Towards a theory of topological order: from FQH states to spin liquids

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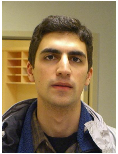
Z.-C. Gu



M. Levin

M. Barkeshli

Z.-H. Wang

To Believe that there is a problem

- We used to believe that symmetry breaking describe all phases and phase transitions. Landau, 1937
- We build a comprehensive theory based
  - Order parameter
  - Ginzburg-Landau theory
  - Group theory
- The discovery of FQH state teaches us that **symmetry breaking orders are not every thing. New kind of orders exist.**
  Different FQH states have the same symmetry, but they still represent the different phases:

$$\Psi_{1/3} = \prod (z_i - z_j)^3 \mathrm{e}^{-\frac{1}{4}\sum |z_i|^2}, \quad \Psi_{1/5} = \prod (z_i - z_j)^5 \mathrm{e}^{-\frac{1}{4}\sum |z_i|^2}.$$

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The new orders (and because it is new)

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- Topological entanglement entropy and spectrum can describe the topological order. Kitaev & Preskill 06, Levin & Wen 06, Haldane 08 (Can be probed by quantum noise Klich & Levitov 08)

Towards a comprehensive theory of topological order

- The description and characterization of symmetry breaking orders using order parameters and group theory play a key role in developing a comprehensive theory of symmetry breaking order.

  Even though some those characterizations of topological order were proposed 20 years ago, we have not been able to use them to develop a comprehensive theory of topological order.

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- Finding easy-to-use ways to describe and characterize topological order is the key in developing a comprehensive theory of topological order.

  In this talk, we will discuss two new ways to describe and characterize topological order based on

- ground state wave functions
- fixed-point Lagrangian (tensor network)

Topological order in FQH states and Pattern of zeros

- Filling fraction $\nu = 1/m$ Laughlin state $\Psi_{1/m} = \prod(z_i - z_j)^m$ is characterized by the m^{th} zero as we bring two electrons together.
- Generalizing that, we bring a electrons together in a wave function: Let $z_i = \lambda\xi_i + z^{(a)}$, $i = 1, 2, \cdots, a$

$$\Phi(\{z_i\}) = \lambda^{S_a} P(\xi^1, ..., \xi^a; z^{(a)}, z_{a+1}, z_{a+2}, \cdots) + O(\lambda^{S_a+1})$$

- The sequence of positive integers $\{S_a\}$ characterizes the FQH wave function in the first Landau level and is called the pattern of zeros.

The pattern of zeros $\{S_a\}$ is a quantitative way to describe and characterize the (chiral) topological order in FQH states Wen & Wang 08, Barkeshli & Wen 08

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- Topological properties, such as filling fraction, ground state degeneracy on genus g surfaces, quasiparticle charges and quantum dimensions, number of quasiparticle types, the fusion algebra of quasiparticles, can all be calculated from such a quantitative characterization.

Towards a classification of Topological order in FQH states

- Not all sequences of integers $\{S_a\}$ can correspond to a FQH wavefunction. Only those that satisfy, for any a, b, c,

$$S_{a+b} - S_a - S_b \geq 0$$
$$S_{a+b+c} - S_{a+b} - S_{b+c} - S_{a+c} + S_a + S_b + S_c = \text{even} \geq 0$$

can correspond to FQH wavefunctions.
Finding all those sequences may lead to a classification of topological orders in FQH states

- Relation to 1D CDW picture: Seidel & Lee 06, Bergholtz etal 06, Bernevig & Haldane 07, Seidel & Yang 08, Ardonne etal 08

view $l_a = S_a - S_{a-1}$ as the orbital occupied by a^{th} electron $\rightarrow$ occupation distribution n_l=number of $l_a = l$.

$$Z_2 : (S_2, S_3, ...) = (0, 2, 4, 8, ...) \quad (n_l) = (20|20|20|...)$$
$$Z_5^{(2)} : (S_2, S_3, ...) = (0, 2, 6, 10, ...) \quad (n_l) = (20102000|20102000|...)$$

An application of $\{S_a\}$ characterization

- A quasiparticle γ in a FQH state can also be quantitatively characterized by pattern of zeros $\{S_{\gamma;a}\}$:
  Let $\Psi_\gamma(\xi, z_i)$ be a FQH wavefunction with a quasiparticle γ at ξ, then $S_{\gamma;a}$ is the order of zero of $\Psi_\gamma(\xi, z_i)$ when we bring a electrons to ξ.
- $\{S_{\gamma;a}\}$ must satisfies:

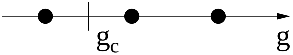
$$S_{\gamma;a+b} - S_{\gamma;a} - S_b \geq 0,$$
$$S_{\gamma;a+b+c} - S_{\gamma;a+b} - S_{\gamma;a+c} - S_{b+c} + S_{\gamma;a} + S_b + S_c \geq 0$$

- The above equations have many solutions, and each solution correspond to a type of quasiparticle.

