let's look closer at $d\sigma_{\mathcal{Q}}(Q, m, z)$:

For $Q >> m$ that function contains large logs:

$$\alpha_s^n \ln^k \left(\frac{m^2}{Q^2} \right) \ ; \ k \leq n$$

to all orders in α_s!

One resums classes of such logs by writing:

$$d\sigma_{\mathcal{Q}}(Q, m, z) = \sum_a \widehat{d\sigma}_a(Q, \mu, z) \otimes D_{a \to \mathcal{Q}}(\mu, m, z)$$

where:

- $\widehat{d\sigma}_a$ - usual coefficient function for producing parton a,
- $D_{a \to \mathcal{Q}}(\mu, m, z)$ - Perturbative Fragmentation Function,
- PFF is a process independent solution of the DGLAP equation.

- PFF satisfy the initial condition:

$$D_{a\to\mathcal{Q}}(\mu=\mu_0,m,z) = D^{\rm ini}_{a\to\mathcal{Q}}(\mu_0,m,z)$$

Note: if $\mu_0 \sim m$, then $D^{\rm ini}$ can be computed perturbatively:

$$\begin{aligned} D^{\rm ini}_{a\to\mathcal{Q}}(\mu_0,m,z) &= \delta_{a\mathcal{Q}}\delta(1-z) + \frac{\alpha_s}{2\pi}d^{(1)}_{a\to\mathcal{Q}} + \left(\frac{\alpha_s}{2\pi}\right)^2 d^{(2)}_{a\to\mathcal{Q}} + \dots \\ &= \mathrm{LL} \quad + \quad \mathrm{NLL} \quad + \quad \mathrm{NNLL} \quad + \dots \end{aligned}$$

- $d^{(1)}$ - computed by Mele and Nason (1991)
- We have evaluated $d^{(2)}_{a\to\mathcal{Q}}$ for $a=\mathcal{Q},\ \overline{\mathcal{Q}},\ q,\bar{q}$.
- The case $a=\mathrm{gluon}$ in progress.

How to compute $D^{\rm ini}$?

Recall: It is a process independent (but prescription dependent) quantity.

Approach 1: $D^{\rm ini}$ can be extracted from any process from the relation:

$$d\sigma^{\rm f.o.}_{\mathcal{Q}}(Q,m,z) = \sum_a \widehat{d\sigma}^{\overline{\rm MS}}_a(Q,\mu_0,z) \otimes D^{\rm ini}_{a\to\mathcal{Q}}(\mu_0,m,z) + \mathcal{O}(m^2/Q^2)$$

where:

- $d\sigma_{\mathcal{Q}}(Q,m,z)$ - F.O. distribution for a particle with mass m,
- $\widehat{d\sigma}^{\overline{\rm MS}}_a(Q,\mu_0,z)$ - for a particle with zero mass. Note: the collinear divergences are subtracted in $\overline{\rm MS}$ scheme,

  However: Such derivation is impractical beyond NLO.

  Remark: if $D^{\rm ini}$ is known, then one can obtain $d\sigma^{\rm f.o.}(m)$ from $\widehat{d\sigma}^{\overline{\rm MS}}$ up to power corrections m^2/Q^2.

Approach 2: A process independent derivation of $D^{\rm ini}$

Previously applied to NLO by Keller and Laenen; Cacciari and Catani.

From the factorization of short- and long-distance physics, one can write:

$$\begin{aligned} d\sigma(m,Q,z) &= \widetilde{d\sigma}(Q,\mu_0,z) \otimes \widetilde{D}^H(\mu_0,m,z) \\ d\sigma(Q,\epsilon,z) &= \widetilde{d\sigma}(Q,\mu_0,z) \otimes \widetilde{D}^L(\mu_0,\epsilon,z) \end{aligned}$$

Where:

- $d\sigma(m,Q,z)$ - finite as long as $m > 0$
- $d\sigma(Q,\epsilon,z) = \widehat{d\sigma}_b^{\overline{\rm MS}}(Q,\mu_0,z) \otimes \Gamma_{ba}^{\overline{\rm MS}}(\epsilon,\mu_0,z)$
- $\widetilde{d\sigma}(Q,\mu_0,z)$ - describes radiation at large transverse momentum $>>$ m. Therefore is the same for both $m \neq 0$ and $m = 0$ case.
- $\widetilde{D}^{L,H}$ - radiation at low transverse momentum from massless (massive) particle.

