Presented at KITP "Low Dimensional Electron Systems" workshop, June 10, 2009

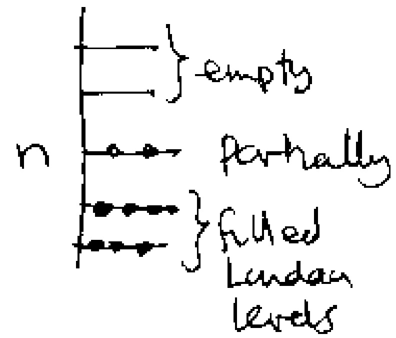
Hall viscosity and intrinsic metric of incompressible FQHE "Hall Fluids"

F. D. M. Haldane,
Princeton University

preprint is
arXiv:0906.1894

- Hall fluids with and without isotropy
- essential "fluid mechanics" of Hall fluids
- Intrinsic electric dipole moment and the stress anomaly at edges between Hall fluids
- Relation to the small-q limit of the "guiding center structure factor"
- Calculation by adiabatic variation of periodic BC's

Effective Hamiltonian for FQHE



$$[R_i^a, R_i^b] = i\ell^2\epsilon^{ab}$$

$$H^{\text{eff}} = \underbrace{\int \frac{d^2q\,\ell^2}{2\pi} V_c(\underline{q}) \left(f_n(\underline{q})\right)^2 \sum_{i<j} e^{i\vec{q}\cdot(\vec{R}_i - \vec{R}_j)}}_{\text{leading term}}$$

form factor

guiding centers.

(+ Landau-level mixing terms
Three body terms, etc.)

$$V_c(\underline{q}) = \frac{e^2}{4\pi\epsilon_0\epsilon|q|_c} \qquad |q|_c^2 = g_c^{ab} q_a q_b$$

Coulomb metric

$$f_n(q) = L_n\left(\frac{|q|_m^2 \ell^2}{2}\right) e^{-\frac{1}{4}|q|_m^2\ell^2} \qquad |q|_m^2 = g_m^{ab} q_a q_b$$

galilean metric

IN GENERAL (E.G. WITH "Tilted field")

$$g_m^{ab} \neq g_c^{ab} \longleftarrow \text{from dielectric tensor}$$

g_m^{ab} $\longleftarrow$ From Galilean mass tensor

NOT ROTATIONALLY INVARIANT!

- MOST TREATMENTS OF FQHE ASSUME ROTATIONAL INVARIANCE WITH some metric g^{ab} $\left(g^{ab}_c = g^{ab}_m = g^{ab} \right)$

- generator of rotations of guiding centers

$$L^z(g) = g_{ab}\Lambda^{ab}$$

$\left(\Lambda^{ab} = \Lambda^{ba}\right)$

$$\boxed{\Lambda^{ab} = \frac{1}{4\ell^2}\sum_i \{R^a_i, R^b_i\}}^{*} \leftarrow$$

(Three) generators of linear transformations of guiding centers

SO(2,1)
"Lorentz group"
Lie Algebra

$$[\Lambda^{ab}, \Lambda^{cd}] = \frac{i}{2}\left(\epsilon^{ac}\Lambda^{bd} + \epsilon^{ad}\Lambda^{bc} + \epsilon^{bc}\Lambda^{ad} + \epsilon^{bd}\Lambda^{ac}\right)$$

Casemir is $(-\det\Lambda) = \left(\frac{\Lambda^{11}+\Lambda^{22}}{2}\right)^2 + \left(\Lambda^{12}\right)^2 - \left(\frac{\Lambda^{11}-\Lambda^{22}}{2}\right)^2$

* will play a key role in the general treatment without rotational invariance

- If rotational invariance is present

$$H^{\text{eff}}_{(g)} = \sum_m \underset{\substack{\uparrow\\ \text{pseudopotentials}}}{V_m} \int \frac{d^2q\,\ell^2}{2\pi}\, \overset{\substack{\text{Laguerre}\\ \text{polynomial}\\ \downarrow}}{L_m}(|q|_g^2\ell^2)\, e^{-\frac{1}{2}|q|_g^2\ell^2} \sum_{i<j} e^{i\vec{q}\cdot(\vec{R}_i-\vec{R}_j)}$$

metric dependence

- For $V_m = \begin{cases} V_1 & m=1 \\ 0 & \text{otherwise} \end{cases}$

  and $\nu = N/N_{orb} = 1/3$

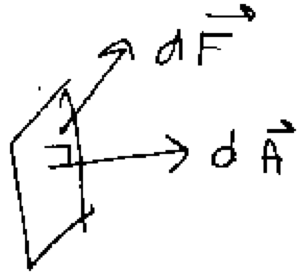
$$H^{\text{eff}}(g)\,|\Psi^L_{1/3}(g)\rangle = 0$$

Laughlin $\nu = 1/3$ state

WHICH IS A CONTINUOUS FUNCTION OF g!

