Pfaffian statistics through the 1D coherent state representation

KITP, February 24, 2009

Alexander Seidel

Collaborators:

D.-H. Lee (Berkeley)

Kun Yang (FSU/NHMFL)

J. Moore (Berkeley)

J.M. Leinaas (Univ. Oslo)

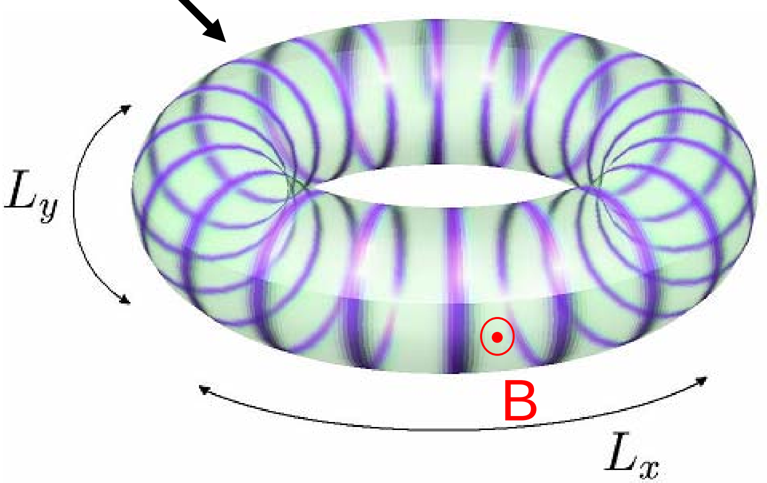
H. Fu (Brown)

Washington University in St. Louis

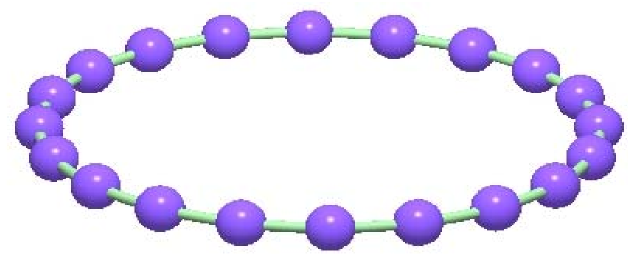
A one-dimensional language for fractional quantum Hall systems

Lowest Landau level:

L flux quanta



L *"lattice"* sites



Wavefunctionslook complicated when expressed in the Landau level basis.
Still, one can construct a simple 1d language for quantum Hall states:

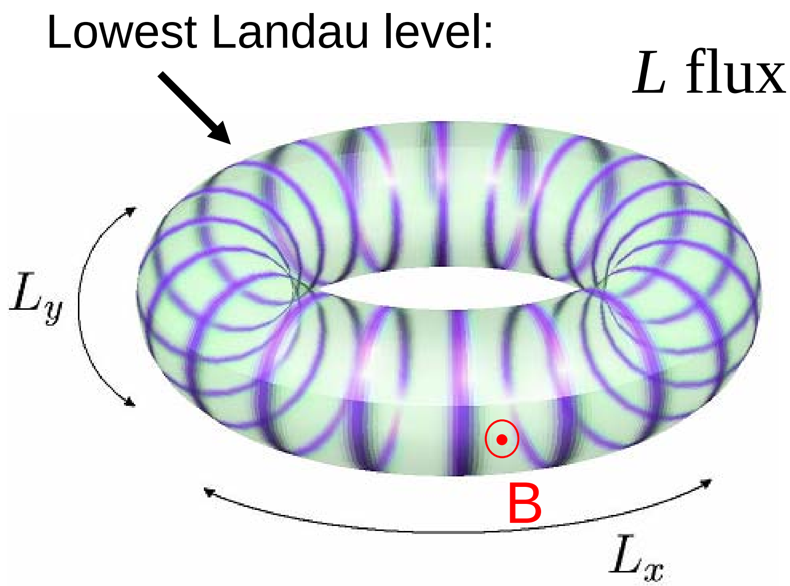
A. S. et. al, PRL 95 '05, PRL 97 '06

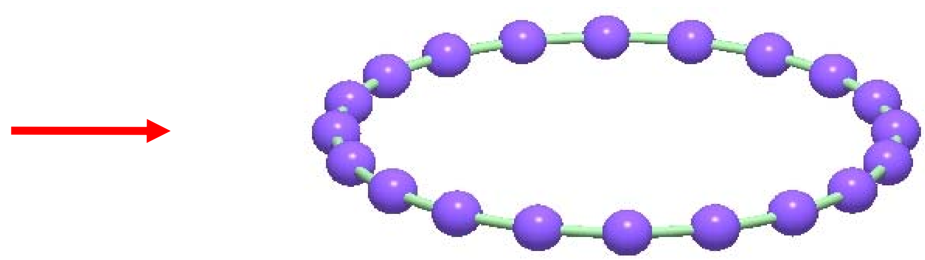
E. Bergholtz et. al, J. Stat. Mech '06, PRB 74 '06

F.D.M. Haldane, talk at 06 March meeting
B.A. Bernevig, F.D.M. Haldane, PRL 100 ‘08

X.-G. Wen, Z. Wang, PRB 77 ’08

A one-dimensional language for fractional quantum Hall systems





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} Thin torus/cylinder limit +adiabatic continuity

F.D.M. Haldane, talk at 06 March meeting
B.A. Bernevig, F.D.M. Haldane, PRL 100 '08

} Jack polynomials

X.-G. Wen, Z. Wang, PRB 77 '08

"pattern of zeros"

What can be (potentially) learned from adiabatic continuity

- ✓ Patterns efficiently encode some defining quantum numbers of underlying quantum Hall states: fractional charges, number of topological sectors …

This talk

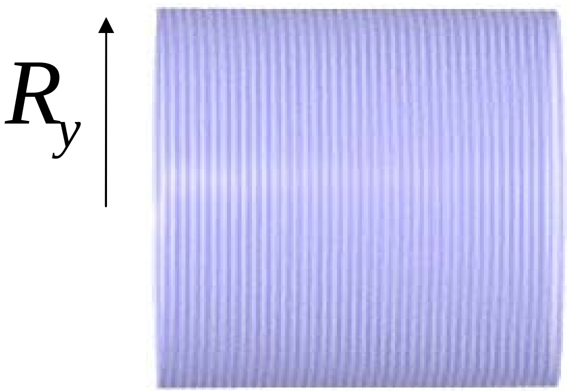
- ❑ Study thin "cylinder version" of critical points/behavior between different quantum Hall phases.

  → e.g. (331)->Pfaffian, A.S., K. Yang, PRL 100, '08 ,
  "non-trivial" thin cylinder limit of Haldane-Rezayi state (unpublished)

- ✓ Provide direct way to extract information about (non-abelian) statistics from wavefunctions. (*)

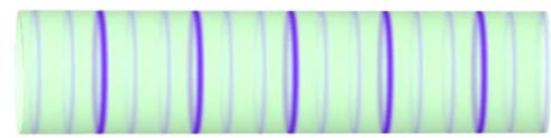
(*) cf. N. Read, PRB 79 ,'09; arXiv:0807.3107

