Part I:
Theory of the many-body localization transition in one dimension

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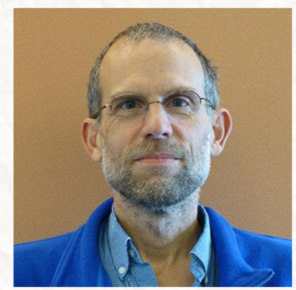
Collaborators: Ronen Vosk (WIS), David Huse (Princeton)



R. Vosk, D. A. Huse and EA, PRX **5**, 031032 (2015)

erc UQUAM ultracold quantum matter

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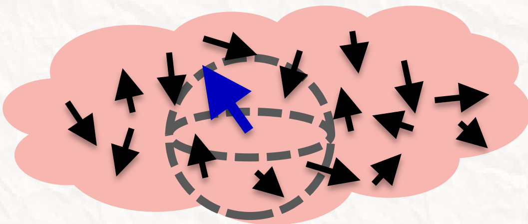
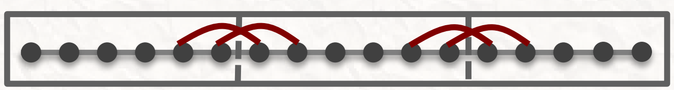
Why the transition is interesting

<u>Many-body localized</u>



Quantum coherent dynamics

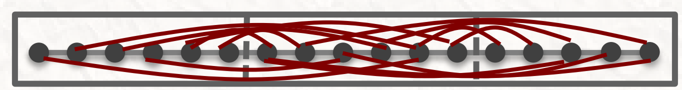
Area law eigenstate entanglement



<u>Thermalizing</u>

"Classical" dynamics

Volume-law eigenst. entanglement



The many-body localization transition:

1. Sharp interface between quantum and classical worlds
2. Fundamental change in entanglement pattern.
   More radical than in any known transition.

Can we understand the critical point?

Outline

- Toy model of the MBL critical point
- RG approach to the MBL transition
  - Universal dynamics
  - Entanglement sclaing
- Part II:
  Many-body delocalization via marginally localized phonons
  (S. Banerjee and EA arXiv:1511.03676)

Rough criterion for MBL (T=∞)

$$H = \sum_i V_i S_i^z + \sum_{\langle ij \rangle} J^z S_i^z S_j^z + J^x S_i^x S_j^x$$

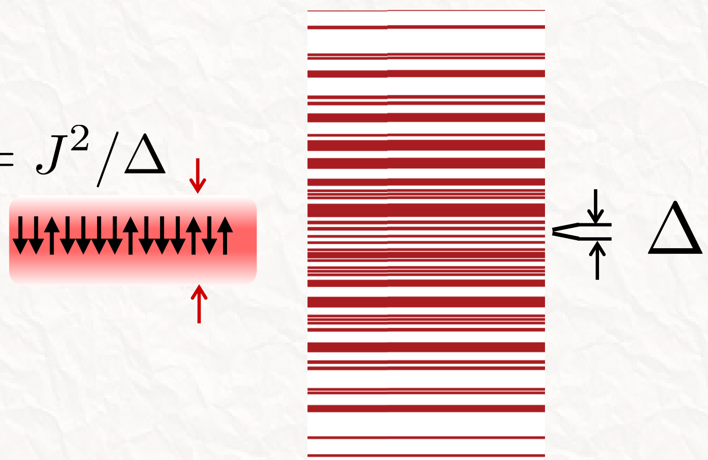
Matrix element to move between typical configurations of L spins:

$$J(L) \sim J^z \left(J^x/\Delta_0\right)^L \equiv J^z e^{-L/\xi_*}$$

$$\Delta(L) \sim \frac{\Delta_0}{2^L} = \Delta_0 e^{-L\ \ln 2}$$

$$g(L) \equiv \frac{\Gamma(L)}{\Delta(L)} = \left(\frac{J(L)}{\Delta(L)}\right)^2$$

$$\Gamma = J^2/\Delta$$



<u>Delocalized phase:</u> $g(L) \gg 1$

Resonance condition = condition for the system to serve as it's own bath:

Rough criterion for MBL (T=∞)

