Coupled wire model of symmetric Majorana surfaces of topological superconductors

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To appear soon

Outline

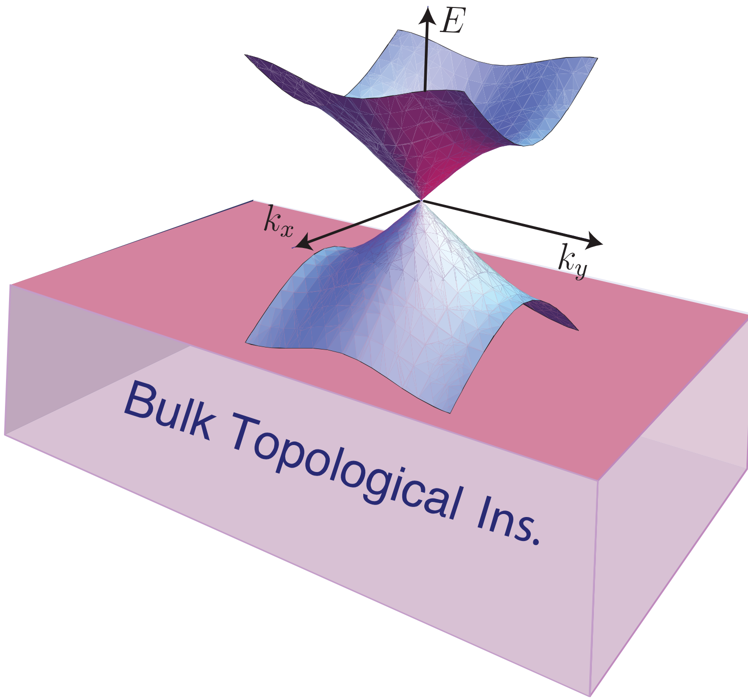
- Introduction
  - Surface States of 3D topological insulator and superconductors
  - Coupled wire construction
- Symmetric 4-fermion gapping terms
  - $so(N)_1$ current algebra and fractionalization
  - Gapping terms for N even
  - Gapping term for N odd: special case N=9

$$so(9)_1 \supseteq so(3)_3 \times so(3)_3$$

- Topological order
  - The abelian D_n-series, Ising-like B_n-series, and $SO(3)_3$ -like states
  - 32-fold extension of the 16-fold $SO(N)_1$ -series
  - Comment on E_8 extension of $so(16)_1$ and connection to Z_{16} classification

Gapless surface states

- 3D Topological insulator (Z_2 classified)



$$\mathcal{H}_c = iv\,\boldsymbol{\psi}^T \partial\!\!\!/ \boldsymbol{\psi}$$

$$\partial\!\!\!/ = \partial_y \sigma_x + \partial_x \sigma_y$$

Massless Dirac (complex) fermion

$$\boldsymbol{\psi} = (\psi^{\uparrow}, \psi^{\downarrow})$$

Surface Dirac cone protected by TRS (Kramers theorem)

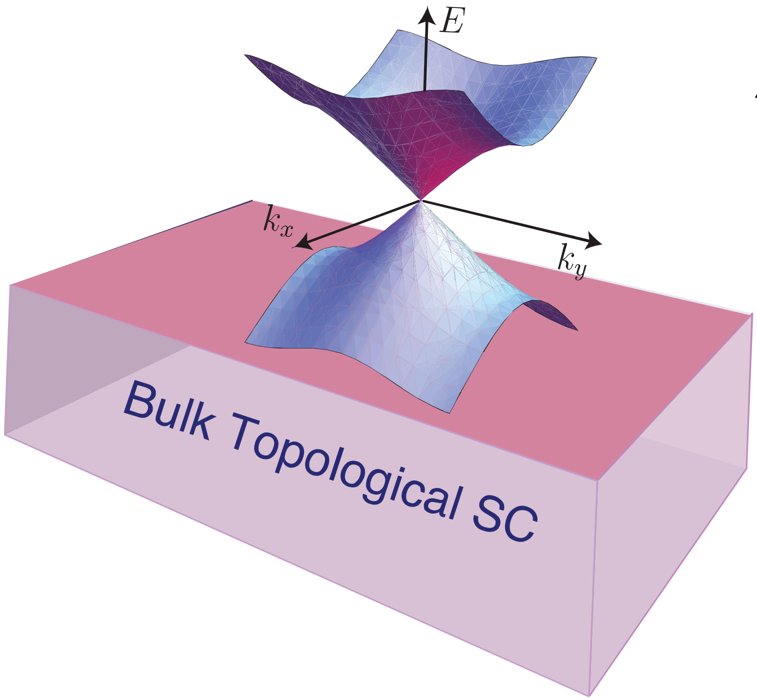
- Time reversal symmetry (TRS): T anti-unitary, $T^2 = -1$

$$TH(\mathbf{k})T^{-1} = H(-\mathbf{k})$$

Moore, Balent , 07
Fu, Mele, Kane, 07
Roy, 09

Gapless surface states

- 3D Topological superconductor ($\mathbb{Z}$ classified)



$$\mathcal{H}_c = \sum_{a=1}^{N} iv\, \boldsymbol{\psi}_a^T \not{\partial} \boldsymbol{\psi}_a$$

$$\not{\partial} = \partial_y \tau_x + \partial_x \tau_y$$

N - component massless Majorana (real) fermion

$$\boldsymbol{\psi}_a = (\psi_a^1, \psi_a^2)$$

Surface Majorana protected by TRS (Kramers theorem) and winding number

- Particle-hole symmetry (PHS): C anti-unitary, $C^2 = 1$

$$CH_{\rm BdG}(\mathbf{k})C^{-1} = -H_{\rm BdG}(-\mathbf{k})$$

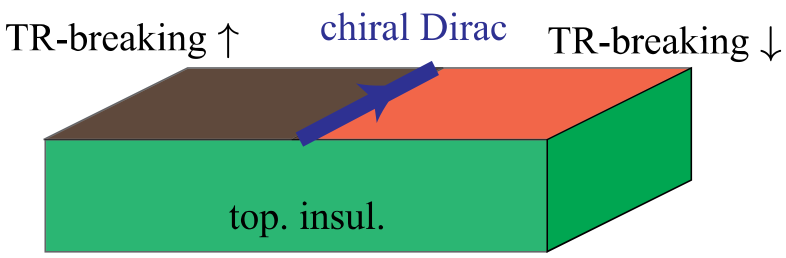
- Time reversal symmetry (TRS): T anti-unitary, $T^2 = -1$

$$TH_{\rm BdG}(\mathbf{k})T^{-1} = H_{\rm BdG}(-\mathbf{k})$$

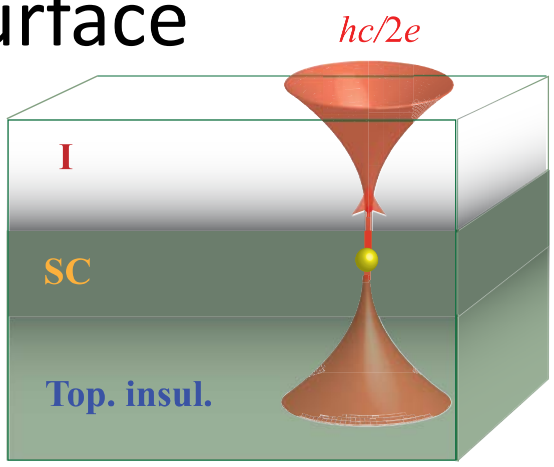
Schnyder, et.al., 08;
Kitaev, 08;
Qi et.al., 09

Gapped Top. Insul. Surfaces

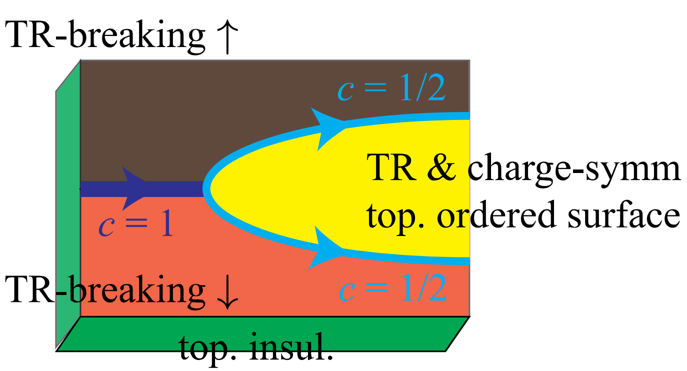
- TR-breaking surface



- Charge breaking surface



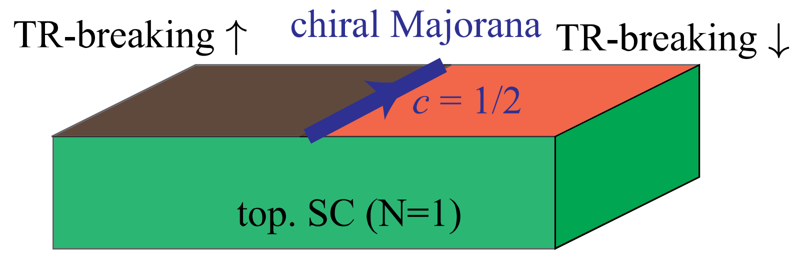
- Symmetric surface with topological order