Thin torus/cylinder limit of quantum Hall states



Example: ν=1/3 Laughlin state on cylinder

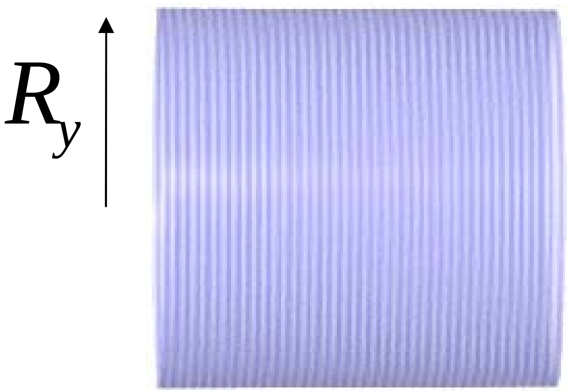
$\longrightarrow$ $1/R_y$

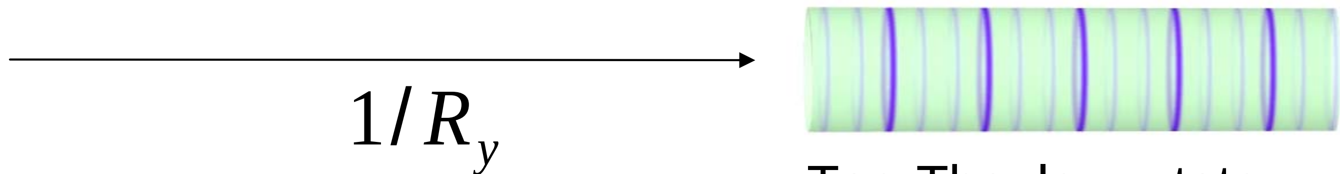


Tao-Thouless state on thin cylinder

Haldane, Rezayi PRB 50 '94

Thin torus/cylinder limit of quantum Hall states





Example: ν=1/3 Laughlin state on cylinder

Tao-Thouless state on thin cylinder

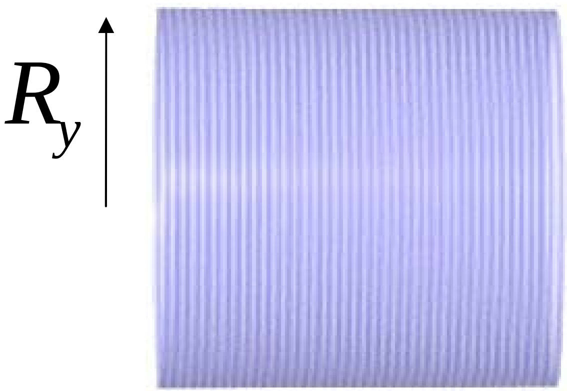
Haldane, Rezayi PRB 50 '94

$$H_{TK} = \int d^2r d^2r' \left[\nabla^2 \delta(r-r')\right] \psi^\dagger(r)\psi^\dagger(r')\psi(r')\psi(r)$$

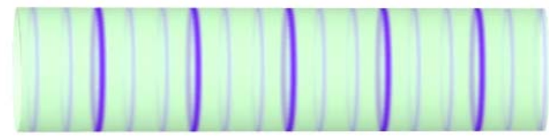
$$\sim \sum_{R,x,y} e^{-(x^2+y^2)/R_y^2}\, c^\dagger_{R-x} c^\dagger_{R+x} c_{R-y} c_{R+y}$$

(expressed in lowest Landau level basis)

Thin torus/cylinder limit of quantum Hall states



$\xrightarrow{\quad 1/R_y \quad}$



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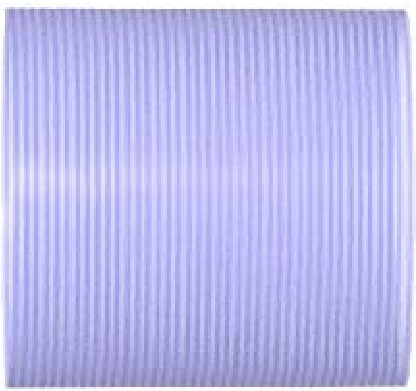
$$\sim \sum_{R,x,y} e^{-(x^2+y^2)/R_y^2}\, c^\dagger_{R-x} c^\dagger_{R+x} c_{R-y} c_{R+y}$$

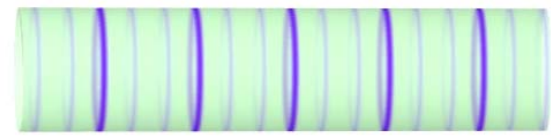
$$\xrightarrow[R_y \to 0]{} \sum_n \left[e^{-\frac{1}{2}R_y^{-2}}\, c^\dagger_n c_n c^\dagger_{n+1} c_{n+1} + e^{-2R_y^{-2}}\, c^\dagger_n c_n c^\dagger_{n+2} c_{n+2}\right]$$

Thin torus/cylinder limit of quantum Hall states

R_y



$1/R_y$



Example: ν=1/3 Laughlin state on cylinder

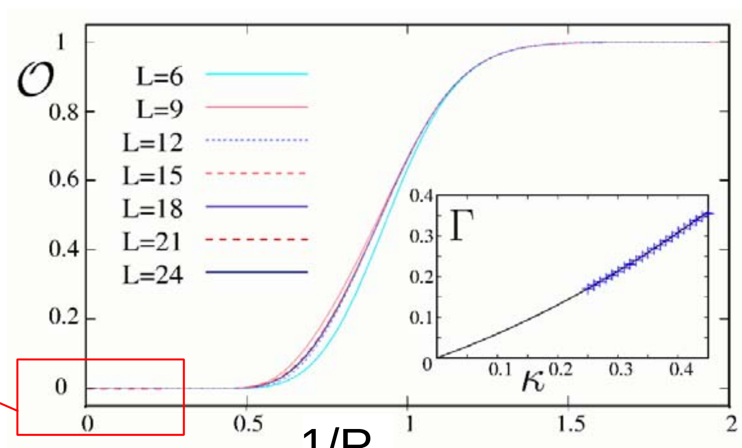
Tao-Thouless state on thin cylinder

Haldane, Rezayi PRB 50 '94

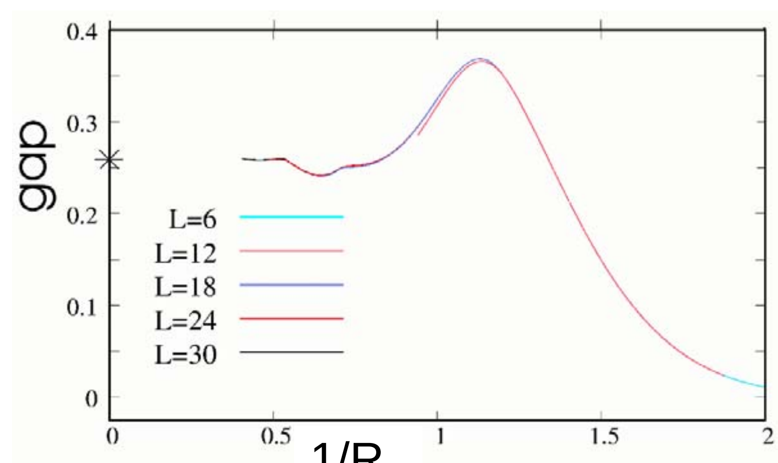
<u>adiabatically connected!</u>

A.S. et al, PRL 2005

O~ exp(-cR_y^2)



$1/R_y$



$1/R_y$

A one-dimensional language for fractional quantum Hall systems

Use thin cylinder limits as labels for the complicated bulk states

ν=1/3Laughlin liquid: →
100100100100100100
010010010010010010
001001001001001001

Laughlin quasi-hole: → 010010010001001001

q=1/3 domain-wall

Laughlin quasi-particle: → 001001001010010010

q=-1/3 anti-domain-wall

- Laughlin quasi-particle as “dressed” domain wall

The ν=1 Pfaffianstate

ν=1 Pfaffian:

$$\Psi = Pf\left[\frac{1}{z_i - z_j}\right]\prod_{(ij)}(z_i - z_j)\ \exp[-\sum_k |z_k|^2/4]$$

(G. Moore, N. Read, Nucl. Phys. B, '91)

The ν=1 Pfaffianstate

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$$\Psi = Pf\left[\frac{1}{z_i - z_j}\right] \prod_{(ij)} (z_i - z_j) \ \exp[-\sum_k |z_k|^2/4]$$

(G. Moore, N. Read, Nucl. Phys. B, '91)

Thin cylinder patterns:

→ 202020202020202020
→ 020202020202020202
→ 111111111111111111

3 ground states!

2-quasi-hole states ↓

0202020202020**1**0202020202020
0202020|1111111111111|0202020
1111111|0202020202020|1111111

A charge 1 excitation in the 2020 grouns state....
... can decay into twocharge 1/2excitations !