  $\{S_a\}$ **and** $\{S_{\gamma;a}\}$ **are quantitative characterizations of FQH state and their quasiparticles. Such quantitative characterizations allow us to calculation topological properties quantitatively.**

Characterize topo. orders through fixed-point Lagrangian

- We may want to use Lagrangian to characterize phases, but
  - some times similar Lagrangian correspond to the same phase
  - some times similar Lagrangian correspond to the different phases



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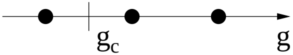
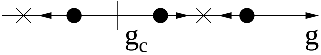


- A RG idea: under the RG transformation, a Lagrangian flows to fixed-point Lagrangian. It is the fixed-point Lagrangian that characterizes phase.



Limitations of standard RG approach

But the RG approach appear not to apply to topological phases:
If a bosonic system is in a topological phase, then its low energy effective Lagrangian (the fixed-point Lagrangian) can be

- A pure gauge theory with $G = Z_2,\ Z_n,\ U(1),\ SU(2), ...$
- A Chern-Simons gauge theory with any G
- A QED ($U(1)$ gauge theory + massless fermions)
- A QCD ($SU(2)$ gauge theory + massless fermions)
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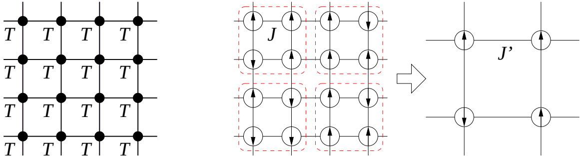
How can the RG flow of a bosonic Lagrangian $L(\varphi)$ with a scaler field φ produces such rich class of fixed-point Lagrangian with gauge fields and fermionic fields?

Tensor renormalization group: a new RG approach

After a discretization, we can rewrite any space-time path integral at a tensor-trace over a tensor network

$$\int D\varphi(x,t)e^{-L(\varphi)} = \sum_{\{\alpha_i\}} \prod T_{\alpha_i\alpha_j\alpha_k\alpha_l} \equiv \text{tTr} \otimes T$$



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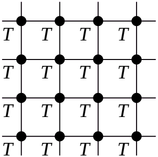

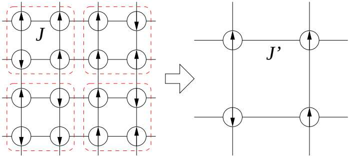

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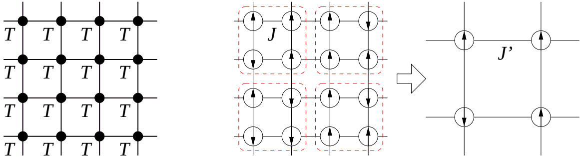

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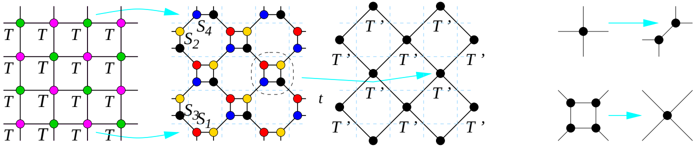
- Unfortunately, according to the principle of "no-free lunch", calculating the tensor-trace is an NP hard problem. Schuch etc 07
- But Levin and Nave discovered a principle of "free lousy lunch": if you are willing to accept some errors, calculating the tensor-trace has only polynomial complexity.

Filtering out local entanglements



- Levin and Nave's implementation of TRG flow $T \to T'$ has a small problem: the resulting fixed-point tensor is not isolated. Local entanglements are not completely removed
- Topological order = pattern of long range entanglements

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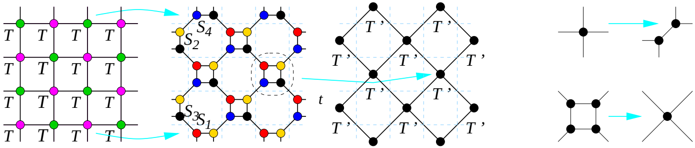