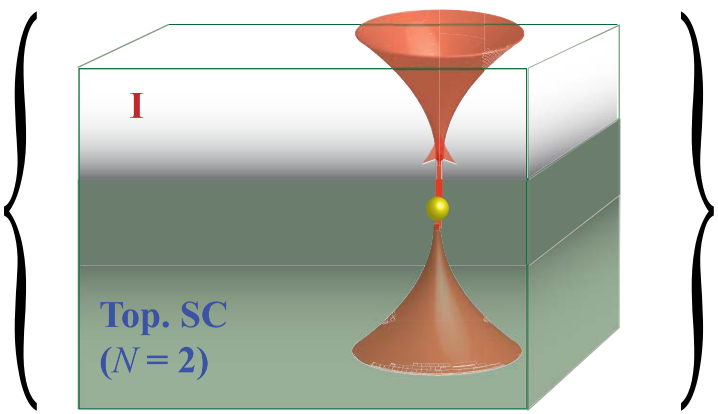
Wang, Potter, Senthil, 14;
Metlitski, Kane, Fisher, 13;
Chen, Fidkowski, Vishwanath, 14;
Bonderson, Nayak, Qi, 14

Gapped Top. SC Surfaces

- TR-breaking surface



- Trivial symmetric $N = 16$ surface



$\times 8$

Z_8 classification in 1D

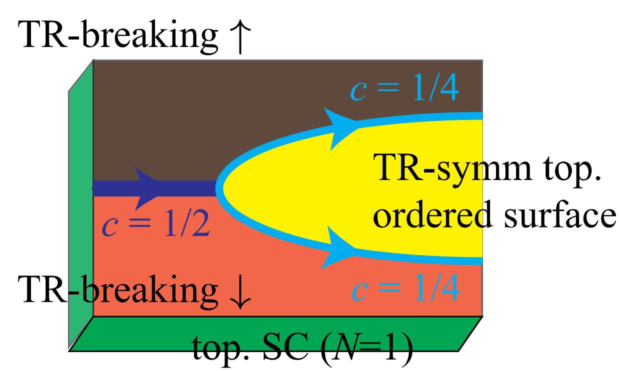
Fidkowski, Kitaev, 10, 11;
Turner, Pollmann, Berg, 11;
Chen, Gu, Wen, 11

Z_{16} classification in 3D

Metlitski, Fidkowski, Chen, Vishwanath, 14;
Wang, Senthil, 14;
Kapustin, Thorngren, Turzillo, Wang, 14

- Top. symmetric $N < 16$ surface

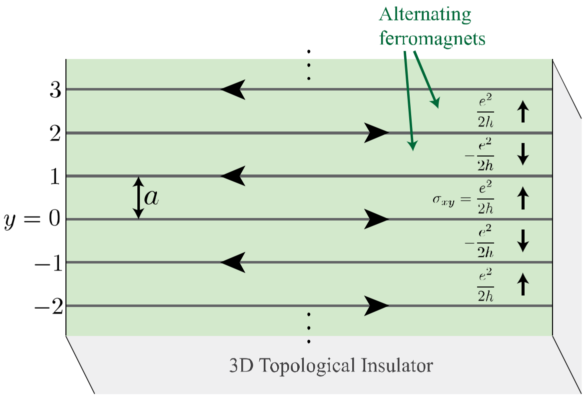
Fidkowski, Chen, Vishwanath, 13



What interaction?

Coupled wire model of top. insul. surface states

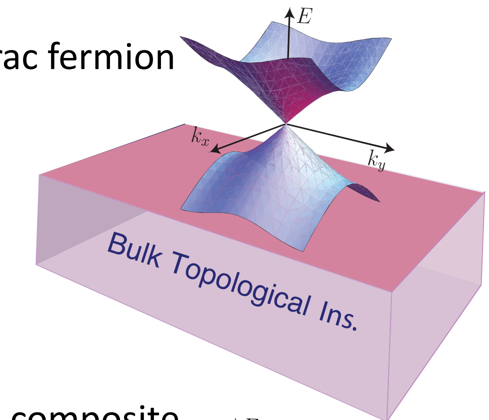
D. F. Mross, A. Essin, and J. Alicea, Phys. Rev. X **5**, 011011 (2015)

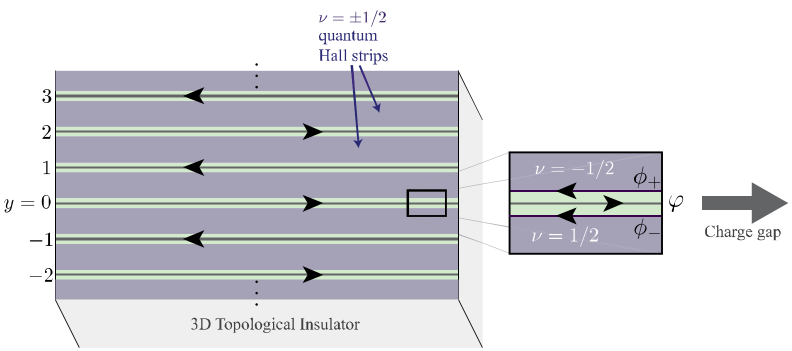


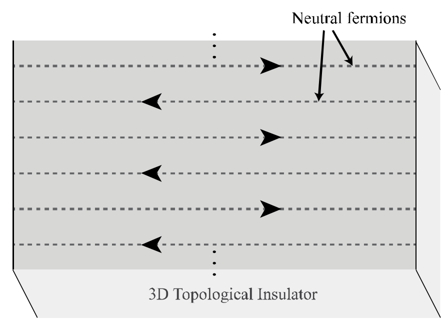
$$H=\sum_y(-1)^y\int_x\left[-i\hbar v_x\psi_y^\dagger\partial_x\psi_y-t(\psi_y^\dagger\psi_{y+1}+\text{H.c.})\right].$$

$$\tilde{\mathcal{T}}:\psi_y\to(-1)^y\psi_{y+1}.$$

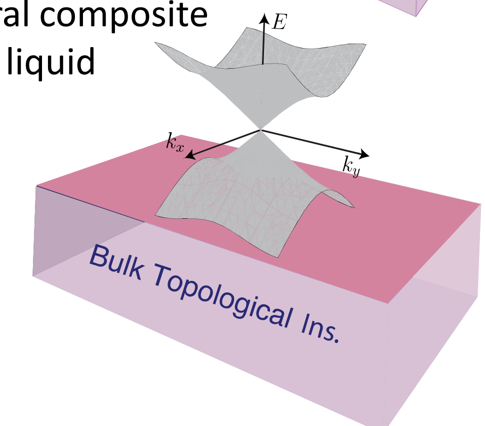
charged Dirac fermion

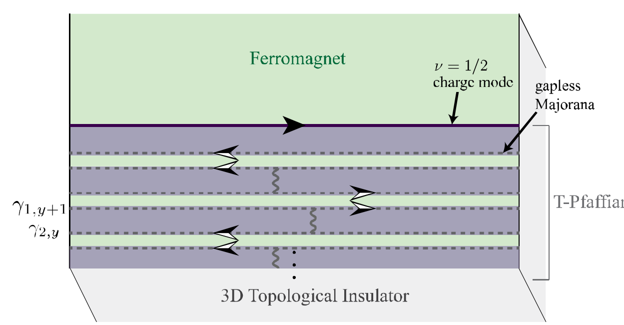




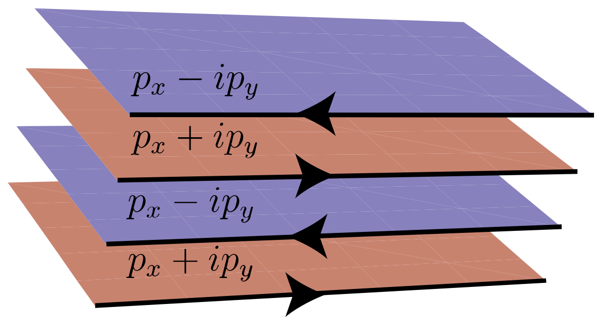


neutral composite
Dirac liquid

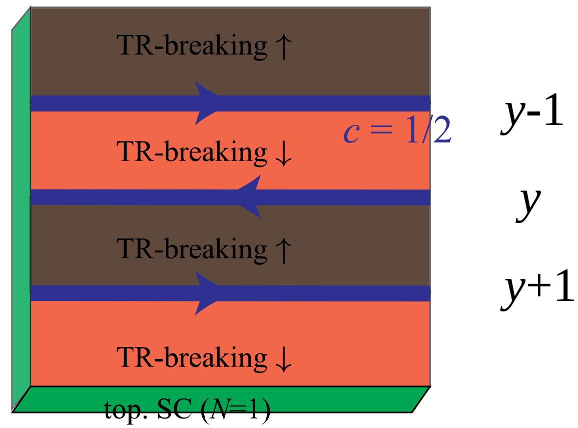




Coupled wire model of a surface Majorana



Stack of p±ip or
Kitaev honeycomb B phase

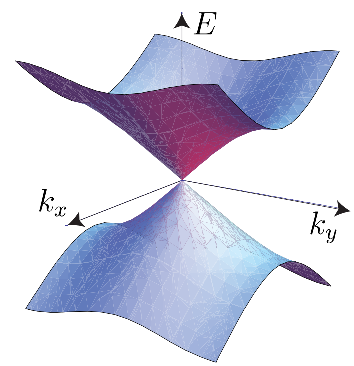


$$\mathcal{H}_0 = \sum_{y=-\infty}^{\infty} i v_{\mathsf{x}} (-1)^y \boldsymbol{\psi}_y^T \partial_x \boldsymbol{\psi}_y + i v_{\mathsf{y}} \boldsymbol{\psi}_y^T \boldsymbol{\psi}_{y+1}$$

$$\mathcal{T} : \psi_y \to (-1)^y \psi_{y+1} \qquad \mathcal{T}^2 = (-1)^F \hat{t}_y$$