(same for the 1111 ground state)

A.S., D.-H. Lee, PRL '06
for ν=1/2: c.f. E. Bergholtz et al. PRB 2006

Pfaffiandegeneracies

Degeneracy of quasi-hole states:

Four-domain-wall states:

0202020|1111111|0202020|1111111|0202020

0202020|111111|020202020|111111|0202020

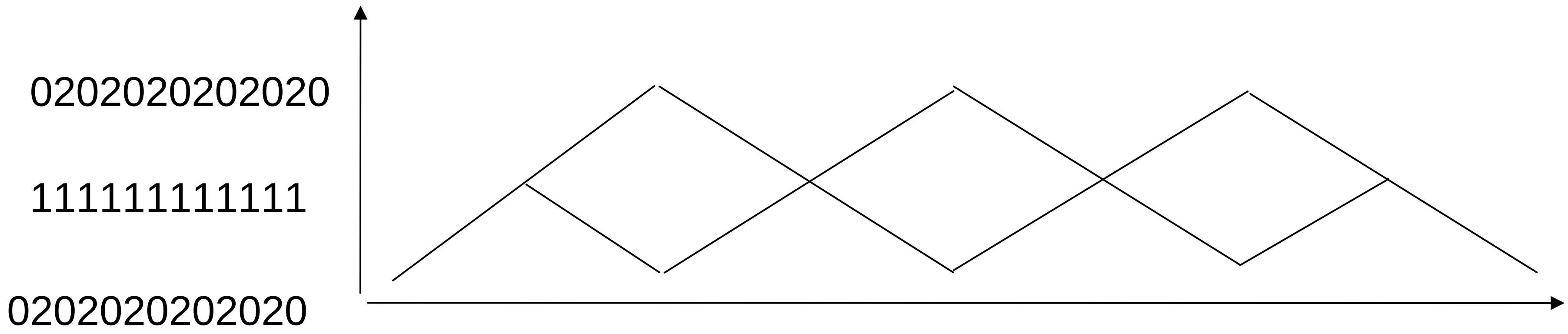
2-fold degeneracy!

Two different out-of-phase "middle strings" are possible

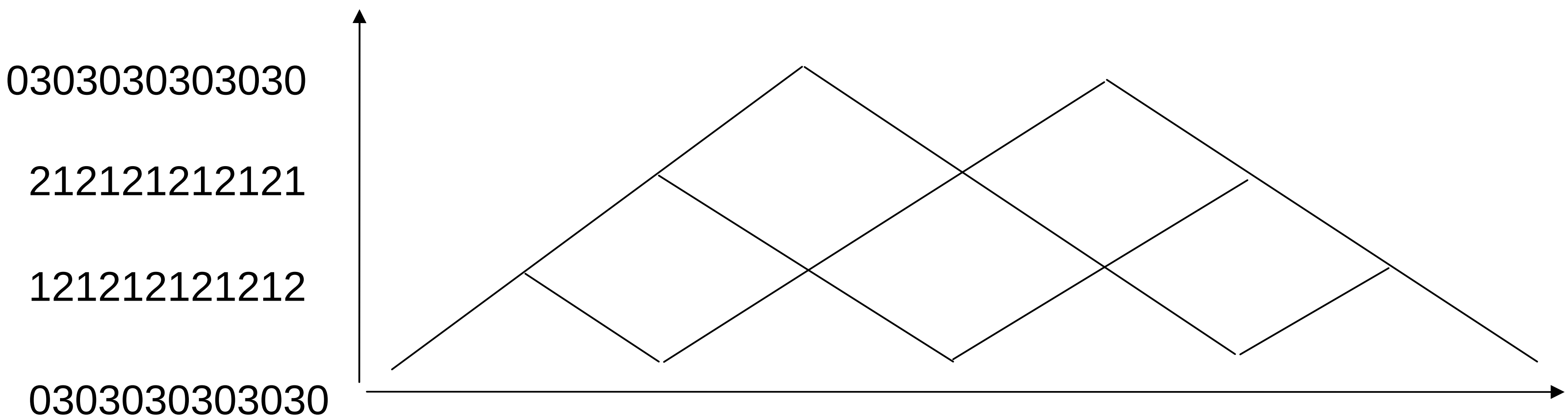
Generalize to 2n hole-type domain-walls (fixing the boundary strings): 2^{n-1} sectors

cf. C. Nayak, F. Wiczek, Nucl .Phys. B '96

Bratteli diagrams out of CDW-Patterns



Bratteli diagram for the Moore-Read

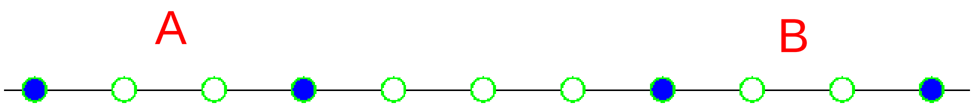


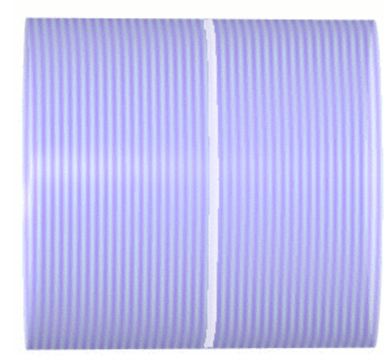
Bratteli diagram for the k=3 Read-Rezayi
cf. J.K. Slingerland, F.A. Bais, Nucl. Phys. B ‘01

More on domain walls+fusion rules: E. Ardonne, arxiv:0809.0389

How to understand braiding statistics in a one-dimensional language?

How to understand braiding statistics in the one-dimensional language: ν=1/m Laughlin state





domain wall position

$|x)$

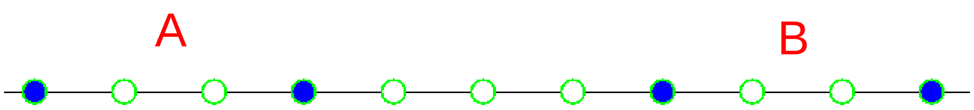
$\hat{S}$

evolve adiabatically

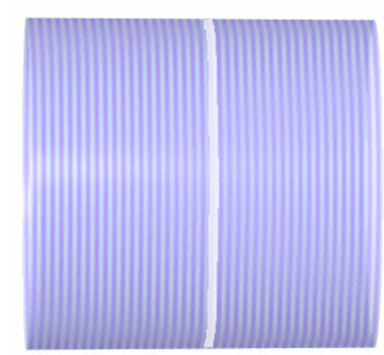
$|x\rangle = \hat{S}|x)$

delocalized Laughlin

How to understand braiding statistics in the one-dimensional language: ν=1/m Laughlin state



domain wall position



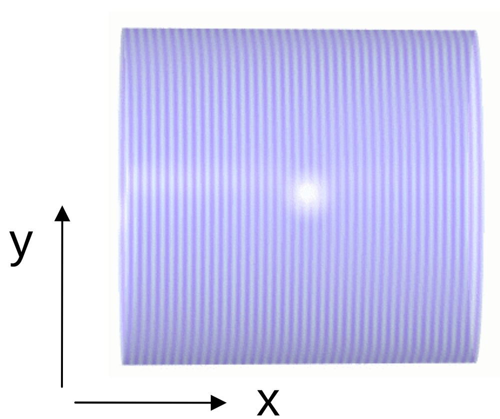
$|x)$ $\xleftrightarrow{\hat{S}}$ $|x\rangle = \hat{S}|x)$

evolve adiabatically

delocalized Laughlin

$$x = (c + mn)/R_y$$

labels m ground state sectors

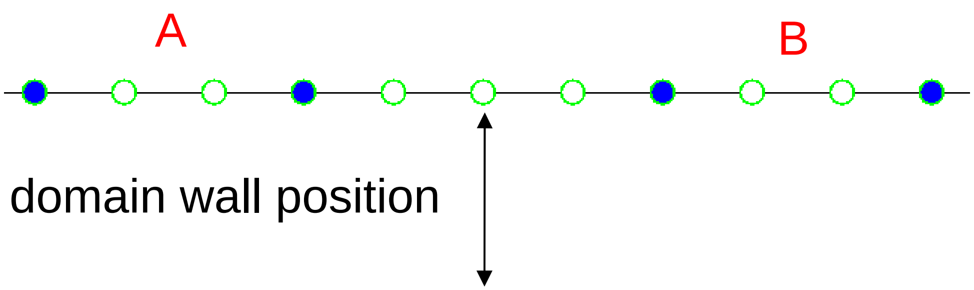


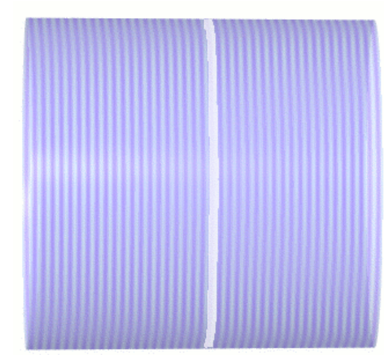
$$|\psi^c_{h_x,h_y}\rangle = \sum_x e^{\frac{i}{m}h_y x - \frac{1}{2m}(x-h_x)^2}|x\rangle$$