The metric $\underline{\underline{g}}$ is the ~~The~~ "hidden" variational parameter of the Laughlin state!

Review: Stress in isotropic fluids



$$dF^i = \sigma^{ij} dA^j$$

Force exerted by fluid on one side of a 'cut' along surface element $d\vec{A}$ on fluid on other side

$$\boxed{\sigma^{ij} = \sigma^{ji} \quad (\text{isotropic fluid})}$$

Linear response

$$\sigma^{ij} = p\delta^{ij} + \eta^{ijkl}\nabla^k v^l + O(v^2)$$

where the velocity field is defined by

$$J^i = \rho v^i$$ particle current

ρ ← particle density

$$\boxed{\partial_t \rho + \nabla^i J^i = 0}$$

Momentum density

$\pi^i = m\rho v^i$ if Galilean invariant

$$\boxed{\partial_t \Pi^i + \nabla^j \sigma^{ij} = 0}$$ Momentum conservation

"Hall Viscosity" (isotropic fluids)

stress tensor

$$\sigma^{ij} = \sigma^{ji} = p\delta^{ij} + \tfrac{1}{2}\eta^{ijkl}\left(\nabla^k v^l + \nabla^l v^k\right)$$

Symmetric

hydrostatic pressure
(only term when $\vec{V}(r)=0$
(velocity field)

$$\eta^{\overset{\text{sym}}{\frown}{ij}\,\overset{\text{sym}}{\frown}{kl}} = \eta^{jikl} = \eta^{ijlk}$$

In 2D isotropic fluid; Three terms

$$\eta^{ijkl} = \underbrace{\eta_L\,\delta^{ij}\delta^{kl}}_{\text{longitudinal}} + \eta_s\left(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} - \delta^{ij}\delta^{kl}\right)$$

Shear visosity
(traceless)

$$\sum_i \eta_s^{iijk} = 0$$

$$+\tfrac{1}{2}\eta^{(A)}\left(\epsilon^{ik}\delta^{jl} + \epsilon^{il}\delta^{jk} + \epsilon^{jk}\delta^{il} + \epsilon^{jl}\delta^{ik}\right)$$

" Hall " or dissipationless Viscosity
ODD UNDER TIME REVERSAL

η_L, η_S
even under Time reversal
dissipative

ηA
odd under T-reversal
non dissipative

- In an isotropic 2D fluid with "spinning" particles

$$\eta_A = \frac{1}{2}\underset{\substack{\uparrow\\ \text{particle}\\ \text{density}}}{\rho}\ \underset{\substack{\uparrow\\ \text{"intrinsic" 2D}\\ \text{angular momentum}\\ \text{per particle}}}{\overline{\ell^z}}$$

- In the Integer quantum Hall effect $\overline{\ell^z} = (n+\frac{1}{2})\hbar$ for particles in the Landau level with kinetic energy $(n+\frac{1}{2})\hbar\omega_c$ (Avron, Seiler + Zograf, 1995)

ℓ = 'magnetic length'

$\ell^2 = \left|\frac{\hbar}{eB}\right|$

$$\eta_A = \frac{\hbar}{4\pi\ell^2}\sum_n \nu_n s_n$$

$s_n = n + \frac{1}{2}$

ν_n = (integer) filling of Landau level n.

$$\nu_{tot} = \sum_n \nu_n$$

$$\boxed{\sigma^{xy} = \frac{\nu e^2}{2\pi\hbar} \ \text{Hall conductance}}$$

Read (2009) extends This to FQHE.

$$\Psi_{\text{Laughlin}} = \underbrace{\prod_{i<j} (z_i - z_j)^m}_{} \prod_i e^{-1/4 |z_i|^2/\ell^2}$$

degree of polynomial in each variable z_i is

$$N_\Phi = m(N-1)$$

In general $N_\Phi = \nu^{-1} N - \mathcal{S} \leftarrow$ "Shift"