One can combine the previous results to get:

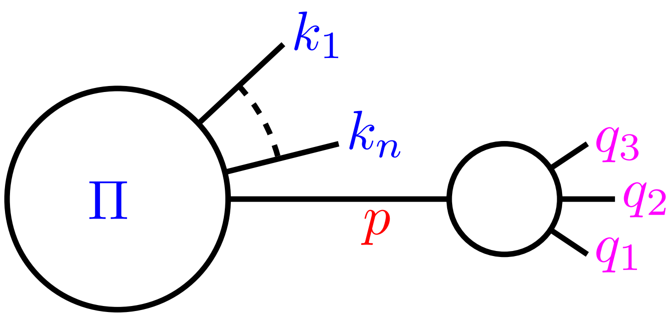
$$\widetilde{D}^H = \widetilde{D}^L \otimes \left(\Gamma^{\overline{\rm MS}}\right)^{-1} \otimes D^{\rm ini}$$

Note: The above equation contains only process independent quantities. Therefore to compute the initial condition one needs the factors $\widetilde{D}^{L,H}$.

Recall: Both $\widetilde{D}^{L,H}$ describe radiation of transverse energy at $p_T \sim m$.

In the limit of small p_T the collinear kinematics dominates.

Therefore: Consider a hard-scattering process Π where particles with momenta $k_1, \dots, k_n, q_1, q_2, q_3$ are produced.



The momenta $q_{1,2,3}$ are collinear i.e. $q_1 + q_2 + q_3 = p + \mathcal{O}(q_T)$

We work in physical gauge $A^\mu n_\mu = 0$. In such case no contribution from interference diagrams. As a result the collinear splitting effectively decouples from the rest of the process.

To derive $\widetilde{D}^{L,H}$ one also uses the factorization of both matrix elements and phase space in the collinear limit :

$$|M^{(n+3)}(k_1,\ldots,k_n,q_1,q_2,q_3)|^2 = |M^{(n+1)}(k_1,\ldots,k_n,p)|^2 W(q_1,q_2,q_3)$$

and

$$\mathrm{dPS}^{(n+3)}(k_1,\ldots,k_n,q_1,q_2,q_3) = \mathrm{dPS}^{(n+1)}(k_1,\ldots,k_n,p)\,\mathrm{d}\Phi^{\mathrm{coll}}(q_2,q_3)$$

The functions $\widetilde{D}^{L,H}$ can now be obtained from integration of the factor W over the momenta of the unobserved collinear partons, i.e. $\mathrm{d}\Phi^{\mathrm{coll}}(q_2,q_3)$.

Evaluation of the function W.

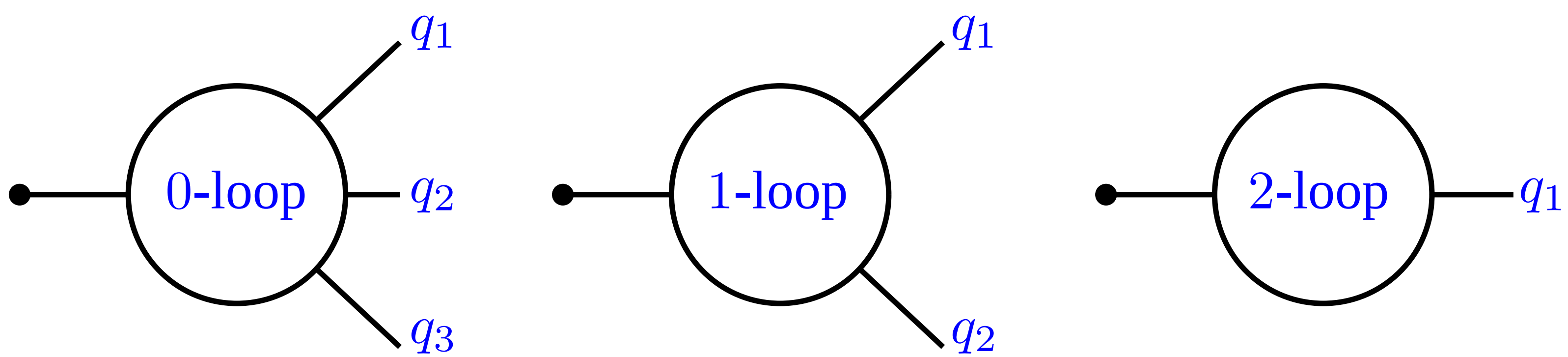
We are interested in the spin-averaged case. Then, for:

- final heavy quark $\mathcal{Q}$ with momentum q_1, and
- decaying fermion of any flavor,

we have:

$$W \sim \mathrm{Tr}\left[\, \not{n} V \right]$$

where V is a matrix in spinor indexes; it is obtained from the squares of the following three types of diagrams (evaluated in a physical gauge):



The functions W are often referred to as "splitting" functions. For the tree-level zero mass case they have been derived by Campbell and Glover; Catani and Grazzini.