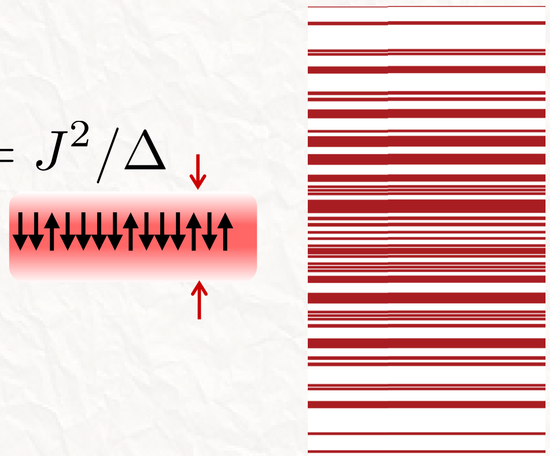
$$H = \sum_i V_i S_i^z + \sum_{\langle ij \rangle} J^z S_i^z S_j^z + J^x S_i^x S_j^x$$

Matrix element to move between typical configurations of L spins:

↓↓↑↓↓↓↓↑↓↓↓↑↓↑ → ↑↓↑↓↑↑↑↓↑↑↓↓↑↑

$$J(L) \sim J^z \left(J^x/\Delta_0\right)^L \equiv J^z e^{-L/\xi_*}$$

$$\Gamma = J^2/\Delta$$



$$\Delta(L) \sim \frac{\Delta_0}{2^L} = \Delta_0 e^{-L \ln 2}$$

$$g(L) \equiv \frac{\Gamma(L)}{\Delta(L)} = \left(\frac{J(L)}{\Delta(L)}\right)^2$$

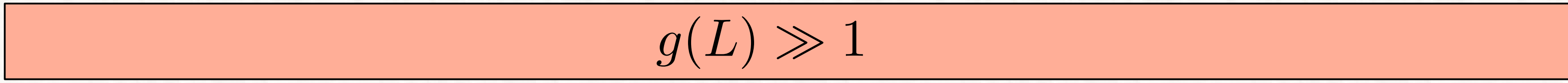
Localized phase: $g(L) < 1$

requires $\xi_* < 1/\ln 2$

Does it mean non-diverging localization length and 1st order transition? **NO!**

Toy model of the critical point

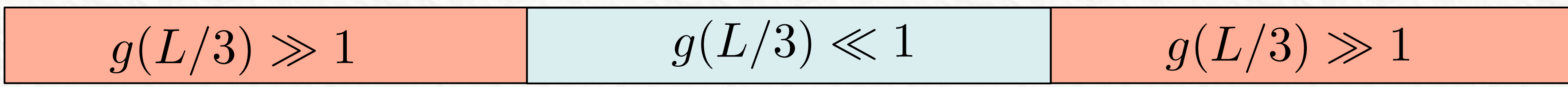
We want a thermal system of length L:



Now consider 3 subsystems of length L/3.

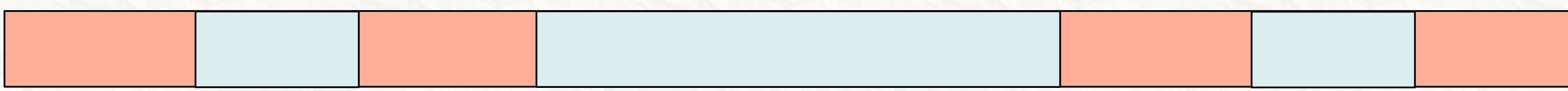
Must they all individually have g(L/3)>>1?

No! The minimal configuration should be something like this:

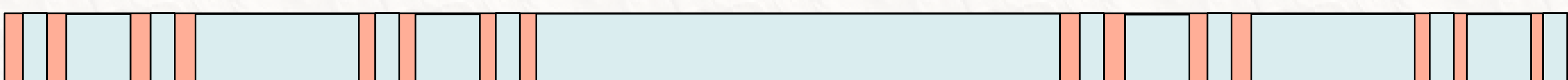


The thermal sides are then just able to thermalize the middle.