Read identifies $\boxed{\overline{\ell^z} = \frac{1}{2} \mathcal{S} \hbar}$

For $\nu = 1/m$ Laughlin in LLL: $\mathcal{S} = m, \ \nu = 1/m$

$$\boxed{\eta_A = \frac{\hbar}{4\pi\ell^2} \times \frac{1}{2}}$$

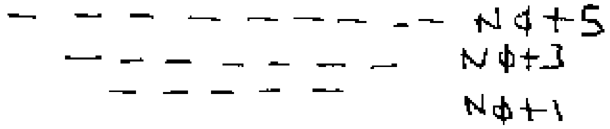
In Second Landau level, Read's formula gives

$$\eta_A = \frac{\hbar}{4\pi\ell^2}\left(\underbrace{\frac{1}{2} + \frac{1}{2} + \frac{1}{m}\frac{(m+2)}{2}}_{3/2+1/m}\right)$$

(FQHE states in Landau level n:

$$N_\Phi + 2n = \nu^{-1}N - S$$
)

argument based on Spherical geometry



$N\phi + 2n + 1$ states in Landau level with index n on sphere

- We can simplify these formulas with a form that seperates the "Hall viscosity contributions from cyclotron motion and guiding center dynamics.

N_{orb} = # of orbitals in landau level

$$\boxed{N = \nu N_{orb} + \gamma_0}$$

γ is "shift" per flux, not shift per particle

$$\eta_A = \frac{\hbar}{4\pi\ell^2}\Big(\underbrace{\sum_n \nu_n s_n}_{\text{Contribution from cyclotron motion}} + \underbrace{\gamma_0}_{\text{contribution from guiding centers}} \Big)$$

$s_n = n + \tfrac{1}{2}$
ν_n = filling factor of Landau level n.

- $\gamma = 0$ if guiding center dynamics is frozen (Integer QHE)
- $\gamma \rightarrow -\gamma$ under particle-hole transformation of partially-occupied Landau levels.

- How do these formulas change for a non-rotationally invariant FQHE state where ~~[illegible]~~ $\overline{L^z}$ makes no sense?

- In this case there is **no** fundamental metric. We **cannot** use a formula like

$$\sigma^{ab} = \frac{1}{A}\left\langle \frac{\delta H(g)}{\delta g_{ab}} \right\rangle$$

There is no "g_{ab}"

if rotational symmetry is absent

that computes the stress tensor as a functional derivative with respect to the metric

(used by Avron et al, Read, etc)

Back to basics

- Set up a formulation of Stress without invoking a "metric".

- Need a covariant formulation that distinguishes spatial coordinates

  $r^a \leftarrow$ upper indices $a = 1, 2$

  and reciprocal space quantities

  $q_a \leftarrow$ lower indices $a = 1, 2$

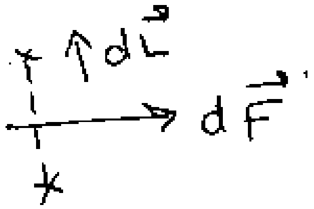
$$\underline{q} \cdot \underline{r} = q_a r^a$$ make sense without any metric

but $\underline{r} \cdot \underline{r} = g_{ab} r^a r^b$ does not!

① Stress tensor is $\sigma^{a}_{\ b}(r)$

NOT symmetric!
(indices dont match)

$$\underset{\substack{\uparrow\\ \text{lower}}}{dF_a} = \underset{\substack{\uparrow\\ \text{lower}}}{\sigma_a^{\ \overset{\text{upper}}{b}}}\ \underset{\substack{\uparrow\\ \text{antisymmetric}\\ \text{symbol}}}{\epsilon_{bc}}\, dL^c$$



Cut along line segment dL^a

② $P = \sigma^{a}_{\ a}(r) = 0$ (Scalar)

Hall fluids have $\underline{\underline{\text{NO}}}$ internal pressure!

- We also specialize to a Hall fluid subject to a non-translationally invariant slowly varying potential $V(\underline{r})$

$$H = \int d^2r \, \underbrace{h_0(r)}_{\substack{\text{Short range} \\ \text{part of interaction} \\ \text{That gives rise to} \\ \text{Incompressibility}}} + V(r) \, \underbrace{\rho(r)}_{\substack{\text{guiding center} \\ \text{density}}}$$

$\uparrow$ a slowly varying local potential (includes effect of long range part of Coulomb's force)

- $$\frac{\partial \rho}{\partial t} + \nabla_a \overset{\text{Particle cu}}{J^a} = 0$$

$$\frac{\partial \Pi_a}{\partial t} + \nabla_b \underset{\text{stress}}{\sigma^b_a} + \rho \nabla_a V = 0$$

- $\sigma^a_b \nabla_a J^b = 0$ dissipationless condition

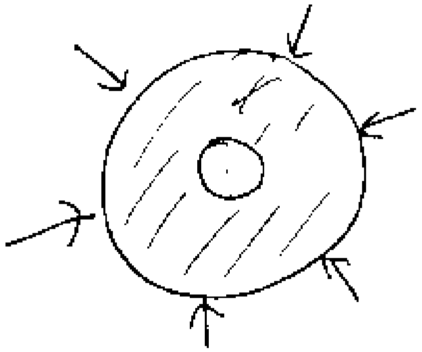
- $P = \sigma^a_a = 0$ (no pressure!)