Antiferrormagnetic
time reversal

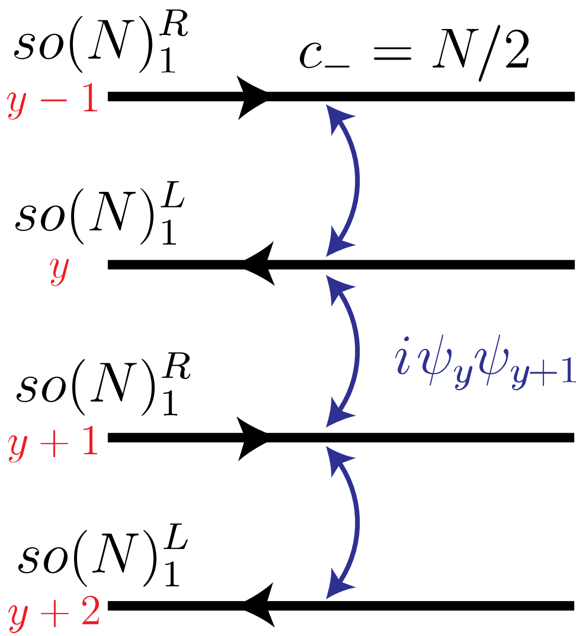
Translation y to y+2



$$H^0_{\mathrm{BdG}}(\mathbf{k}) = 2v_{\mathsf{x}} k_x \tau_x + v_{\mathsf{y}} \left[-\sin k_y \tau_y + (1 - \cos k_y) \tau_z \right]$$

Protected by TRS (Kramers theorem)

Coupled wire model of N Majorana's



$$\boldsymbol{\psi}_y = (\psi_y^1, \dots, \psi_y^N)$$

$$\mathcal{H}_0 = \sum_{y=-\infty}^{\infty} i v_{\mathsf{x}} (-1)^y \boldsymbol{\psi}_y^T \partial_x \boldsymbol{\psi}_y + i v_{\mathsf{y}} \boldsymbol{\psi}_y^T \boldsymbol{\psi}_{y+1}$$

$$\mathcal{T}\left(\sum_{a=1}^{N} \alpha_a \psi_y^a\right)\mathcal{T}^{-1} = (-1)^y \sum_{a=1}^{N} \alpha_a^* \psi_{y+1}^a \qquad \mathcal{T}^2 = (-1)^F \hat{t}_y$$

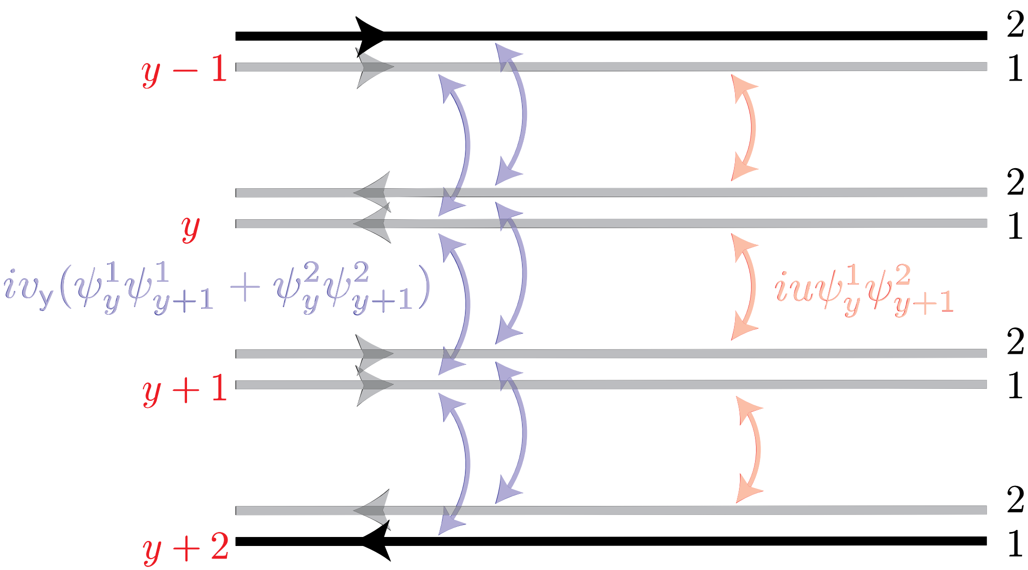
- “Non-symmorphic” chiral symmetry

$$T_{\mathbf{k}} T_{-\mathbf{k}} = -e^{ik_y} \qquad C T_{\mathbf{k}} = T_{-\mathbf{k}} C \qquad \Pi_{\mathbf{k}} = i C T_{\mathbf{k}}$$

$$\Pi_{\mathbf{k}} H_{\mathrm{BdG}}(\mathbf{k}) = -H_{\mathrm{BdG}}(\mathbf{k}) \Pi_{\mathbf{k}} \qquad \boxed{\Pi_{\mathbf{k}}^2 = e^{-ik_y}}$$

- New classification: Z_2

Trivial gapping term for $N = 2$



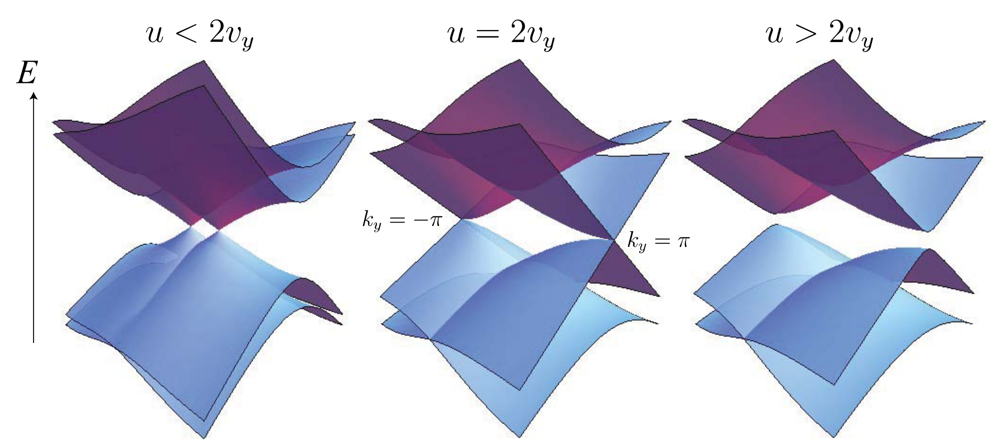
$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\rm bc}$$

$$\mathcal{H}_0 = \sum_{y=-\infty}^{\infty} iv_{\rm x}(-1)^y \boldsymbol{\psi}_y^T \partial_x \boldsymbol{\psi}_y + iv_{\rm y}\boldsymbol{\psi}_y^T \boldsymbol{\psi}_{y+1}$$

$$\mathcal{H}_{\rm bc} = iu \sum_{y=-\infty}^{\infty} \psi^1_y \psi^2_{y+1}$$

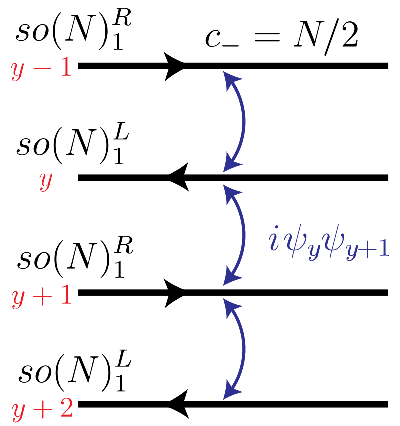
$$H^{\rm bc}_{\rm BdG}(\mathbf{k}) = \frac{u}{2}\left[(1-\cos k_y)\sigma_x\tau_z + (1+\cos k_y)\sigma_y\tau \right.$$
$$\left. - \sin k_y(\sigma_y\tau_z + \sigma_x\tau_y)\right]$$

Ch = 1



$so(N)_1$ WZW CFT (Review)

- $SO(N)$ symmetry and Current operators



$$\psi_y^a \to O_b^a \psi_y^b$$

$$\boldsymbol{\psi}_y = (\psi_y^1, \dots, \psi_y^N)$$

$$J^\beta(z) = \frac{i}{2}\boldsymbol{\psi}(z)^T t^\beta \boldsymbol{\psi}(z) = \frac{i}{2}\sum_{ab}\psi^a(z)t^\beta_{ab}\psi^b(z)$$

$$= i\psi^a(z)\psi^b(z), \quad \text{for } 1 \le a < b \le N \qquad z = e^{\tau + ix}$$

- Current algebra and energy momentum

$$\psi^a(z)\psi^b(w) = \frac{\delta^{ab}}{z-w} + \dots$$

$$J^\beta(z)J^\gamma(w) = \frac{\delta^{\beta\gamma}}{(z-w)^2} + \sum_\delta \frac{if_{\beta\gamma\delta}}{z-w}J^\delta(w) + \dots$$