Now apply this reasoning to each of the two thermal sides to get:



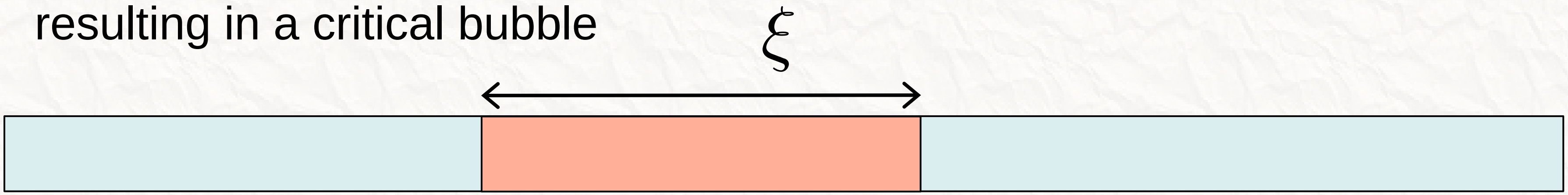
And iterate:



Toy model of the critical point

Critical system is a Cantor set of bare thermal regions
with fractal dimension: $d_f \approx \ln 2 / \ln 3$
This should be just enough to thermalize the whole system!

Fluctuation in the tuning parameter (bare disorder)
resulting in a critical bubble

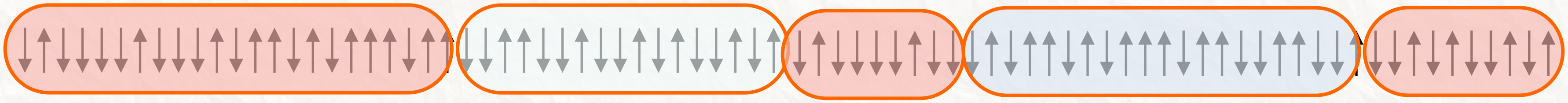


$$\delta\Delta \sim \frac{1}{\sqrt{N_{\text{special}}}} \sim \frac{1}{\xi^{d_f/2}}$$

$\Rightarrow$ $\xi \sim (\Delta - \Delta_c)^{-2/d_f}$ $\qquad$ $\nu = 2/d_f \approx 3.2$

RG approach to the MBL transition

Vosk, Huse and E.A. arXiv:1412.3117



Spin chain fragmented into puddles of different types:
incipient insulators and incipient metals.
Modeled as coupled random matrices:

$\Delta_i , \Gamma_i \quad \Rightarrow \quad g_i = \Gamma_i / \Delta_i$

Δ_i Mean level spacing in the block

$\Gamma_i^{-1} = \tau_i$ Time for entanglement to spread across the block

$g_i \ll 1$ "insulating block"

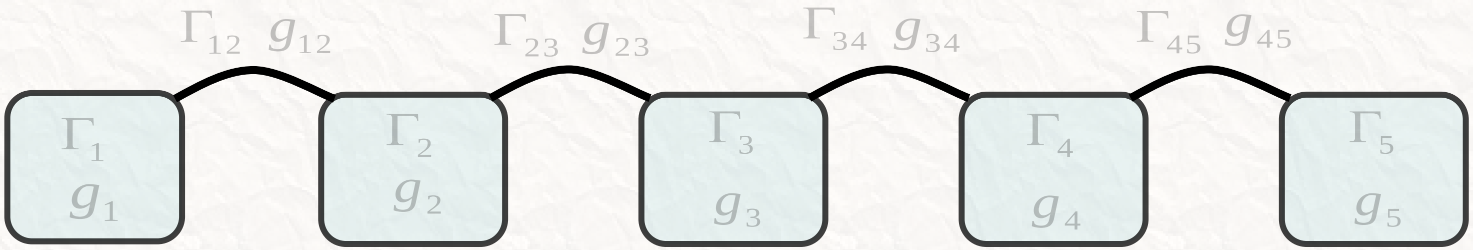
(Poisson level statistics)

$g_i \gg 1$ "thermalizing block"

(Wigner-Dyson statistics)

RG flow: itteratively join matrices that entangle with each other at running cutoff scale. At the end of the flow we are left with one big block that is either insulating or thermalizing

Starting point for RG: chain of coupled blocks



Meaning of
the link variables:

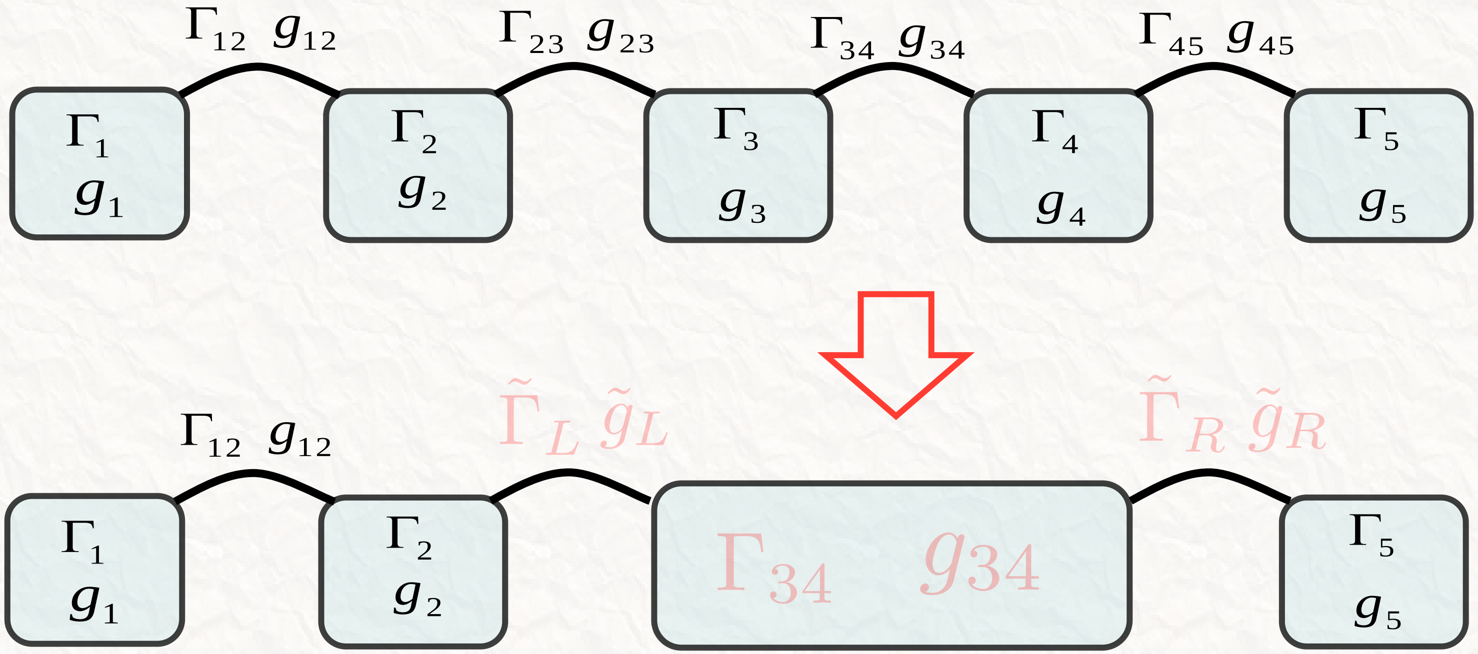
Γ_{ij} = far end-to-end entanglement rate of adjacent blocks
(Γ of the two blocks if they were considered as a single block)

Δ_{ij} = Mean level spacing of the two block system $\Delta_{ij} \sim 2^{-l_{ij}}$

$g_{ij} >> 1$ → 'effective' link ('thermalizing')

$g_{ij} << 1$ → 'ineffective' link ('insulating')