$$\Pi_a(r) = \int \frac{d^2q}{(2\pi)^2} \tilde{\Pi}_a(q) \, e^{-i\vec{q}\cdot r}$$

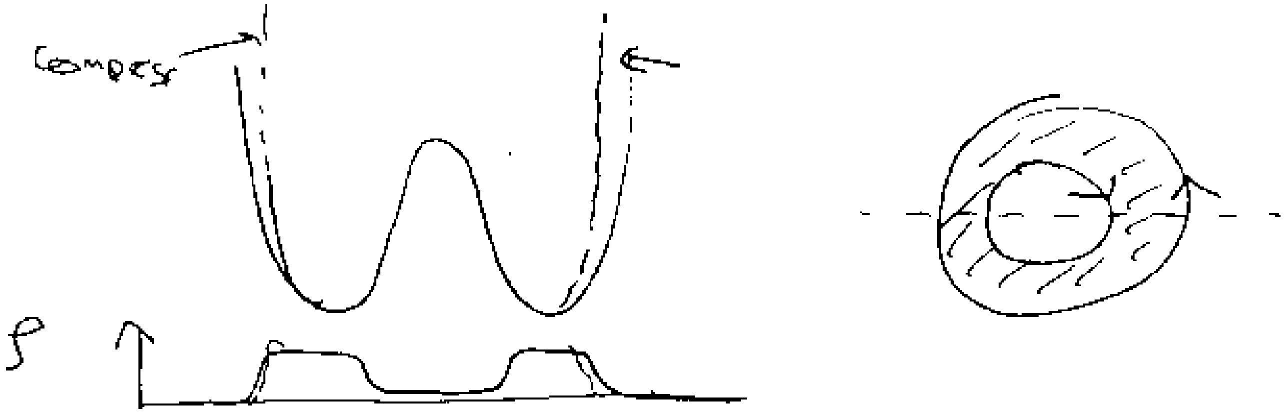
$$\tilde{\Pi}_a(\underline{q}) = \sum_i e^{i\frac{q}{2}\cdot R_i} \frac{\hbar}{\ell^2} \epsilon_{ab} R_i^b \, e^{i\frac{q}{2}\cdot R_c}$$

Momentum density

hydrostatic pressure



- Cavity inside a simple incompressible fluid is compressed when pressure is applied to exterior surface.
- Not true for Hall fluid



When the exterior wall is compressed, the edge current increases to oppose the applied force.

No Force is transmitted to internal boundary!

② Velocity field in Hall fluid is drift velocity

$$v^a = \frac{\ell^2}{\hbar}\epsilon^{ab}\nabla_b V$$

$$\nabla_a v^a = 0$$ (incompressible)

$$J^a = \rho v^a \qquad \rho = \frac{\nu}{2\pi\ell^2}$$

$$\boxed{J^a = \frac{\nu}{2\pi\hbar}\epsilon^{ab}\nabla_b V}$$ linear response

③ Stress is response to non-uniform electric field $\nabla_a\nabla_b V$

$$\boxed{\sigma^a{}_b = \frac{\gamma}{2\pi}\epsilon_{be}\Gamma_A^{\underbrace{ae}\,\underbrace{cd}}\nabla_c\nabla_d V}$$