$so(N)$ at level 1

$$T(z) = \frac{1}{2(N-1)}\mathbf{J}(z)\cdot\mathbf{J}(z) = -\frac{1}{2}\boldsymbol{\psi}(z)^T\partial_z\boldsymbol{\psi}(z)$$

$$T(z)T(w) = \frac{c_-/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \dots$$

$$c_- = N/2$$

$$I_T \approx c_- \frac{\pi^2 k_B^2}{6h} T^2$$

$so(N)_1$ WZW CFT (Review)

- Primary field content

$$J^{\beta}(z)V_{\lambda}^{a}(w) = -\sum_{b=1}^{N}\frac{(t_{\lambda}^{\beta})_{ab}}{z-w}V_{\lambda}^{b}(w)+\dots$$

λ = irreducible representation of $so(N)$

$$T(z)\mathbf{V}_{\lambda}(w) = \frac{h_{\lambda}}{(z-w)^{2}}\mathbf{V}_{\lambda}(w) + \frac{\partial_{w}\mathbf{V}_{\lambda}(w)}{z-w}+\dots$$

h_λ = conformal (scaling) dimension

N even

λ	1	ψ	s_+	s_-
irred. rep.	trivial	vector	even spinor	odd spinor
scal. dim. h_λ	0	1/2	$N/16$	$N/16$
quant. dim. d_λ	1	1	1	1

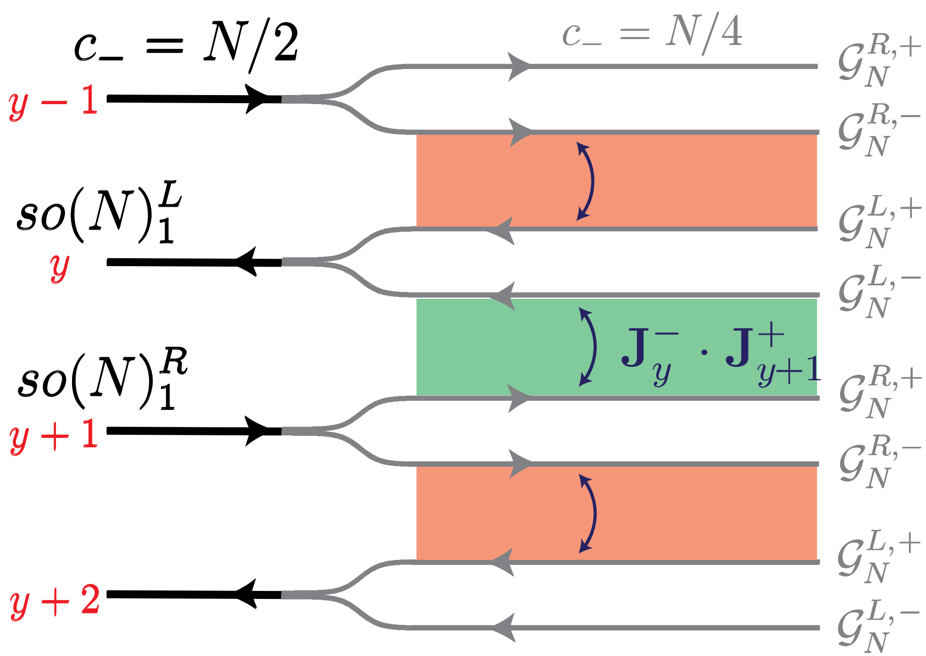
$$s_{\pm}\times\psi = s_{\mp}$$

N odd

λ	1	ψ	σ
irred. rep.	trivial	vector	spinor
scal. dim. h_λ	0	1/2	$N/16$
quant. dim. d_λ	1	1	$\sqrt{2}$

$$\sigma\times\sigma = 1+\psi$$

General gapping scheme



- Fractionalization (conformal embedding)

$$so(N)_1 \supseteq \mathcal{G}_N \times \mathcal{G}_N \qquad T_{so(N)_1} = T_{\mathcal{G}_N^+} + T_{\mathcal{G}_N^-}$$

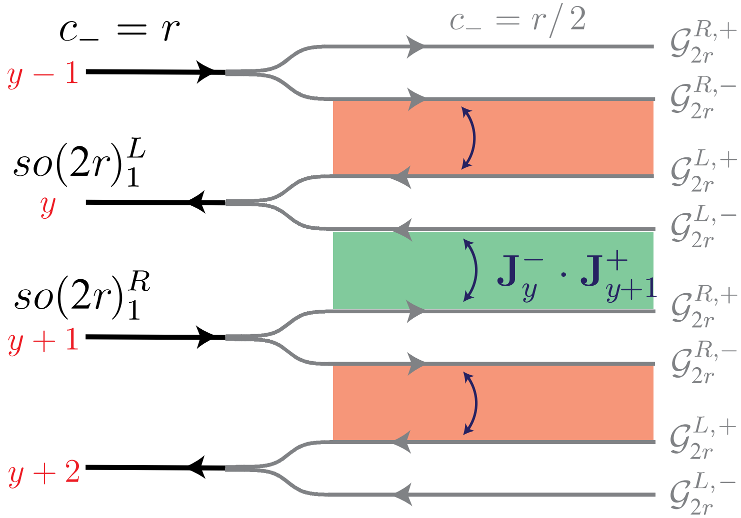
- Gapping 4-fermion interaction

2-fermion backscattering

$$\mathcal{H}_{\text{int}} = u \sum_{y=-\infty}^{\infty} \mathbf{J}^y_{\mathcal{G}_N^-} \cdot \mathbf{J}^{y+1}_{\mathcal{G}_N^+}$$

Gapping even Majorana's

$N = 2r$



$$so(2r)_1 \supseteq \mathcal{G}_{2r}^+ \times \mathcal{G}_{2r}^- = so(r)_1^+ \times so(r)_1^-$$

$$\psi^1, \ldots, \psi^r \qquad \psi^{r+1}, \ldots, \psi^{2r}$$

$$T_{so(r)_1^+} = -\frac{1}{2}\sum_{a=1}^{r} \psi^a \partial \psi^a, \quad T_{so(r)_1^-} = -\frac{1}{2}\sum_{a=r+1}^{2r} \psi^a \partial \psi^a$$

$$T_{so(2r)_1} = -\frac{1}{2}\sum_{a=1}^{2r} \psi^a \partial \psi^a = T_{so(r)_1^+} + T_{so(r)_1^-}$$

$$\mathcal{H}_{\text{int}} = u \sum_{y=-\infty}^{\infty} \sum_{1 \leq a < b \leq r} \psi_y^{r+a} \psi_y^{r+b} \psi_{y+1}^a \psi_{y+1}^b$$

- Symmetric under antiferrormagnetic time reversal

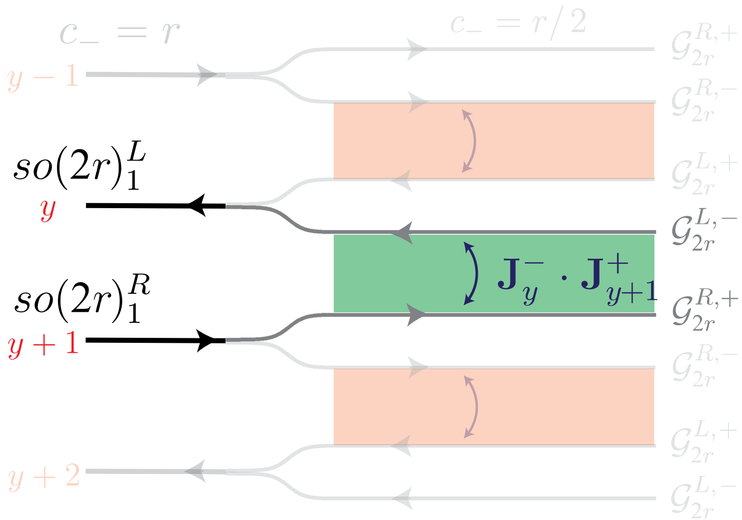
$$\mathcal{T} : \psi_y \to (-1)^y \psi_{y+1}$$

- Marginally relevant

$$\frac{du}{dl} = +4\pi(r-2)u^2$$

Gapping even Majorana's

$N = 2r = 4n$



$O(r)$ Gross-Neveu (GN) model

$$\mathcal{H}_{\mathrm{GN}} = -\frac{u}{2}\left(\boldsymbol{\psi}_R \cdot \boldsymbol{\psi}_L\right)^2$$

$$\psi^{r+a}_y = \psi^a_R \text{ and } \psi^a_{y+1} = \psi^a_L, \text{ for } a = 1, \ldots, r.$$

Bosonization

$$c^j_{R/L} = (\psi^{2j-1}_{R/L} + i\psi^{2j}_{R/L})/\sqrt{2} \sim e^{i\widetilde{\phi}^j_{R/L}}$$

$$j = 1, \ldots, n$$

$$\mathcal{H}_{GN} \sim u \sum_{j=1}^{n} \partial_x \widetilde{\phi}^j_R \partial_x \widetilde{\phi}^j_L - u \sum_{j_1 \neq j_2} \sum_{\pm} \cos\left(2\Theta^{j_1} \pm 2\Theta^{j_2}\right)$$