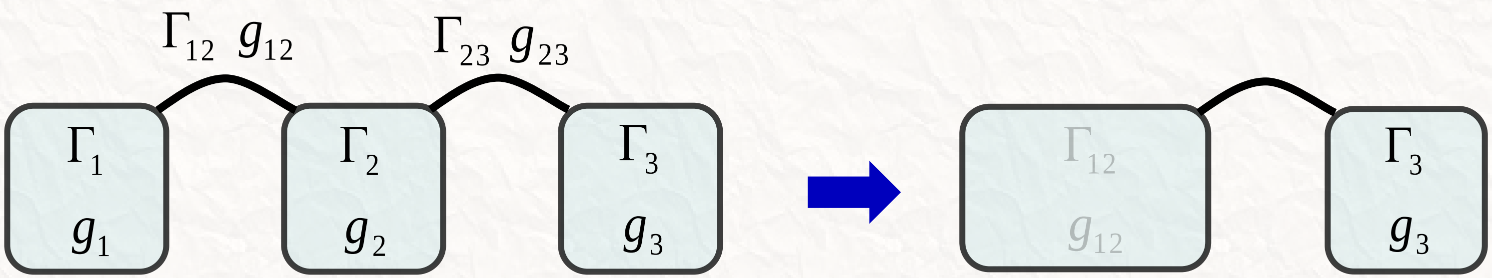
Schematics of the RG



Join blocks which entangle with each other on the fastest scale.
Then compute renormalized couplings to the left and right.

Computing the flow will tell us whether we end up with one big thermalizing matrix ($g \gg 1$) or a big insulator ($g \ll 1$) at large scales

RG scheme



The simplest limits:

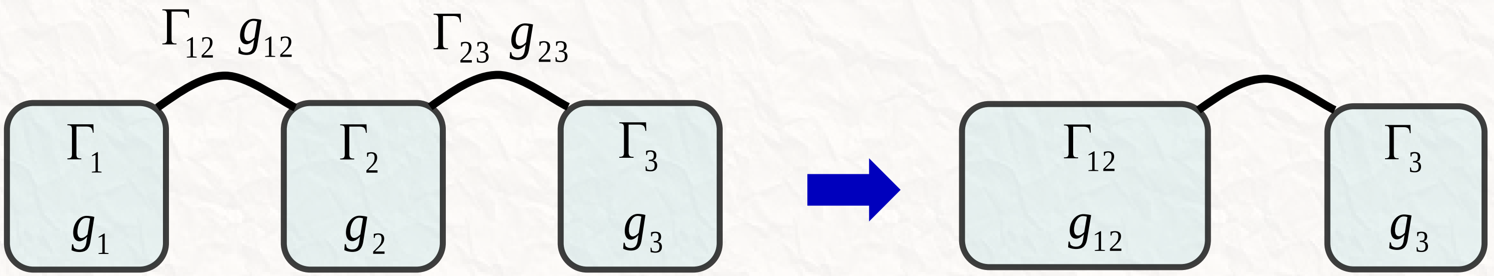
(i) Two 'insulating' links, i.e. g_{12}<<1 and g_{23}<<1 $\Rightarrow$ $\Gamma_R = \frac{\Gamma_{12}\Gamma_{23}}{\Gamma_2}$

Can be derived for insulators from first principles.
But also simply understood by taking a log of the two sides:

$$\log \tau_{tot} = \log \tau_{12} + \log \tau_{23} - \log \tau_2$$

$$l_{tot} = l_{12} + l_{23} - l_2$$

RG scheme



The simplest limits:

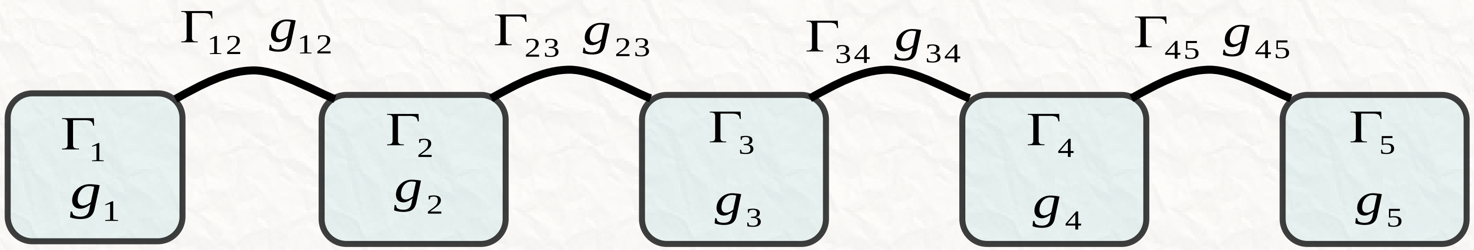
(ii) Two thermalizing links, i.e. $g_{12}, g_{23} >> 1$