symmetric

antisymmetric symbol

Symmetric because P = 0

④ dissipation → free

$$\rightarrow \quad \sigma^a_b \nabla_a J^b = 0$$

or $$\Gamma_A^{abcd} (\nabla_a \nabla_b V)(\nabla_c \nabla_d V) = 0$$

$$\rightarrow \boxed{\Gamma_A^{abcd} = - \Gamma_A^{cdab}}$$

or

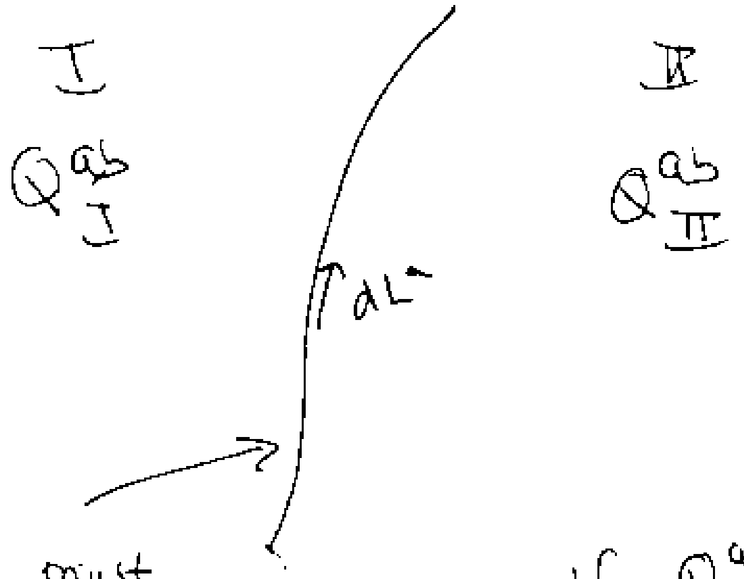
$$\frac{1}{2\pi} \Gamma_A^{abcd} = \frac{1}{2} \left(\epsilon^{ac} Q^{bd} + \epsilon^{ad} Q^{bc} + \epsilon^{bc} Q^{ad} + \epsilon^{bd} Q^{ac} \right)$$

↑

a symmetric tensor

What is its physical significance?

Boundary between two Hall fluids



← uniform in a Hall fluid phase, discontinuous across boundary

Static boundary, must be on equipotential

$$dL^a \nabla_a V = 0$$

- If Q^{ab} (and Γ^{abcd} here) is discontinuous, there is a <u>net</u> <u>stress</u> force

$$dF_a = \epsilon_{ae}(\Delta\Gamma^{abcd})\nabla_c\nabla_d V \epsilon_{bf}\, dL^f$$

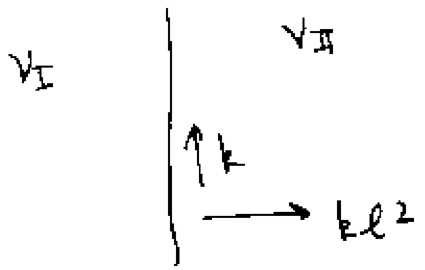
on the boundary.

to balance it, there must be an electric dipole moment

$$\boxed{dP^a = e\Delta Q^{ab}\epsilon_{bc}\, dL^c}$$

$$\boxed{dP^a = e\Delta Q^{ab}\epsilon_{bc}dL^c}$$

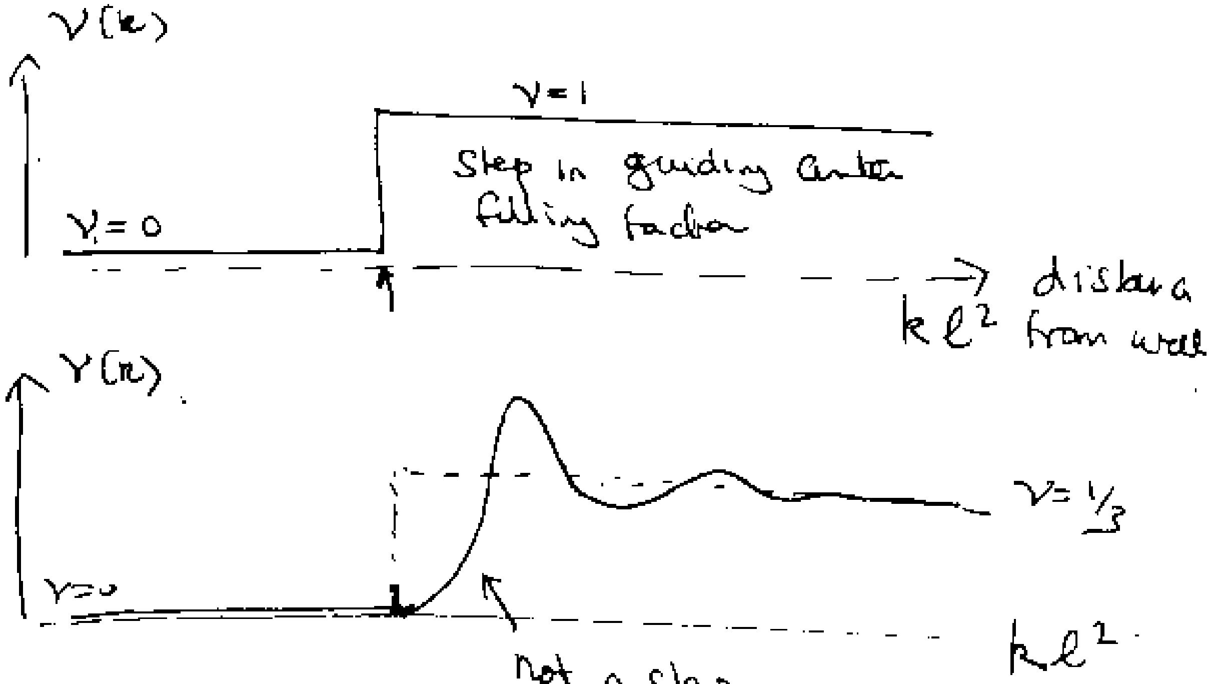
- Intrinsic dipole moment on an (unreconstructed) boundary between two distinct Hall fluids with different Q^{ab} tensors.