We generalized their results in order to include the mass m of the heavy quark $\mathcal{Q}$.

Comments:

We have $W \sim \mathrm{Tr}\,[\,\not{n}V\,]$:

- n is the light-like gauge vector (it also enters the Sudakov parametrization),
- $\mathrm{Tr}\,[\,\not{n}\ldots\,]$ is a projector that extracts the leading behavior of the tensor V in the collinear limit. The form of that projector follows from power counting arguments in the collinear limit.

To obtain the functions $\widetilde{D}^{L,H}$ one has to integrate W over the momenta of the collinear particles:

- the tree-level case:

$$\widetilde{D}^{L,H}(\mu_0, m, z) \sim \int^{\mu_0} [dq_2][dq_3] W^{(\mathrm{tree})} \delta[1 - z - (nq_2) - (nq_3)]$$

- The one-loop case:

$$\widetilde{D}^{L,H}(\mu_0, m, z) \sim \int^{\mu_0} [dq_2] W^{(1-\mathrm{loop})} \delta[1 - z - (nq_2)]$$

- The two-loop case is $\sim \delta(1-z)$. The (ϵ-dependent) constant can be fixed from the fermion number conservation condition:

$$\int_0^1 dz \left(D^{\mathrm{ini}}_{\mathcal{Q}/\mathcal{Q}}(z) - D^{\mathrm{ini}}_{\overline{\mathcal{Q}}/Q} \right) = 1.$$

  We use that condition at order $\mathcal{O}(\alpha_s^2)$ but, as a check, we have evaluated the pure virtual contributions at order $\mathcal{O}(\alpha_s)$.

Practical issues in the evaluation of $\widetilde{D}^{L,H}$.

- Many integrals have to be evaluated in the case $m = 0$ since $\widetilde{D}^{L}_{a\to b}$ with $a, b = \mathcal{Q}, \overline{\mathcal{Q}}, q, \overline{q}, g$.
- Asymmetric real and virtual integrations (the upper limits are respectively μ_0 and ∞).

One can solve both problems if one extends to infinity the upper limit of the integration over the transverse momenta.

One gains a lot from that:

- $\widetilde{D}^{L}_{a\to b} = \delta_{ab}\delta(1-z)$ to all orders in α_s.

  Reason: all integrals for the case $m = 0$ are scaleless i.e. vanish.
- Now complete symmetry between real and virtual integrations .

  Therefore one can evaluate them simultaneously .

Final working expressions for $D^{\rm ini}$:

Using the previous results we get:

$$D^{\rm ini}_{a\to\mathcal{Q}}(\mu_0,m,z) = \sum_b \Gamma^{\overline{\rm MS}}_{ab}(\mu_0,\epsilon,z)\otimes \widetilde{D}^H_{b\to\mathcal{Q}}(\mu_0,m,\epsilon,z)$$

and the contributions to $\widetilde{D}^H_{b\to\mathcal{Q}}(\mu_0,m,\epsilon,z)$ are of the form:

- Tree-level:

$$\widetilde{D}^H \sim \int^\infty [dq_2][dq_3]W^{(\rm tree)}\delta[1-z-(nq_2)-(nq_3)]$$

- One-loop:

$$\widetilde{D}^H \sim \int^\infty [dq_2][dk]W^{(1-\rm loop)}\delta[1-z-(nq_2)]$$

Above, $q_{2,3}$ are real momenta, while k is virtual.

Various components to PFF and the participating sub-processes at tree-level:

I. $D^{\rm ini}_{\mathcal{Q}\to\mathcal{Q}}$:

- $\mathcal{Q} \to \mathcal{Q} + g + g$,
- $\mathcal{Q} \to \mathcal{Q} + q + \overline{q}$,
- $\mathcal{Q} \to \mathcal{Q} + \mathcal{Q} + \overline{\mathcal{Q}}$.

II. $D^{\rm ini}_{\overline{\mathcal{Q}}\to\mathcal{Q}}$:

- $\overline{\mathcal{Q}} \to \mathcal{Q} + \overline{\mathcal{Q}} + \overline{\mathcal{Q}}$.