Remember: $[x, y] \propto i$

localized Laughlin quasi hole == “coherent state”

How to understand braiding statistics in the one-dimensional language: ν=1/m Laughlin state





$$|x)$$

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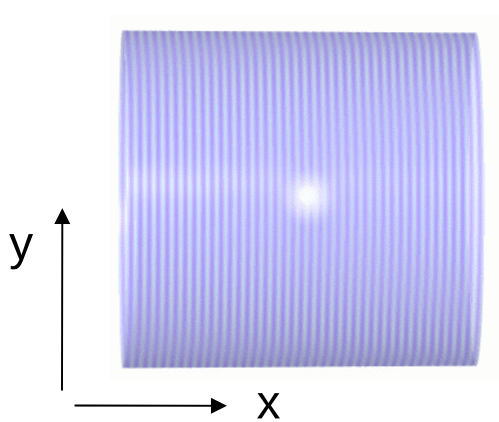


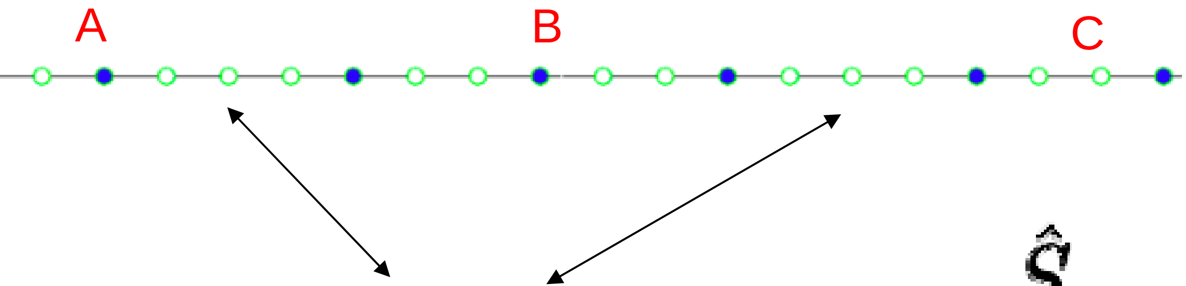
$$|\psi^c_{h_x,h_y}\rangle = \sum_x e^{\frac{i}{m} h_y x - \frac{1}{2m}(x-h_x)^2} |x\rangle$$

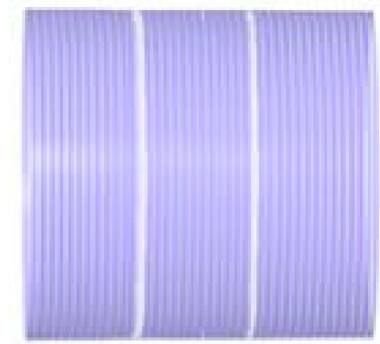
$$\propto (x|\psi^c_{h_x,h_y}\rangle$$

localized Laughlin quasi hole == “coherent state”

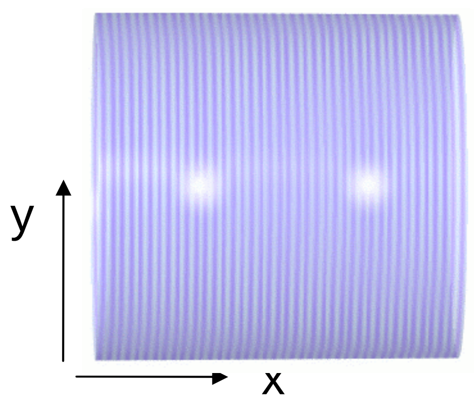
since $(x|x') \propto \delta_{x,x'}$

How to understand braiding statistics in the one-dimensional language: ν=1/m Laughlin state





$$|x_1, x_2) \xleftrightarrow[\text{evolve adiabatically}]{\hat{S}} |x_1, x_2\rangle = \hat{S}|x_1, x_2)$$



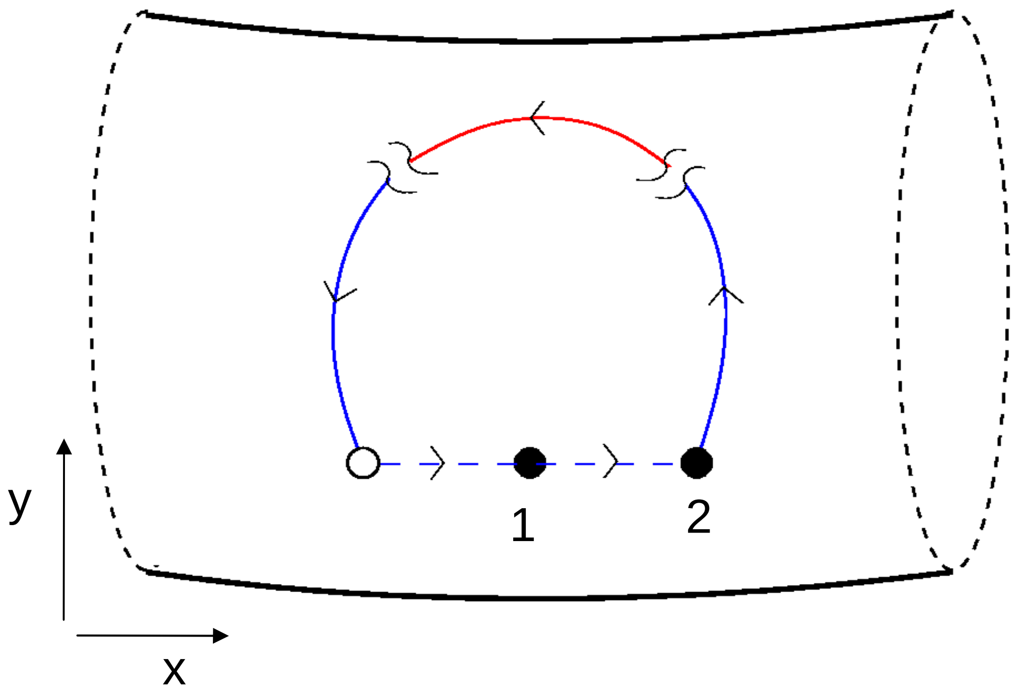
labels m ground state sectors

$$|\psi^{c}_{h_{1x},h_{1y},h_{2x},h_{2y}}\rangle = \sum_{x_1<x_2} e^{\frac{i}{m}h_{1y}x_1 - \frac{1}{2m}(x_1-h_{1x})^2} \times e^{\frac{i}{m}h_{2y}x_2 - \frac{1}{2m}(x_2-h_{2x})^2} |x_1, x_2\rangle$$