momentum along wall = distance from (straight) wall

Integer Hall effect

fractional case



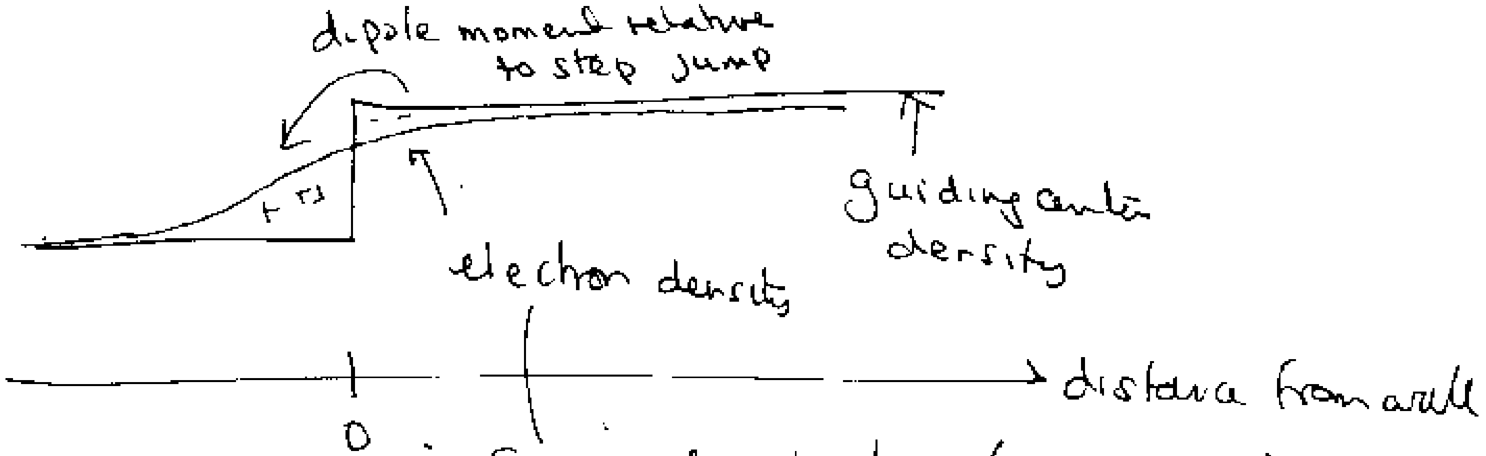
Step in guiding center filling factor

not a step. but a non-trivial profile $\nu(k)$

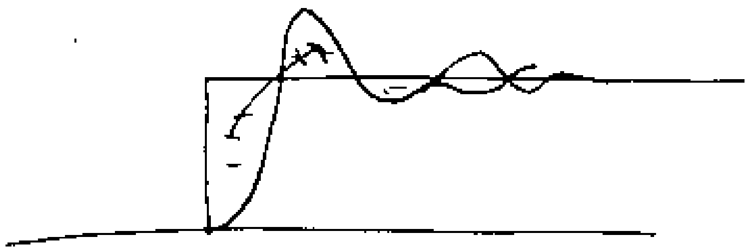
Calculation of dipole moment:

2D electron density (fourier transformed) $= \sum_{i=1}^{N} \sum_{n} f_n(q)\, \delta_{n,n_i}\, e^{i\vec{q}\cdot\vec{R_i}}$

$\uparrow$ landau levels $\quad \uparrow$ form factor $\quad \uparrow$ guiding center



Jump is smeared out by (gaussian) form factor.



Modified Y(K) is also a dipole relative to step!

The two effects simply add!

$$Q^{ab} = \underbrace{-\frac{1}{4\pi\ell^2}\sum_n \nu_n \nabla_q^a \nabla_q^b f_n(q)}_{\text{"cyclotron motion contribution" (Smearing)}} + \underbrace{Q_0^{ab}}_{\text{guiding center contribution}}$$

$$\downarrow$$

$$\frac{1}{4\pi}\sum_n \nu_n s_n \, g_m^{ab}$$

normalized galilean mass tensor.

for galilean invariant Landau levels, as in Avron et al.