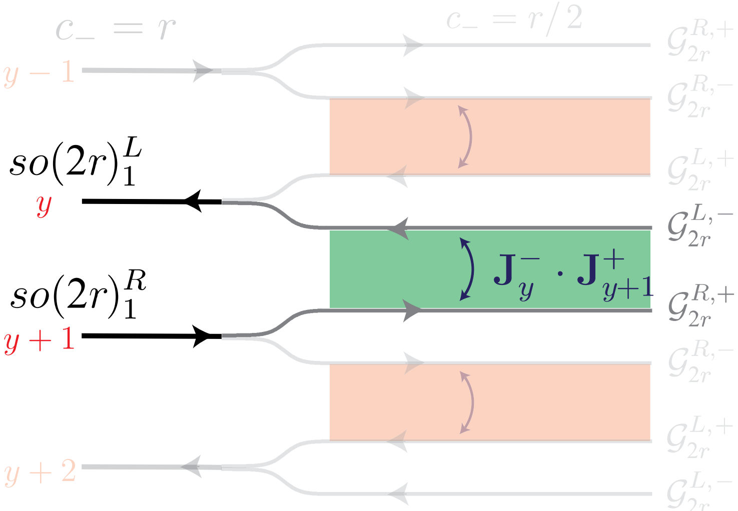
$$= u \sum_{j=1}^{n} \partial_x \widetilde{\phi}^j_R \partial_x \widetilde{\phi}^j_L - u \sum_{\boldsymbol{\alpha} \in \Delta} \cos\left(\boldsymbol{\alpha} \cdot 2\boldsymbol{\Theta}\right)$$

where $2\boldsymbol{\Theta} = (2\Theta^1, \ldots, 2\Theta^n)$ and $2\Theta^j = \widetilde{\phi}^j_R - \widetilde{\phi}^j_L$

$$\left\langle 2\Theta^j(x) \right\rangle = \pi m^j_\psi, \quad m^j_\psi \in \mathbb{Z}.$$

Gapping even Majorana's

$N = 2r = 4n + 2$



$O(r)$ Gross-Neveu (GN) model

$$\mathcal{H}_{\mathrm{GN}} = -\frac{u}{2}\left(\boldsymbol{\psi}_R \cdot \boldsymbol{\psi}_L\right)^2$$

$\psi^{r+a}_y = \psi^a_R$ and $\psi^a_{y+1} = \psi^a_L$, for $a = 1, \ldots, r$.

Bosonization

$$c^j_{R/L} = (\psi^{2j-1}_{R/L} + i\psi^{2j}_{R/L})/\sqrt{2} \sim e^{i\widetilde{\phi}^j_{R/L}}$$

$j = 1, \ldots, n$

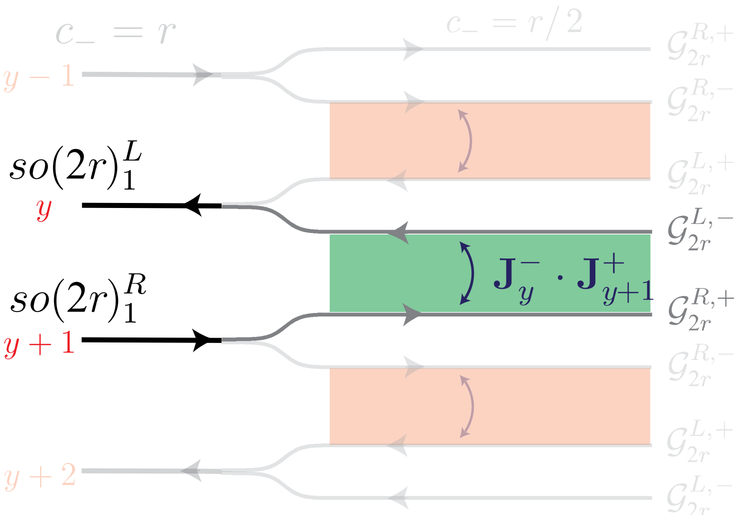
$$\mathcal{H}_{\mathrm{GN}} \sim -u \sum_{\boldsymbol{\alpha} \in \Delta_{so(2n)}} \cos\left(\boldsymbol{\alpha} \cdot 2\boldsymbol{\Theta}\right) - u\left[\sum_{j=1}^{n} \cos\left(2\Theta^j\right)\right] i\psi^r_R \psi^r_L$$

where $2\boldsymbol{\Theta} = (2\Theta^1, \ldots, 2\Theta^n)$ and $2\Theta^j = \widetilde{\phi}^j_R - \widetilde{\phi}^j_L$

$$\langle 2\Theta^j(x)\rangle = \pi m^j_\psi, \quad m^j_\psi \in \mathbb{Z}.$$

$$\mathcal{H}_{\mathrm{GN}} \sim -2n(n-1)u - nu(-1)^{m_\psi} i\psi^r_R \psi^r_L$$

Gapping even Majorana's



Special case: $N = 2r = 4$

$O(2)$ Gross-Neveu model is gapless

Alternative decomposition:

$$so(4)_1 = su(2)^+_1 \times su(2)^-_1$$

$$c^1_y = (\psi^1_y + i\psi^2_y)/\sqrt{2} = e^{i\widetilde{\phi}^1_y} \qquad \widetilde{\phi}^1 = \phi^+ - \phi^-$$

$$c^2_y = (\psi^3_y + i\psi^4_y)/\sqrt{2} = e^{i\widetilde{\phi}^2_y} \qquad \widetilde{\phi}^2 = \phi^+ + \phi^-$$

The $su(2)_1$ current generators:

$$S^I_{\mathsf{z}}(z) = i\sqrt{2}\partial\phi^I(z) \qquad S^I_{\pm}(z) = (S^I_{\mathsf{x}} \pm iS^I_{\mathsf{y}})/\sqrt{2} = e^{i2\phi^I(z)}$$

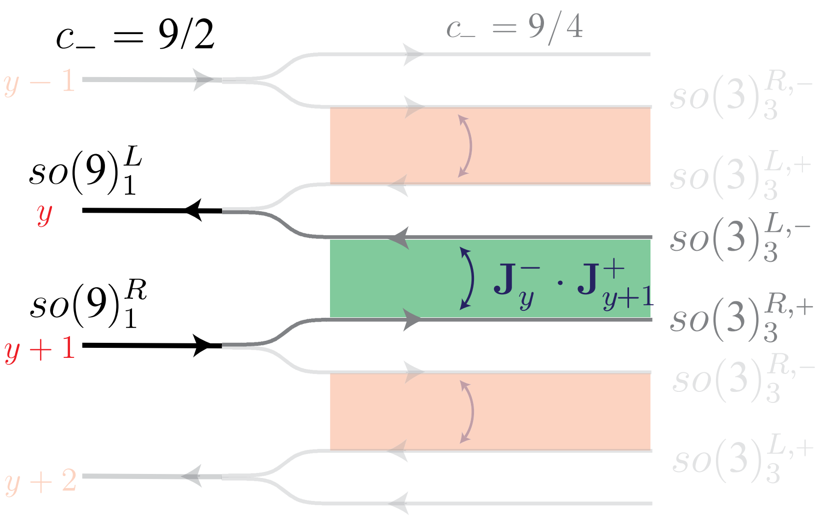
$$S^I_{\mathsf{i}}(z)S^I_{\mathsf{j}}(w) = \frac{\delta_{\mathsf{ij}}}{(z-w)^2} + \frac{i\sqrt{2}\varepsilon_{\mathsf{ijk}}}{z-w}S^I_{\mathsf{k}}(w) + \ldots$$

The gapping Hamiltonian is

$$\mathcal{H}_{\text{int}} = u\sum_{y=-\infty}^{\infty} \mathbf{S}^2_y \cdot \mathbf{S}^1_{y+1} = 2u\sum_{y=-\infty}^{\infty} \partial_x\phi^2_y\partial_x\phi^1_{y+1} - 2\cos\left(4\Theta_{y+1/2}\right),$$

$$4\Theta_{y+1/2} = 2\phi^+_{y+1} - 2\phi^-_y = \widetilde{\phi}^1_{y+1} + \widetilde{\phi}^2_{y+1} + \widetilde{\phi}^1_y - \widetilde{\phi}^2_y.$$

Gapping odd Majorana's



Special case: $N = 9 = 3^2$

Conformal embedding (level rank duality):

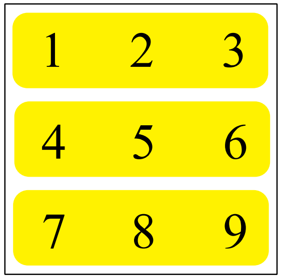
$$so(9)_1 \supseteq so(3)_3 \times so(3)_3$$

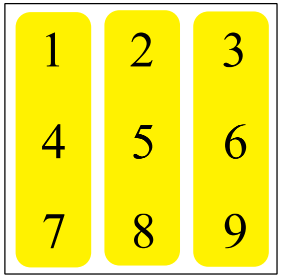
Generator of so(3): $\boldsymbol{\Sigma} = (\Sigma_{\mathrm{x}}, \Sigma_{\mathrm{y}}, \Sigma_{\mathrm{z}})$

$$\Sigma_{\mathrm{x}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \Sigma_{\mathrm{y}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \Sigma_{\mathrm{z}} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