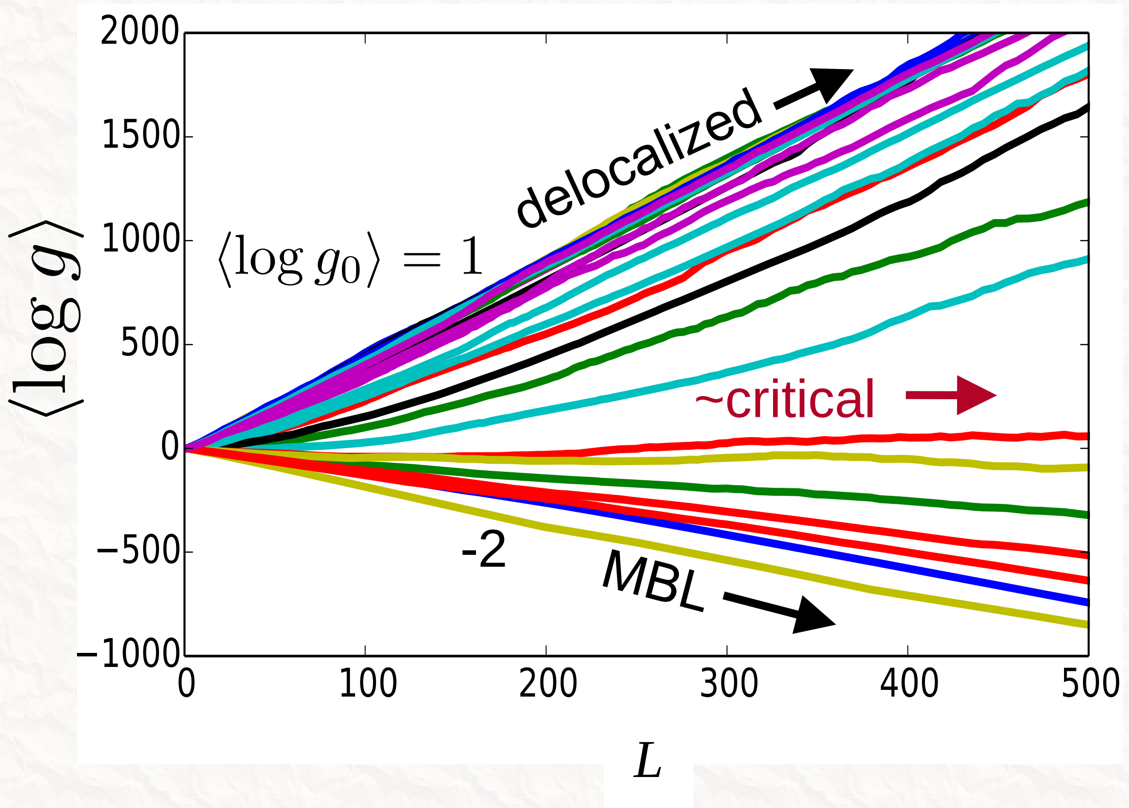
Γ_R cannot be derived perturbatively in this case.
But we know: energy transport is diffusive
and (therefore) entanglement propagates ballistically.

$$\frac{1}{\Gamma_R} = \frac{1}{\Gamma_{12}} + \frac{1}{\Gamma_{23}} - \frac{1}{\Gamma_2}$$

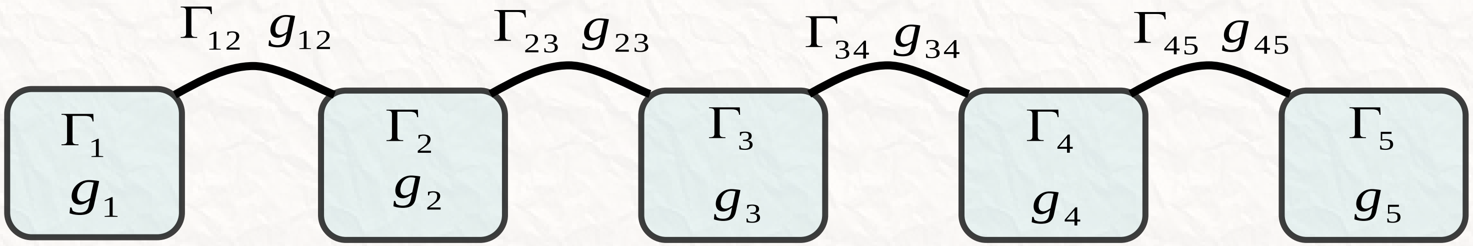
Rules (i) and (ii) apply to interfaces provided $g_{12} << 1, \; g_{23} >> 1$