III. $D^{\rm ini}_{q(\overline{q})\to\mathcal{Q}}$:

- $q(\overline{q}) \to \mathcal{Q} + \overline{\mathcal{Q}} + q(\overline{q})$.

IV. $D^{\rm ini}_{g\to\mathcal{Q}}$ (In progress) :

- $g \to \mathcal{Q} + \overline{\mathcal{Q}} + g$.

Evaluation of the integrals

- We evaluate separately the contributions with different masses in the final state.
- Within each group, we apply IBP identities.
- Algebraically reduce the contributions from all diagrams to ~ 20 Master Integrals.
- MI's contain a single scale (m) and are functions of a single variable (z).
- We also rederived the known results at order α_s to order ϵ
- perform the usual UV renormalization
- Perform collinear subtraction

General form of the result:

$$D_a^{\rm ini}\left(\mu_0, m, z\right) = \sum_{n=0}\left(\frac{\alpha_s(\mu_0)}{2\pi}\right)^n d_a^{(n)}\left(z, \frac{\mu_0}{m}\right).$$

Since PFF is a solution of the DGLAP equation, the general form of the log-terms is known:

$$\begin{aligned} d_a^{(2)}\left(z, \frac{\mu_0}{m}\right) &= \left[\frac{1}{2}P_{ba}^{(0)} \otimes P_{\mathcal{Q}b}^{(0)}(z) + \pi\beta_0 P_{\mathcal{Q}a}^{(0)}(z)\right] \ln^2\left(\frac{\mu_0^2}{m^2}\right) \\ &+ \left[P_{\mathcal{Q}a}^{(1)}(z) + P_{ba}^{(0)} \otimes C_b^{(1)}(z) + 2\pi\beta_0 C_a^{(1)}(z)\right] \ln\left(\frac{\mu_0^2}{m^2}\right) \\ &+ C_a^{(2)}(z) \end{aligned}$$

Note $C_a^{(1)}$ and $C_a^{(2)}$ are integration constants for the DGLAP. Must be evaluated explicitly. They control respectively the NLL and NNLL logs.

Properties of the results:

- The result satisfy the fermion number conservation condition (by construction):

$$\int_0^1 dz \left(D^{\rm ini}_{\mathcal{Q}/\mathcal{Q}}(z) - D^{\rm ini}_{\overline{\mathcal{Q}}/Q} \right) = 1.$$

  However the two functions are not separately integrable due to $1/z$ terms.

- The Non-Singlet term $\sim n_f$ coincides with the known result in the large β_0-limit (Cacciari and Gardi).

- The limit $z \to 1$: our result reproduces the NLL "soft-logs" $\alpha_s^2 \ln^k(1-z)/(1-z)$ for $k = 3, 2, 1$ and $m = \mu_0$ from the known result for the soft-gluon resumed initial condition (Cacciari and Catani). Also from our result one can extract the constant $H^{(2)}$ that is needed to promote the soft-resummation to NNLL accuracy.

Applications for $D^{\rm ini}$: (It is a universal result ...)

I. Fixed order results: one can compute spectra of massive fermions at NNLO (and up to terms $\mathcal{O}(m^2/Q^2)$) from massless calculations. Great simplification beyond NLO! Examples:

- b-spectrum in top decay. Currently known at NLO. Important for precise top-mass measurement at LHC.
- electron spectrum in μ-decay. Currently being measured with high precision.

II. All order resummations of quasi-collinear logs $\ln(m^2/Q^2)$ with NNLL accuracy.

- For that the time-like splitting functions will be needed.
- Particularly relevant for the extraction of the non-perturbative fragmentation function (like $b \rightarrow B$) for bottom and charm. Recall recent analysis of the b-production at the Tevatron.
- Our result will help to significantly reduce the theoretical uncertainty in the extraction of NP fragmentation function.

Conclusions:

- We have calculated all fermion initiated components of the initial condition for the perturbative fragmentation function at order α_s^2 (NNLO), thus extending the PFF formalism to NNLL level.
- We followed a process independent approach for the computation of D^{ini} that exploits the universal behavior of the collinear radiation.
- To evaluate the two-loop integrals we made use of "hot" techniques for multi-loop calculations: IBP, reduction to MI's and their solving.
- I discussed the general properties of our result as well as the checks with partial results existing in the literature.
- I discussed some of the many possible applications of our result:
- Fixed order spectra for heavy particles from massless results,
- Resummations of quasi-collinear logs with NNLL accuracy and accurate extraction of non-perturbative fragmentation function from data.