Landau basis of guiding center states

$$\hat{n} \cdot \vec{R} \, | k \rangle = k\ell \, | k \rangle$$

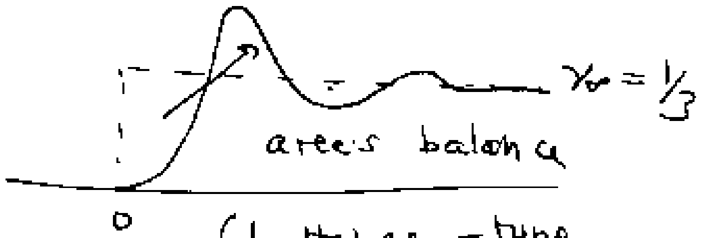
normal to wall

dimensionless

occupation factor $\nu(k)$

(wall is at $\hat{n} \cdot \vec{r} = 0$)

$$\int_0^\infty dk \, (\nu(k) - \nu_\infty) = 0$$



(Luttinger-type sum rule

$$\int_0^\infty \frac{dk}{2\pi} \, k \, (\nu(k) - \nu_\infty) = Q_0^{ab} n_a n_b$$

This is ground state momentum of the translationally-invariant wall.

<u>Cannot change adiabatically</u>

Q_0^{ab} is an adiabatic invariant!

- We can calculate Q_0^{ab} at the edge. How about a bulk calculation?

— Key is the algebra of deformation operators

$$\Lambda^{ab} = \frac{1}{4\ell^2}\sum_i \{R_i^a, R_i^b\}$$

$$= \underset{\uparrow}{\Lambda_0^{ab}} + \frac{1}{8\ell^2}\frac{1}{N}\sum_{ij}\{R_i^a - R_j^a, R_i^b - R_j^b\}$$

Center of mass part (discard)

relative coordinate part.

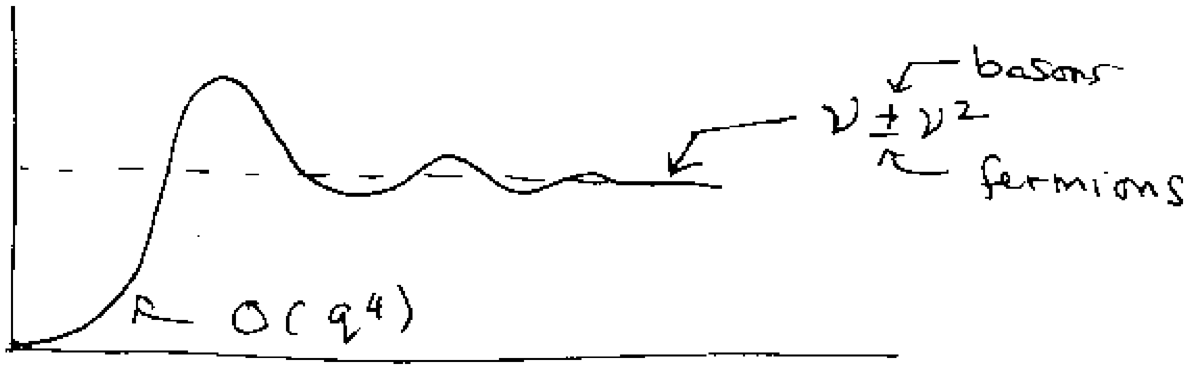
get

$$\boxed{\Gamma_A^{abcd} = \frac{1}{N_{orb}}\langle\Psi_0|\frac{1}{2i}[\Lambda^{ab}, \Lambda^{cd}]\Psi_0\rangle}$$

Another Surprise:

A relation to the guiding-center structure factor

$$\bar{S}(q) = \frac{1}{N_{orb}} \sum_{ij} \langle \Psi_0 | e^{i\vec{q}\cdot R_i - R_j} | \Psi_0 \rangle - \langle \Psi_0 | e^{iq\cdot R_i} | \Psi_0 \rangle \langle \Psi_0 | e^{+iq\cdot R_i} | \Psi_0 \rangle$$



as $q \to 0$ $\quad \bar{S}(q) \to \frac{1}{4}\alpha(q^2\ell^2)^2$ (rotationally invariant states

a fundamental measure of incompressibility.