$$x_1 = (c + mn_1)/R_y,\ x_2 = (c + 1 + mn_2)/R_y$$

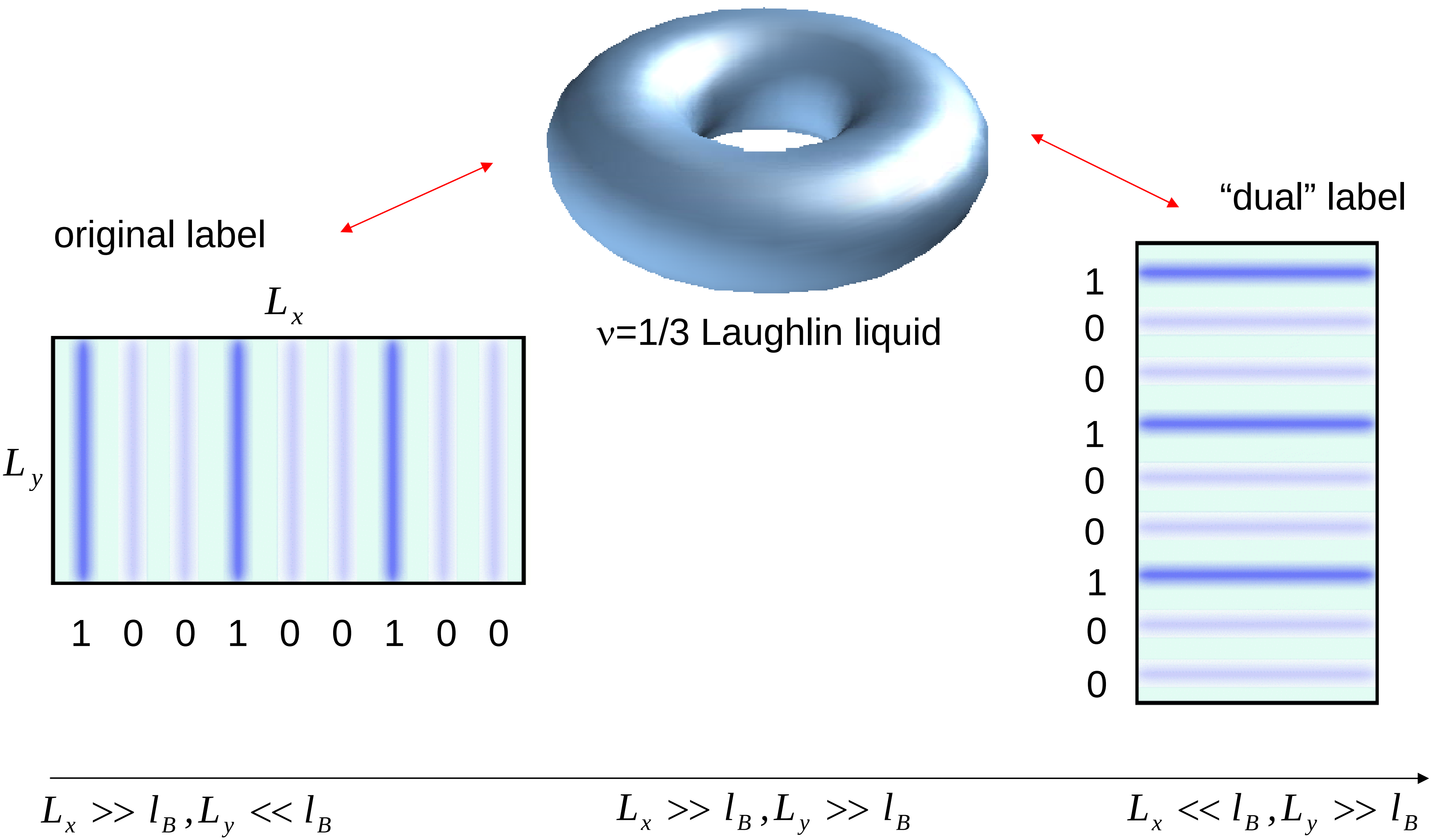
But: only good for $h_{2x} - h_{1x} \ll \ell_b$

What aboutexchange paths ?



$$|h_{2x} - h_{1x}| \gg \ell_b$$ violated !

Duality between thin torus limits (modular invariance)



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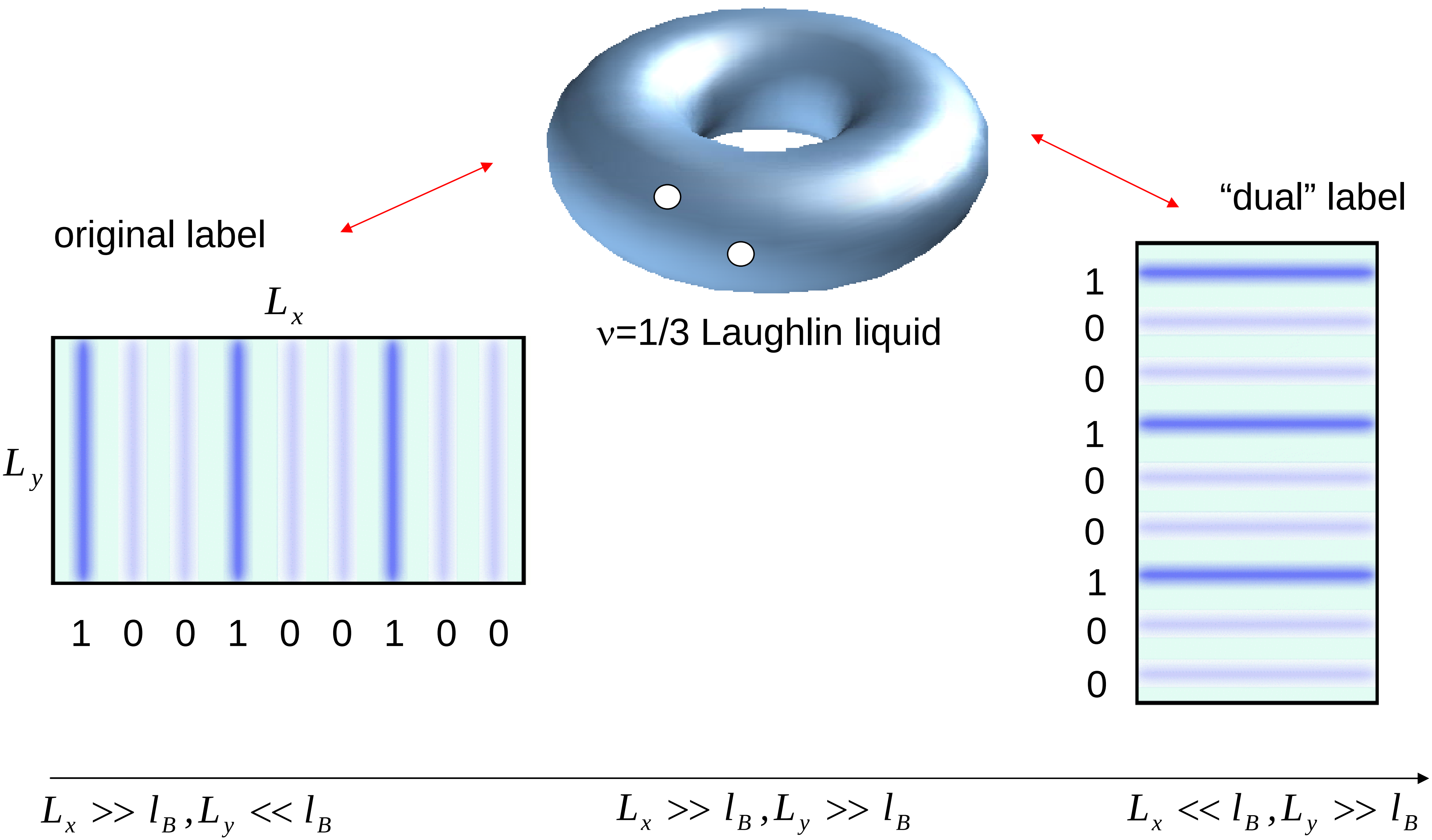


Two hole exchange

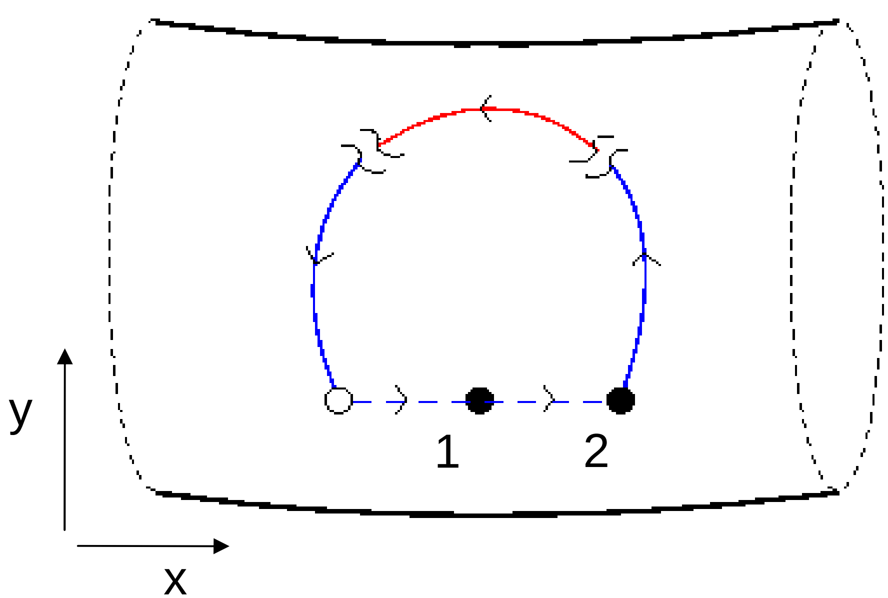
A.S., D.-H.Lee PRB '07

$$|\psi_h\rangle \rightarrow e^{i\gamma}|\psi_h\rangle$$

Berry phase:

$$\gamma = \sum_{\text{segments } c_i} \int_{c_i} \vec{A}\cdot d\vec{s} + \text{"twist"}$$