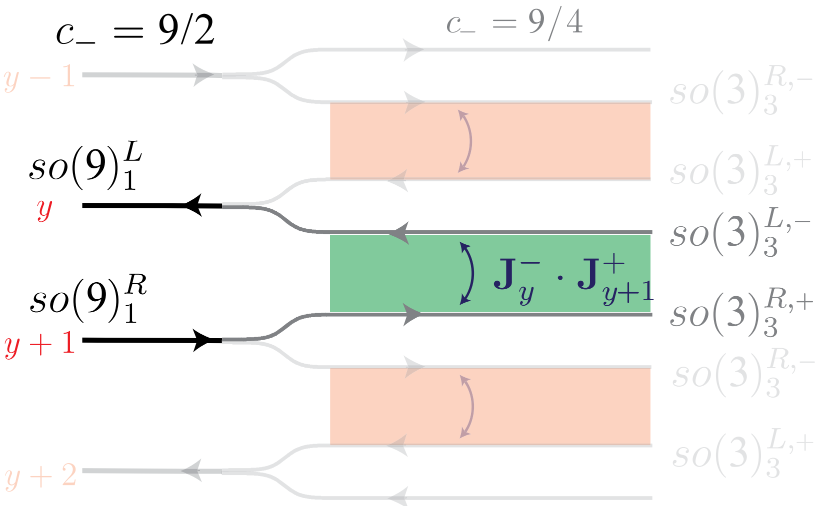
Embedding: $\boldsymbol{\Sigma}^+ = \boldsymbol{\Sigma} \otimes \mathbb{1}_3, \quad \boldsymbol{\Sigma}^- = \mathbb{1}_3 \otimes \boldsymbol{\Sigma}.$

$$\mathbf{J}_{so(3)^{\pm}_3}(z) = \frac{i}{2}\psi^a(z)\boldsymbol{\Sigma}^{\pm}_{ab}\psi^b(z)$$





Gapping odd Majorana's



Special case: $N = 9 = 3^2$

Conformal embedding (level rank duality):

$$so(9)_1 \supseteq so(3)_3 \times so(3)_3$$

level

$$J_\mathrm{i}^{\pm}(z) J_\mathrm{j}^{\pm}(w) = \frac{3\delta_\mathrm{ij}}{(z-w)^2} + \frac{i\varepsilon_\mathrm{ijk}}{z-w} J_\mathrm{k}^{\pm}(w) + \dots$$

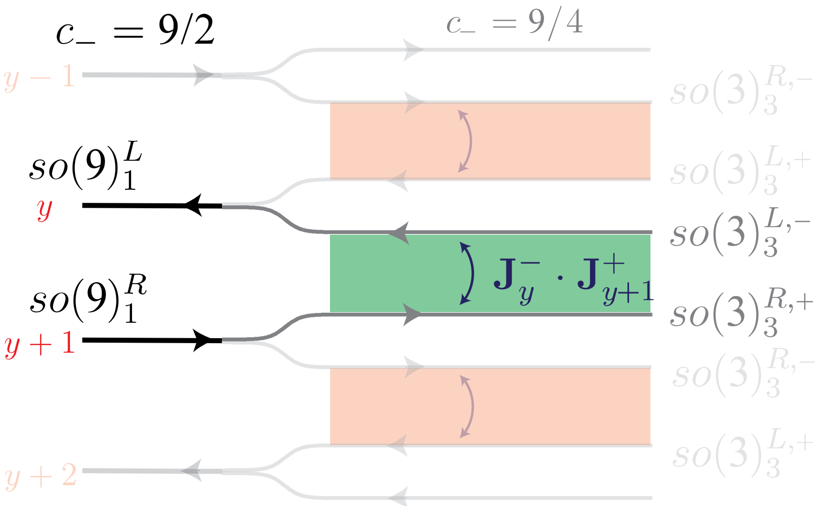
$$T_{so(3)_3^\pm}(z) = \frac{1}{8} \mathbf{J}_{so(3)_3^\pm}(z) \cdot \mathbf{J}_{so(3)_3^\pm}(z).$$

$$= -\frac{1}{4} \sum_{a=1}^{9} \psi^a(z) \partial \psi^a(z) \mp \frac{1}{4} \mathcal{O}_\psi(z)$$

$$\mathcal{O}_\psi(z) = \psi^{1245} + \psi^{1278} + \psi^{4578} + \psi^{1346} + \psi^{1379} + \psi^{4679} + \psi^{2356} + \psi^{2389} + \psi^{5689}$$

$$T_{so(9)_1} = -\frac{1}{2} \sum_{a=1}^{9} \psi^a \partial \psi^a = T_{so(3)_3^+} + T_{so(3)_3^-}. \qquad c_{so(3)_3^\pm} = 9/4.$$

Gapping odd Majorana's



Special case: $N = 9 = 3^2$

4-fermion TR-symmetric interaction:

$$\mathcal{H}_{\text{int}} = u\mathbf{J}^R_{so(3)_3^-} \cdot \mathbf{J}^L_{so(3)_3^+}$$

Further decomposition:

$$so(3)_3 = u(1)_6 \times \text{“}\mathbb{Z}_6\text{”}, \quad \text{“}\mathbb{Z}_6\text{”} = \frac{so(3)_3}{so(2)_3} = \frac{su(2)_6}{u(1)_6}$$

parafermion

$$J_{\pm}^{R/L} = \mp\sqrt{3}e^{\mp i\phi^\rho_{R/L}}\Psi^{\mp}_{R/L}$$

$$\mathcal{H}_{\text{int}} \sim 3u\left[e^{i(\phi^\rho_L - \phi^\rho_R)}\Psi^\dagger_R\Psi_L + h.c.\right]$$

$$c^1_R = \frac{\psi^1_R + i\psi^4_R}{\sqrt{2}}, \quad c^2_R = \frac{\psi^2_R + i\psi^5_R}{\sqrt{2}}, \quad c^3_R = \frac{\psi^3_R + i\psi^6_R}{\sqrt{2}}$$

$$c^1_L = \frac{\psi^1_L + i\psi^2_L}{\sqrt{2}}, \quad c^2_L = \frac{\psi^4_R + i\psi^5_L}{\sqrt{2}}, \quad c^3_L = \frac{\psi^7_L + i\psi^8_L}{\sqrt{2}}$$

$$\Psi_R = \frac{1}{\sqrt{3}}\left(e^{i\phi^{\sigma,1}_R}\psi^7_R + e^{i\phi^{\sigma,2}_R}\psi^8_R + e^{i\phi^{\sigma,3}_R}\psi^9_R\right)$$

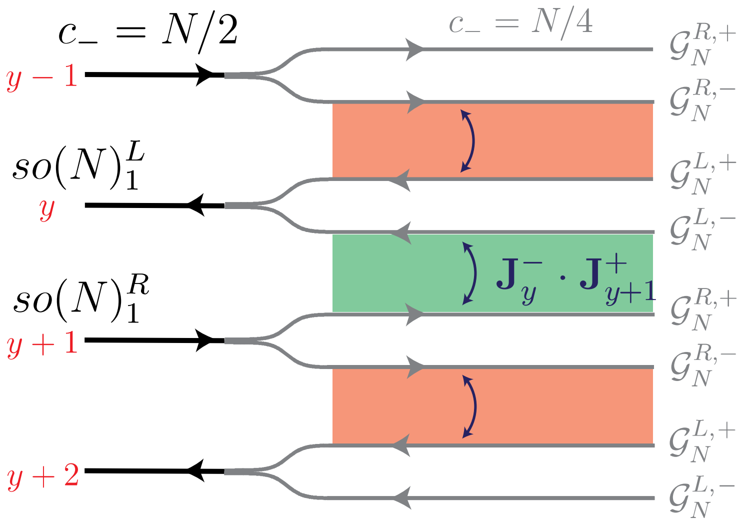
$$\Psi_L = \frac{1}{\sqrt{3}}\left(e^{i\phi^{\sigma,1}_L}\psi^3_L + e^{i\phi^{\sigma,2}_L}\psi^6_L + e^{i\phi^{\sigma,3}_L}\psi^9_L\right)$$

$$c^j_{R/L} \sim \frac{1}{\sqrt{l_0}}\exp\left(i\widetilde{\phi}^j_{R/L}\right) \qquad \phi^\rho_{R/L} = \frac{\widetilde{\phi}^1_{R/L} + \widetilde{\phi}^2_{R/L} + \widetilde{\phi}^3_{R/L}}{3}.$$

$$\phi^{\sigma,j}_{R/L} = \widetilde{\phi}^j_{R/L} - \phi^\rho_{R/L}$$

Gapping odd Majorana's

$N = 9 + 2r$



$$so(N)_1$$

$$\downarrow$$

$$so(9)_1 \times so(2r)_1$$

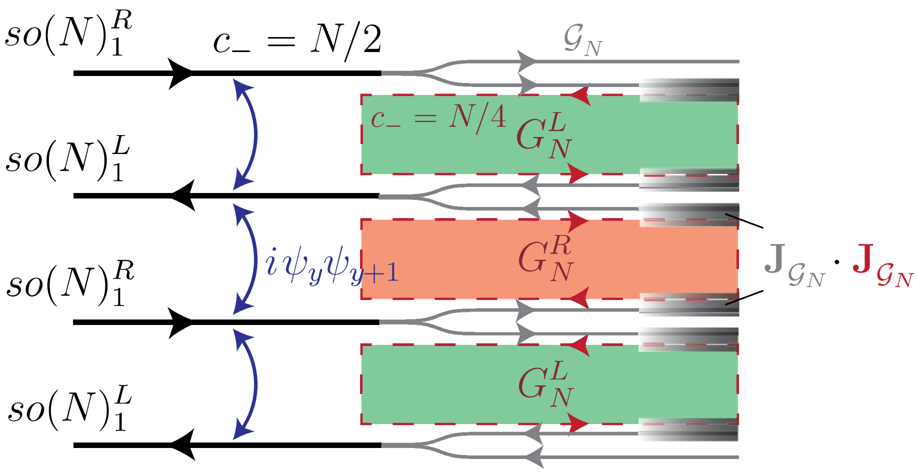
$$so(3)_3 \times so(r)_1 \qquad so(3)_3 \times so(r)_1$$