Girvin MacDonald + Platzman (1985)

Single mode approximation to collective mode

$$|q\rangle = \sum_{i} e^{i\vec{q}\cdot R_i}|\psi_0\rangle$$

$$\Delta E(q) \le \frac{\langle q|H|q\rangle}{\langle q|q\rangle} = \frac{a(q^4)}{\bar{S}(q)} \quad \text{as } q \to 0$$

Excitation energy

gap requires that $S(q) \underset{q\to 0}{\sim} \alpha q^4$

large $\alpha \to$ small gap!

but $\boxed{\alpha \ge |\gamma_0|} \leftarrow Q_0^{ab} = \frac{\gamma_0}{4\pi} g^{ab}$ for a rotationally invariant system.

equality for Laughlin, Moore-Read etc model polynomial states more generally, without Rotational invariance, find

$$\bar{S}(q) \xrightarrow[q\to 0]{} \frac{1}{4} \Gamma_S^{abcd} q_a q_b q_c q_d \ell^4$$

$$\Gamma_S^{abcd} = \frac{1}{N_{orb}} \left(\langle \psi_0 | \tfrac{1}{2} \{ \Lambda^{ab}, \Lambda^{cd} \} | \psi_0 \rangle - \langle \psi_0 | \Lambda^{ab} | \psi_0 \rangle * \langle \psi_0 | \Lambda^{cd} | \psi_0 \rangle \right)$$

$$\boxed{\underbrace{\Gamma_S^{abcd} + i \Gamma_A^{abcd}}_{\text{positive Hermitian}} = \frac{1}{N_{orb}} \left(\langle \psi_0 | \Lambda^{ab} \Lambda^{cd} | \psi_0 \rangle - \langle \psi_0 | \Lambda^{ab} | \psi_0 \rangle \langle \psi_0 | \Lambda^{cd} | \psi_0 \rangle \right)}$$

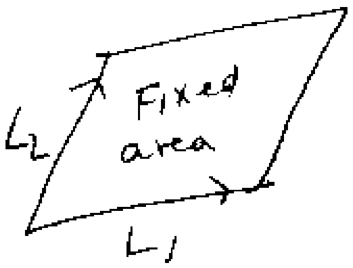
$$\Gamma_S^{abcd} \geq \pm i \, \Gamma_A^{abcd}$$

like a Bogolm'yi Pontryagin bound

Adiabatic variation of Boundary conditions.

$$|\psi_0\rangle \to e^{i \underline{\alpha_{ab}} \Lambda^{ab}} |\psi\rangle$$

3 dimensional parameter Space



$$\left.\begin{array}{l} \left\{\begin{array}{l} \vec{L}_1 \to \vec{L}_1 + x \vec{L}_2 \\ L_2 \to \vec{L}_2 \end{array}\right. \\ \\ \left\{\begin{array}{l} L_1 \to L_1 \\ L_2 \to L_2 + y L_2 \end{array}\right. \\ \\ \left\{\begin{array}{l} L_1 \to e^{z/2} L_1 \\ L_2 \to e^{-z/2} L_2 \end{array}\right. \end{array}\right\}$$

Three distinct deformations of pbc.

(Generalizes modular transformations of Avron, Read for rotationally -invariant systems)

In the 3 dim. space

$$\left\langle \frac{\partial\psi}{\partial x^i} \middle| \frac{\partial\psi}{\partial x^j} \right\rangle = G_{ij} + i F_{ij}$$

Hermitian positive

real symmetric

"metric"

(Fubini-Study) metric

↑ imaginary antisymmetric

Berry curvature

Six parameters

$$\Pi_S^{abcd} = A^{ac}A^{bd} + A^{ad}A^{bc} + B^{ab}B^{cd}$$

Parametrized by two rank-2 symmetric tensors.

A^{ab}, B^{ab} (6 parameters)

F_{12}, F_{23}, F_{31}

↓

$Q_0^{xx}, Q_0^{xy}, Q_0^{xy}$

Π_A^{abca}

Hall viscosity.

Numerics (and exact result for laughlin)
(Moore-Read)

Show $\alpha = |\gamma|$ for Laughlin, moore read etc in <u>Finite Systems</u>

- for finite systems with "realistic" interactions (Coulomb etc) $\alpha > |\gamma|$. IS This a finite size correction so $\lim_{N \to \infty} \alpha - |\gamma| \to 0$ or a real correction to incompressibility due to fluctuations that "dress" the ground state?

Future directions

- dynamics of quantum Hall edges coupled to variations of $V(r,t)$ (beyond The linearized regime)

- numerical studies to see if $S(Q) \propto Q^4$ remains modified by realistic interactions or returns to the minimum value at large sizes.

- $Q^{ab}_\alpha = \frac{\gamma}{4\pi} g^{ab}_\alpha$ <u>defines</u> The incompressibility length scale metric of a Hall fluid. Sign of γ distinguishes hole and particle fluids (Pfaffian vs AntiPfaffian etc).