Berry connection: $\vec{A} = i\langle\psi_h|\nabla_h\psi_h\rangle$



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Berry phase:

$$\gamma = \sum_{\text{segments } c_i} \int_{c_i} \vec{A}\cdot d\vec{s} + \text{"twist"} \quad = \frac{-e}{m} \times \text{flux} \quad + \frac{\pi}{m}$$

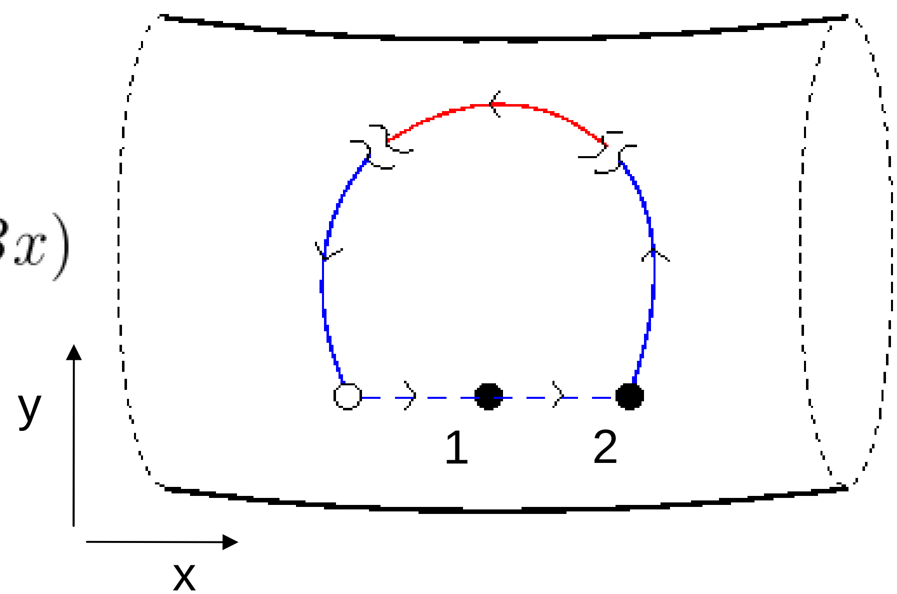
AB-phase

Statistical phase

Berry connection: $\vec{A} = i\langle\psi_h|\nabla_h\psi_h\rangle$

$$\vec{A} = \frac{1}{m}(eBy, 0)$$

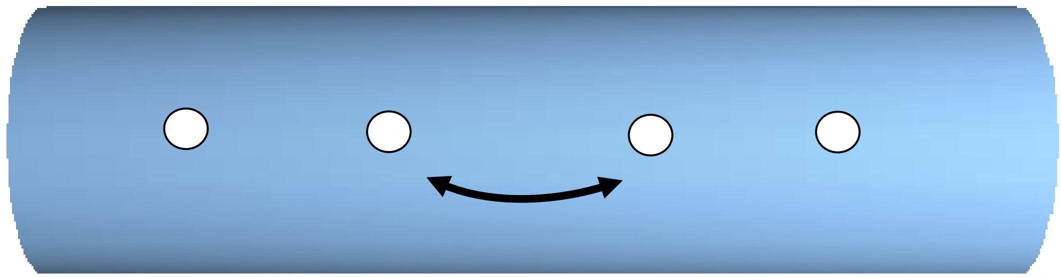
$$\vec{A} = \frac{1}{m}(0, -eBx)$$



$$\vec{A} = \frac{1}{m}(0, -eBx)$$

Pfaffian statistics from a 1d viewpoint

A.S., PRL '08

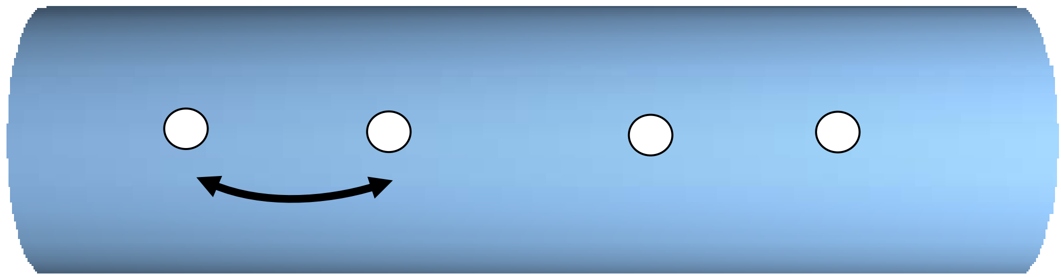


020202020201111111111110202020202011111111111102020202020202

020202020201

Pfaffian statistics from a 1d viewpoint

A.S., PRL '08



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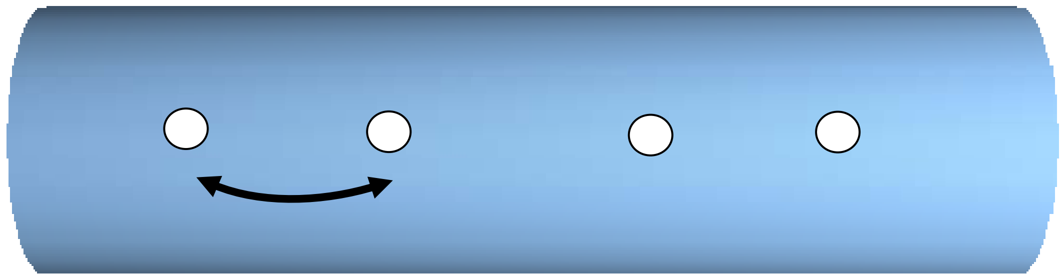
Pfaffian statistics from a 1d viewpoint

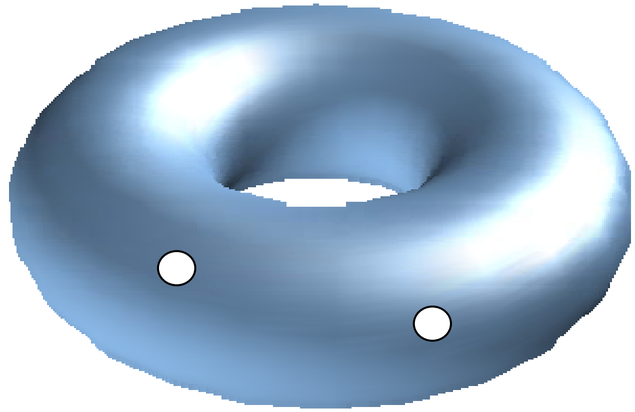
A.S., PRL ’08



02020202020111111111111020202020201111111111102020202020202

In this basis: Every second generator of the braid group is diagonal,
and every <u>other</u> generator in block diagonal with block size 2.

Pfaffian 2-hole states



Sectors:

$c = \begin{cases} 1. & 1111111111111\,020202020202020\,11111111111111 \\ 2. & 11111111111111\,020202020202020\,1111111111111 \\ 3. & 0202020202020\,111111111111111\,02020202020202 \\ 4. & 20202020202020\,111111111111111\,0202020202020 \end{cases}$

$x_2 = (2n_2 - s)/R_y$

1 0

s

$$|\psi(h_1,h_2)\rangle_c = \sum_{x_1,x_2 \text{ in sector } c} \phi(h_1,x_1)\phi(h_2,x_2)|x_1x_2\rangle_c$$

$$\overline{|\psi(h_1,h_2)\rangle}_{\bar{c}} = \sum_{y_1,y_2 \text{ in sector } \bar{c}} \bar{\phi}(h_1,y_1)\bar{\phi}(h_2,y_2)\overline{|y_1y_2\rangle}_{\bar{c}}$$

"dual" basis

$$\phi(h,x) = \exp[\frac{i}{2}h_y x - \frac{1}{4}(x-h_x)^2] \qquad \bar{\phi}(h,y) = \phi(-ih,y)$$

Transition matrices

$$|\psi(h_1,h_2)\rangle_c = u_{c,\bar{c}}(h_1,h_2)\overline{|\psi(h_1,h_2)\rangle}_{\bar{c}}$$

Can diagonalise using transformation properties under magnetic translations:

$$|\psi(h_1,h_2)\rangle_c \longrightarrow |\psi(h_1,h_2)\rangle_{\mu\nu} \quad , \quad \overline{|\psi(h_1,h_2)\rangle}_c \longrightarrow \overline{|\psi(h_1,h_2)\rangle}_{\mu\nu}$$

$\mu,\nu = \pm 1$

where:

$$T_x|\psi(h_1,h_2)\rangle_{\mu\nu} \simeq \mu e^{-i(h_{1y}+h_{2y})/2R_y}|\psi(h_1,h_2)\rangle_{\mu\nu}$$

$$T_y|\psi(h_1,h_2)\rangle_{\mu\nu} \simeq \nu e^{-i(h_{1x}+h_{2x})/2R_x}|\psi(h_1,h_2)\rangle_{\mu\nu}$$

(and similarly for $\overline{|\psi(h_1,h_2)\rangle}_{\mu\nu}$)

$$|\psi(h_1,h_2)\rangle_{\mu\nu} = u_{\mu\nu}(h_1,h_2)\overline{|\psi(h_1,h_2)\rangle}_{\mu\nu}$$

diagonal in ($\mu\nu$) –basis !