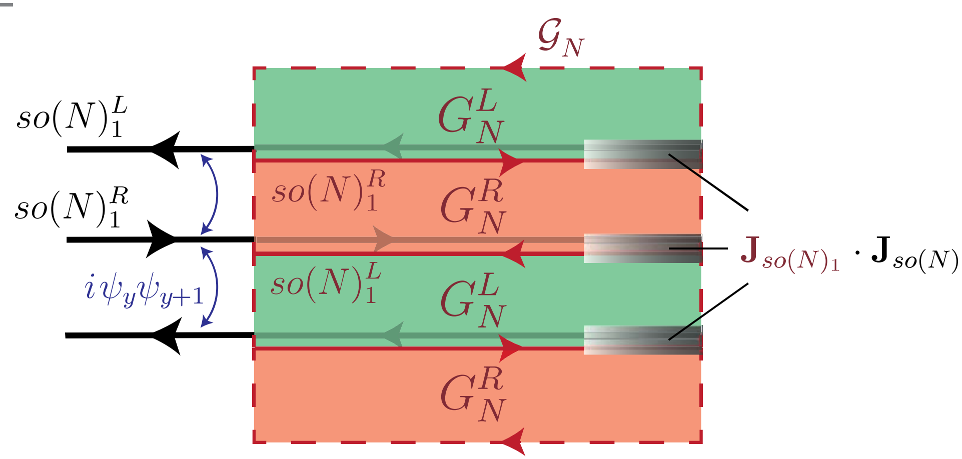
$$\mathcal{G}_N^+ \qquad \mathcal{G}_N^-$$

Negative r = reverse propagating channels

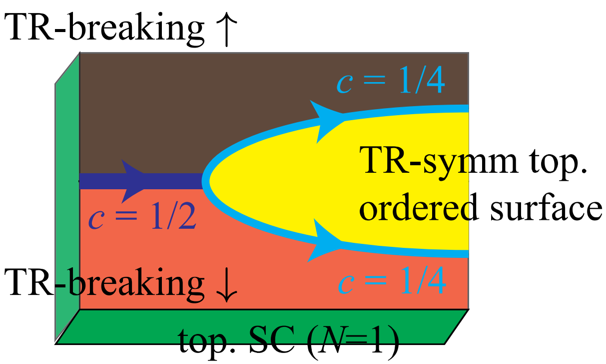
$$\begin{aligned} \mathcal{H}_{\text{int}} &= u\mathbf{J}_{\mathcal{G}_N^+} \cdot \mathbf{J}_{\mathcal{G}_N^-} \\ &= u\mathbf{J}_{so(3)_3^+} \cdot \mathbf{J}_{so(3)_3^-} + u\mathbf{J}_{so(r)_1^+} \cdot \mathbf{J}_{so(r)_1^-} \end{aligned}$$

Gapping by Quantum Hall stripes



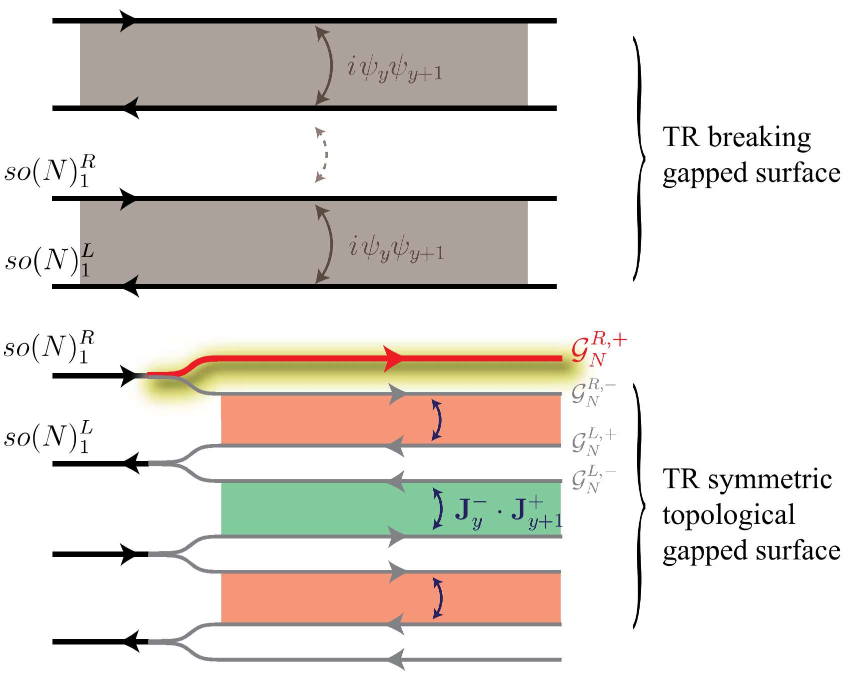


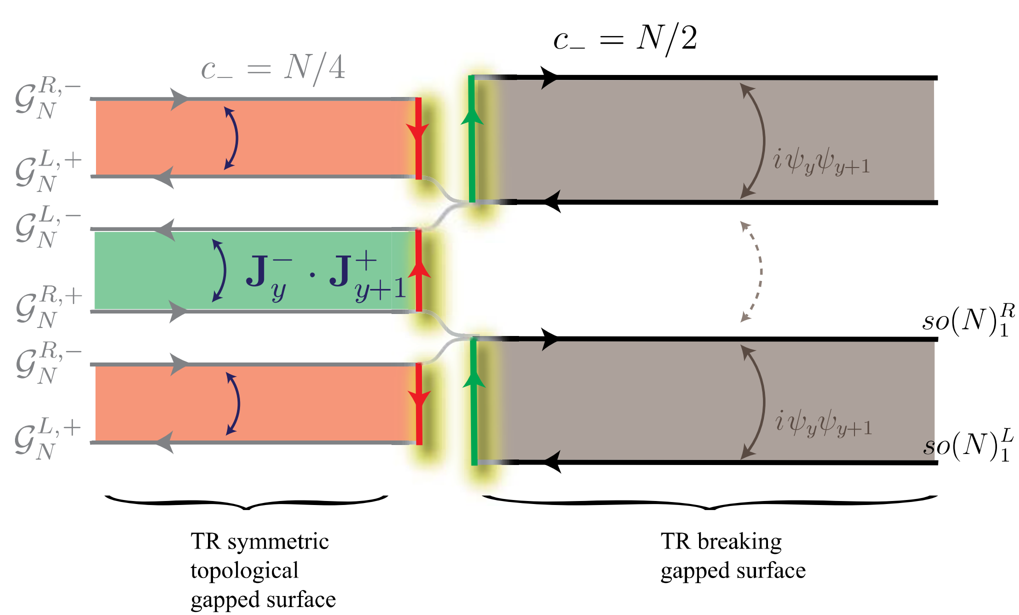
Time reversal breaking interface



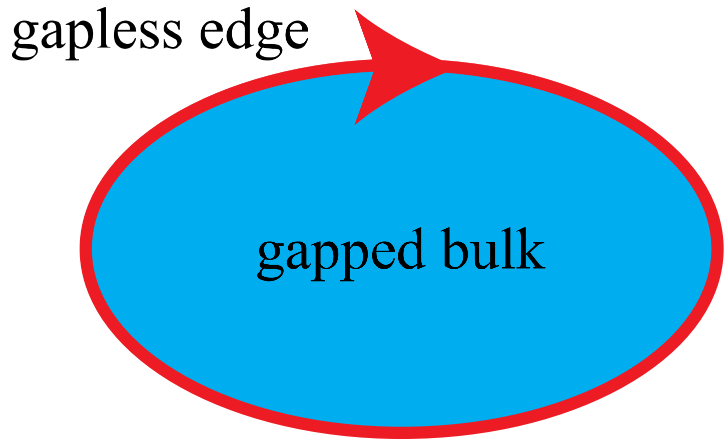
Interface CFT:

$$\mathcal{G}_N = \begin{cases} so(N/2)_1 & \text{for } N \text{ even} \\ so(3)_3 \times so\left(\frac{N-9}{2}\right)_1 & \text{for } N \text{ odd} \end{cases}$$





Bulk boundary correspondence



- Topological order
- Quasiparticles
- Fusion
- Exchange statistics
- Braiding

- Boundary CFT
- Primary fields
- Operator product expansion
- Scaling dimension
- Modular transformation

Topological order

$$G_N = \begin{cases} SO(r)_1, & \text{for } N = 2r \\ SO(3)_3 \boxtimes_b SO(r)_1 & \text{for } N = 9 + 2r \end{cases}$$