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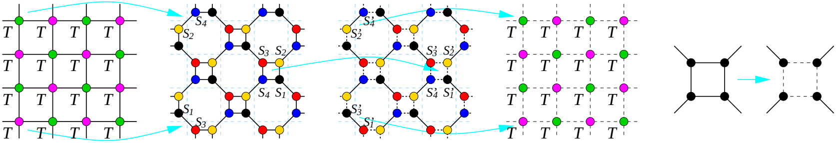
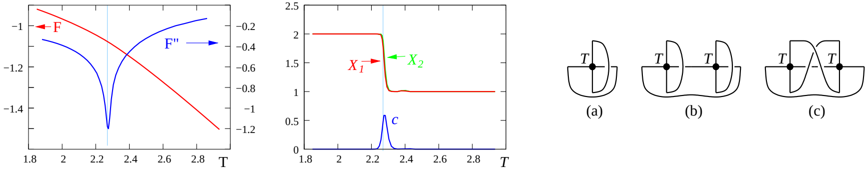


- Tensor entanglement filtering renormalization (TEFR): Gu & Wen 09 Filter out local entanglement but keep long range entanglement

Application of TEFR to 2D statistical Ising model

- Using TEFR and the resulting fixed-point tensor, we can calculate free energy, ground state energy, low energy spectrum, central charge and scaling dimensions, entanglement entropy, correlation functions, *etc* Zhengcheng Gu & Wen 09



- Fixed-point tensors: $T^{high}_{1111} = 1$ and $T^{low}_{1111} = T^{low}_{2222} = 1$.
  Symmetry breaking as direct sum: $T^{low} = T^{high} \oplus T^{high}$

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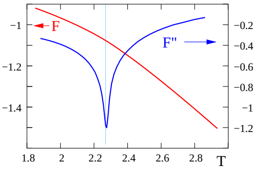

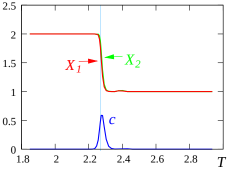

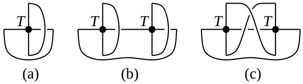

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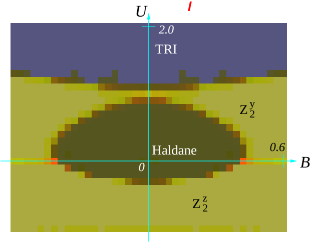
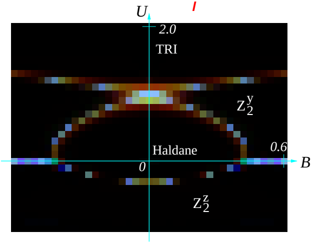
c	h_1	h_2	h_3	h_4
0.49942	0.12504	0.99996	1.12256	1.12403
1/2	1/8	1	9/8	9/8

Computation time: 10 hours on a desktop.

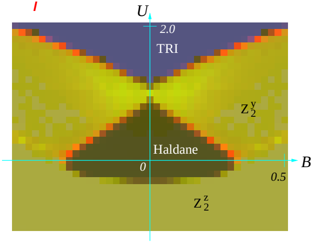
Application of TEFR to spin-1 chain

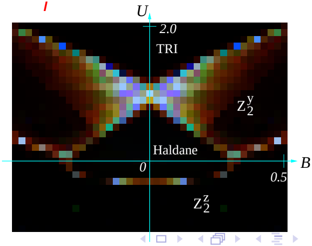
$$H = \sum_i \left(\mathbf{S}_i \cdot \mathbf{S}_{i+1} + U(S_i^z)^2 \right) + B \sum_i S_i^x$$



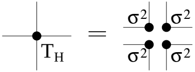


$$H = \sum_i \left(\mathbf{S}_i \cdot \mathbf{S}_{i+1} + U(S_i^z)^2 \right) + \frac{B}{2} \sum_i (S_i^x (S_{i+1}^z)^2 + S_{i+1}^x (S_i^z)^2 + 2S_i^z)$$

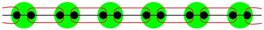


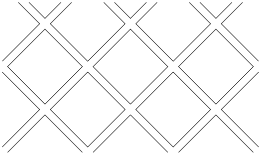


Haldane phase is a symmetry protected topological phase

- Fixed-point tensor for Haldane phase: 

- $T_H + \delta T \to T_H$ if δT has time-reversal, parity and translation symmetry.

  The Haldane phase is a symmetry protected topological phase.
- “Fixed-point” wavefunction: 



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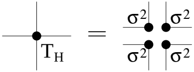
- Fixed-point tensor for Haldane phase: 

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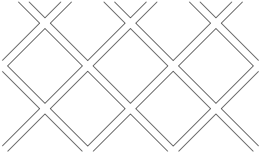

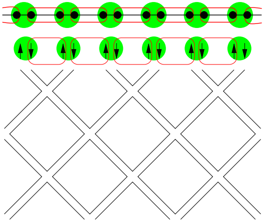

- The boundary spin-1/2 and string order parameter are not good ways to characterize the Haldane phase.

Theory of topological order

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