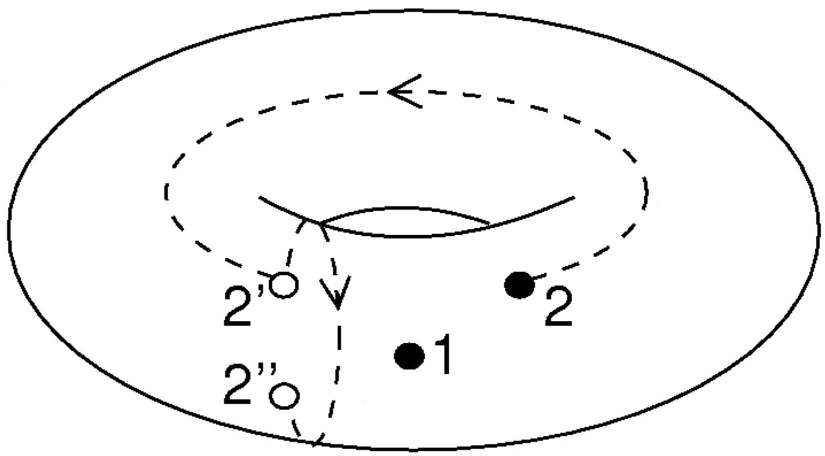
Transition matrices

$$|\psi(h_1, h_2)\rangle_{\mu\nu} = u_{\mu\nu}(h_1, h_2)\overline{|\psi(h_1, h_2)\rangle_{\mu\nu}}$$

from Berry phase

$$u_{\mu\nu}(h_1, h_2) = e^{i\alpha} e^{i(h_{1x}h_{1y} + h_{2x}h_{2y})/2}$$

locally constant phase, depends on μ,ν, and “region”



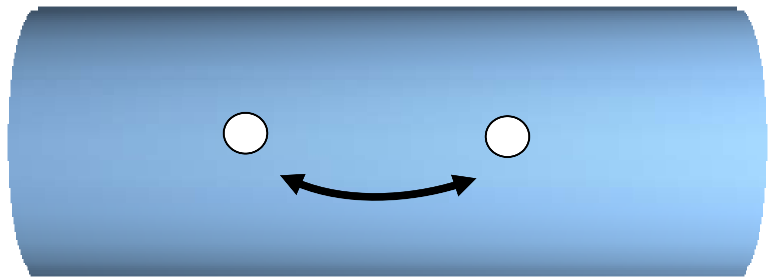
These paths can be used to relate the overall phase factors in various sectors/regions.

The configuration space of the two holes comes in two disjoint pieces, each of which is characterized by a phase α.
Only the difference between these two phases matters.

Determination of remaining parameters

3 parameters thus far undetermined:

- One phase for the even particle number sector,
- one phase for the odd particle number sector,
- and the “shift” parameter s.



20202020202020202020201111111111110202020202020202020202

This process should be diagonal

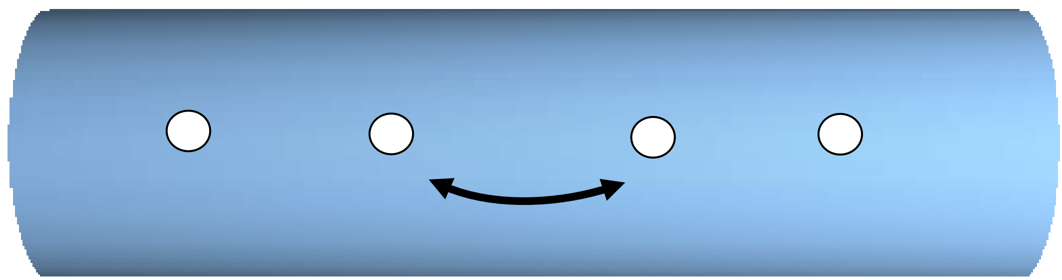
1111111111111111111111110202020202020111111111111111111111111

This process should not depend on particle number parity

→ Parameters sufficiently constrained

Pfaffian statistics from a 1d viewpoint

A.S., PRL ’08



Determined modulo π/4

02020202020111111111110202020202011111111111102020202020202

$$\theta = 3\pi/8$$

$$e^{i\theta}/\sqrt{2}$$ 02020202020111111111110202020202011111111111102020202020202

+

$$ie^{i\theta}/\sqrt{2}$$ 02020202020111111111110202020202011111111111102020202020202

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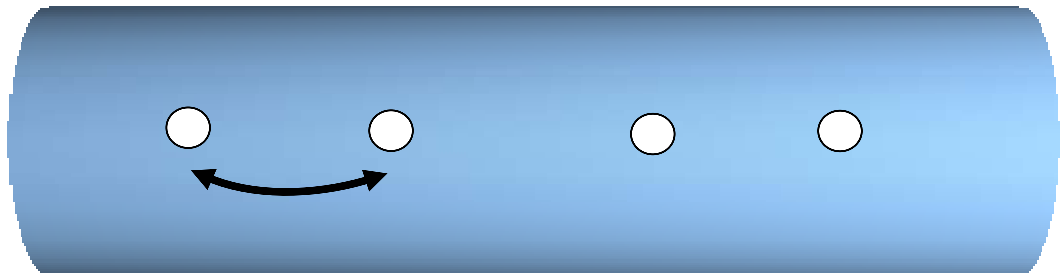


Determined modulo $\pi/4$

02020202020111111111111020202020201111111111102020202020202

$$\theta = 3\pi/8$$

even length 11-string

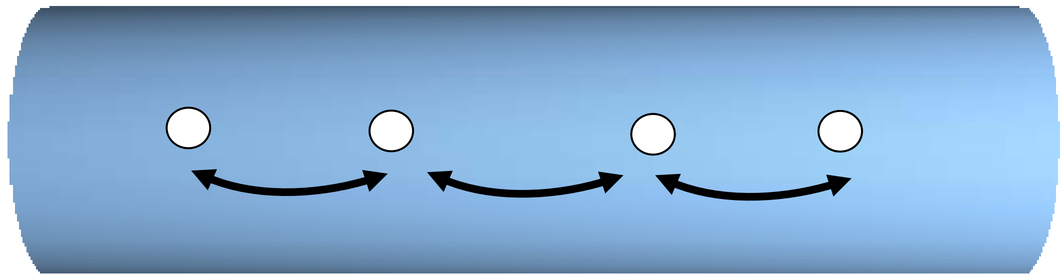
$$e^{i\theta \mp i\pi/4}$$

02020202020111111111111020202020201111111111102020202020202

odd length 11-string

Pfaffian statistics from a 1d viewpoint

A.S., PRL ’08



Determined modulo $\pi/4$

020202020201111111111110202020202011111111111102020202020202

$\theta = 3\pi/8$

$$e^{i\theta} e^{(\pi/4)\eta_j \eta_{j+1}}$$

Equivalent to Majoranafermion representation

C. Nayak, F. Wiczek, Nucl .Phys. B ’96
D. A. Ivanov, PRL, ‘01

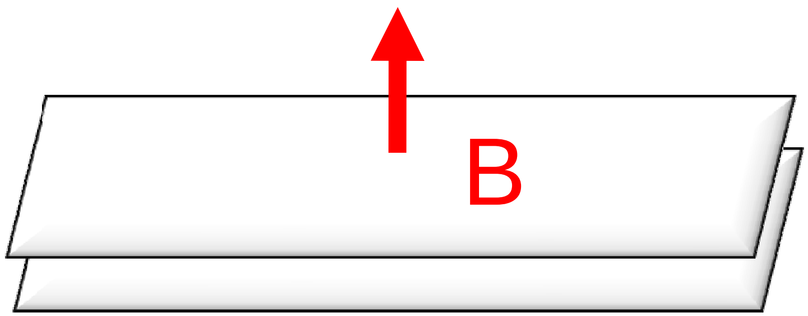
Conclusions

1D labels for fractional quantum Hall states do….