$N = 2r$

$\mathbf{x}$	r even				r odd		
$\mathbf{x}$	1	ψ	s_+	s_-	1	ψ	σ
$d_\mathbf{x}$	1	1	1	1	1	1	$\sqrt{2}$
$\theta_\mathbf{x}$	1	-1	$e^{\pi i r/8}$	$e^{\pi i r/8}$	1	-1	$e^{\pi i r/8}$

$$\psi \times \psi = 1, \quad s_\pm \times \psi = s_\mp$$

$$s_\pm \times s_\pm = \begin{cases} 1, & \text{for } r \equiv 0 \bmod 4 \\ \psi, & \text{for } r \equiv 2 \bmod 4 \end{cases}$$

$$\psi \times \psi = 1, \quad \psi \times \sigma = \sigma, \quad \sigma \times \sigma = 1 + \psi$$

$$S_{SO(r)_1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & e^{i\pi r/4} & -e^{i\pi r/4} \\ 1 & -1 & -e^{i\pi r/4} & e^{i\pi r/4} \end{pmatrix}, \quad \text{for } r \text{ even}$$

$$S_{SO(r)_1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}, \quad \text{for } r \text{ odd}$$

Topological order

$$G_N = \begin{cases} SO(r)_1, & \text{for } N = 2r \\ SO(3)_3 \boxtimes_b SO(r)_1 & \text{for } N = 9 + 2r \end{cases}$$

$N = 9 + 2r$

$\mathbf{x}$	1	α_+	γ_+	β	γ_-	α_-	f
$d_{\mathbf{x}}$	1	$\sqrt{2+\sqrt{2}}$	$1+\sqrt{2}$	$\sqrt{4+2\sqrt{2}}$	$1+\sqrt{2}$	$\sqrt{2+\sqrt{2}}$	1
$\theta_{\mathbf{x}}$	1	$e^{\pi i \frac{3+2r}{16}}$	$e^{i\pi/2}$	$e^{\pi i \frac{15+2r}{16}}$	$e^{-i\pi/2}$	$e^{\pi i \frac{3+2r}{16}}$	-1

r even

$\mathbf{x}$	1	α_+	γ_+	β	γ_-	α_-	f
$d_{\mathbf{x}}$	1	$\sqrt{2+\sqrt{2}}$	$1+\sqrt{2}$	$\sqrt{4+2\sqrt{2}}$	$1+\sqrt{2}$	$\sqrt{2+\sqrt{2}}$	1
$\theta_{\mathbf{x}}$	1	$e^{\pi i \frac{15+2r}{16}}$	$e^{i\pi/2}$	$e^{\pi i \frac{3+2r}{16}}$	$e^{-i\pi/2}$	$e^{\pi i \frac{15+2r}{16}}$	-1

r odd

$$f \times f = 1, \quad f \times \gamma_\pm = \gamma_\mp, \quad f \times \alpha_\pm = \alpha_\mp, \quad f \times \beta = \beta$$
$$\gamma_\pm \times \gamma_\pm = 1 + \gamma_+ + \gamma_-, \quad \alpha_\pm \times \beta = \gamma_+ + \gamma_-$$
$$\beta \times \beta = 1 + \gamma_+ + \gamma_- + f, \quad \beta \times \gamma_\pm = \alpha_+ + \alpha_- + \beta$$

$$\alpha_\pm \times \alpha_\pm = \begin{cases} 1 + \gamma_+, & \text{for } r \equiv 0 \bmod 4 \\ f + \gamma_+, & \text{for } r \equiv 1 \bmod 4 \\ f + \gamma_-, & \text{for } r \equiv 2 \bmod 4 \\ 1 + \gamma_-, & \text{for } r \equiv 3 \bmod 4 \end{cases}$$

$$\alpha_\pm \times \gamma_\pm = \begin{cases} \alpha_+ + \beta, & \text{for } r \text{ even} \\ \alpha_- + \beta, & \text{for } r \text{ odd} \end{cases}$$

Topological order

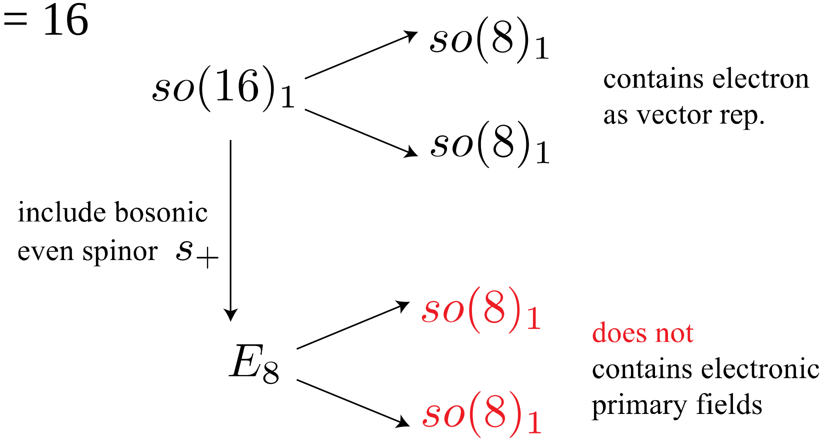
$$G_N = \begin{cases} SO(r)_1, & \text{for } N = 2r \\ SO(3)_3 \boxtimes_b SO(r)_1 & \text{for } N = 9 + 2r \end{cases}$$

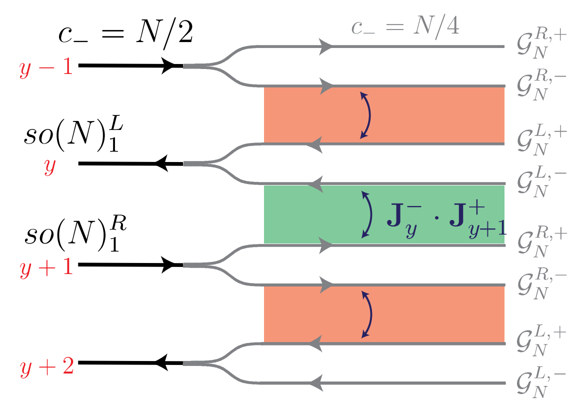
32-fold periodicity

$$G_{N+32} \cong G_N$$

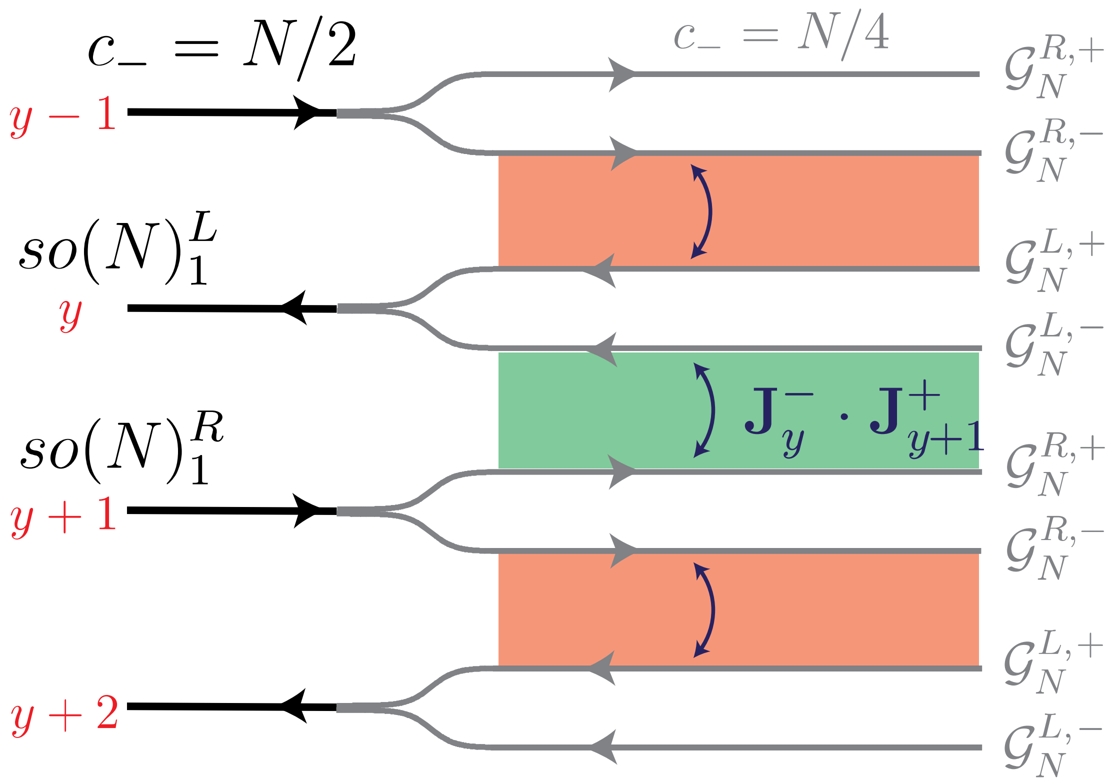
$$G_M \boxtimes_b G_N \cong G_{M+N}$$

Special case: $N = 16$





Conclusion



$$G_N = \begin{cases} SO(r)_1, & \text{for } N = 2r \\ SO(3)_3 \boxtimes_b SO(r)_1 & \text{for } N = 9 + 2r \end{cases}$$