- …efficiently encode fundamental quantum numbers (fractional charges, topological degeneracies…)
- …allow independent derivation of statistics for Laughlin states, Moore-Read states and perhaps others.
- …give rise to simple and picturesque representations for topological sectors and the result of quasi-hole braiding processes

Multicomponent states and critical points: The Halperin (331) state

$$\prod_{i<j}(z_i - z_j)^3 \prod_{I<J}(z_I - z_J)^3 \prod_{i,J}(z_i - z_J) \exp(-\sum_\alpha |z_\alpha|^2/4)$$



For general Halperin (m,m',n) states:
A.S., K Yang, PRL '08

Thin torus patterns:

↑ 0 ↓ 0 ↑ 0 ↓ 0 ↑ 0 ↓ 0 ↑ 0 ↓ 0 x 4

↑ ↓ 0 0 ↑ ↓ 0 0 ↑ ↓ 0 0 ↑ ↓ 0 0 x 4

↑↓ = ↑ ↓ + ↓ ↑

c.f. ν=1/2 Pfaffian:

E. Berholtz et al., PRB '06

1010101010101010 x 2

1100110011001100 x 4

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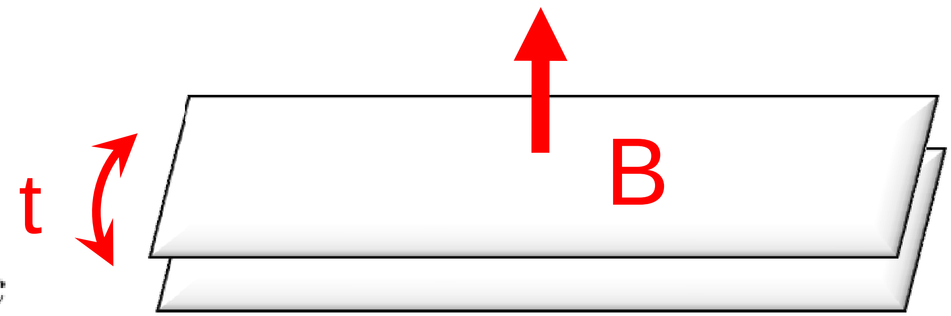
hopping

$\longrightarrow$ t $\hat{\sigma}_x$



Nature of the transition?

(T.-L. Ho, PRL '95

N. Read, E. Rezayi, PRB '96)

Thin torus patterns:

↑ 0 ↓ 0 ↑ 0 ↓ 0 ↑ 0 ↓ 0 ↑ 0 ↓ 0 x 4

(↑ ↓) 0 0 (↑ ↓) 0 0 (↑ ↓) 0 0 (↑ ↓) 0 0 x 4

(↑↓) = ↑↓ + ↓↑

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1010101010101010 x 2

1100110011001100 x 4

Multicomponent states and critical points: The Halperin (331) state

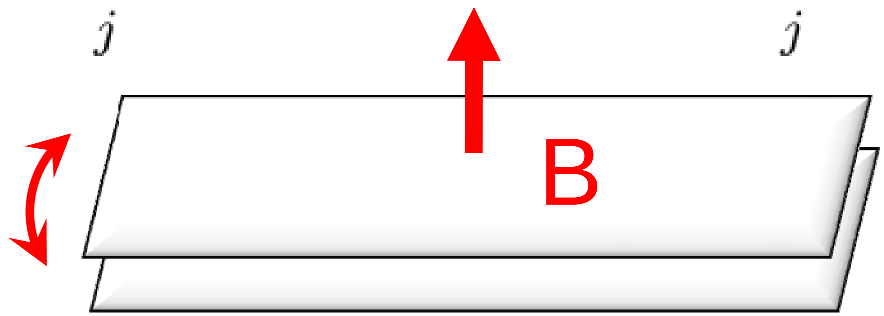
$$H_{\text{pseudo-potential}} + t\sum_j \hat{\sigma}^x_{2j} \longrightarrow H_{\text{eff}}$$

$$H_{\text{eff}} = J_z \sum_j (\hat{\sigma}^z_{2j}\hat{\sigma}^z_{2j+2} + 1) + t\sum_j \hat{\sigma}^x_{2j}$$

Effective Hamiltonian in “Ising” sector is transverse field Ising model!

hopping

$\longrightarrow$ t $\hat{\sigma}_x$



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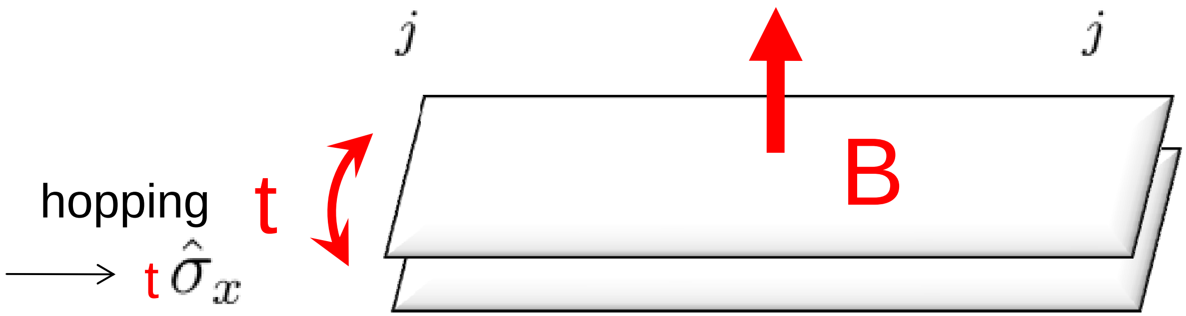

Thin torus patterns:

↑ 0 ↓ 0 ↑ 0 ↓ 0 ↑ 0 ↓ 0 ↑ 0 ↓ 0 x 4

(↑ ↓) 0 0 (↑ ↓) 0 0 (↑ ↓) 0 0 (↑ ↓) 0 0 x 4

(↑ ↓) = ↑ ↓ + ↓ ↑

In the thin torus limit, a transverse field Ising transition takes place at $t=J_z$ in one of the topological sectors.

Multicomponent states and critical points: The Haldane-Rezayi state

$$\Psi_{HR}(z_1^{\uparrow},\ldots,z_{N/2}^{\uparrow},z_1^{\downarrow},\ldots,z_{N/2}^{\downarrow}) = \det\left(\frac{1}{(z_i^{\uparrow}-z_j^{\downarrow})^2}\right)\prod_{\alpha<\beta}(z_\alpha - z_\beta)^2$$

Argued to be critical: N. Read, D. Green, PRB '00

$\nu=1/2$

Thin torus patterns (unpublished):

(↑↓) = ↑↓ - ↓↑

(↑↓) 0 0 (↑↓) 0 0 (↑↓) 0 0 (↑↓) 0 0 x 4

⇅ 0 0 0 ⇅ 0 0 0 ⇅ 0 0 0 ⇅ 0 0 0 x 4

⇅ 0 0 0 ⇅ 0 0 ↑ 0 0 ⇅ 0 0 0 ⇅ 0 0 ↓ 0 0 ⇅ 0 0 0 ⇅ x 2

delocalized

singlet

Multicomponent states and critical points: The Haldane-Rezayi state

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Thin torus patterns (unpublished):

(↑↓) = ↑↓ - ↓↑

(↑↓)00(↑↓)00(↑↓)00(↑↓)00 x 4

⇅000⇅000⇅000⇅000 x 4

⇅000⇅00↑00⇅000⇅00↓00⇅000⇅ x 2

delocalized

singlet

“Special” sector will have gapless excitations in thin torus limit!