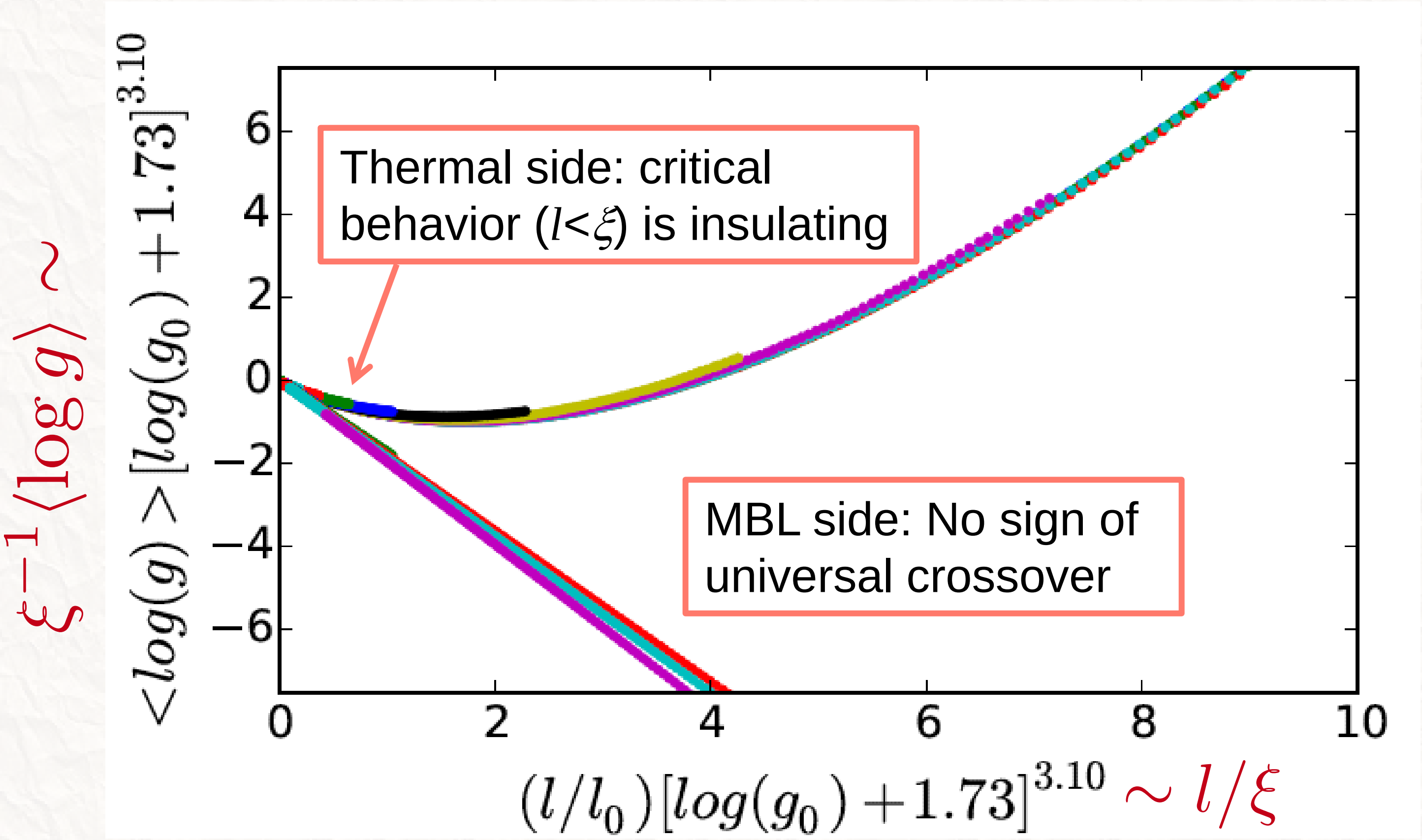
Outcome of the RG flow





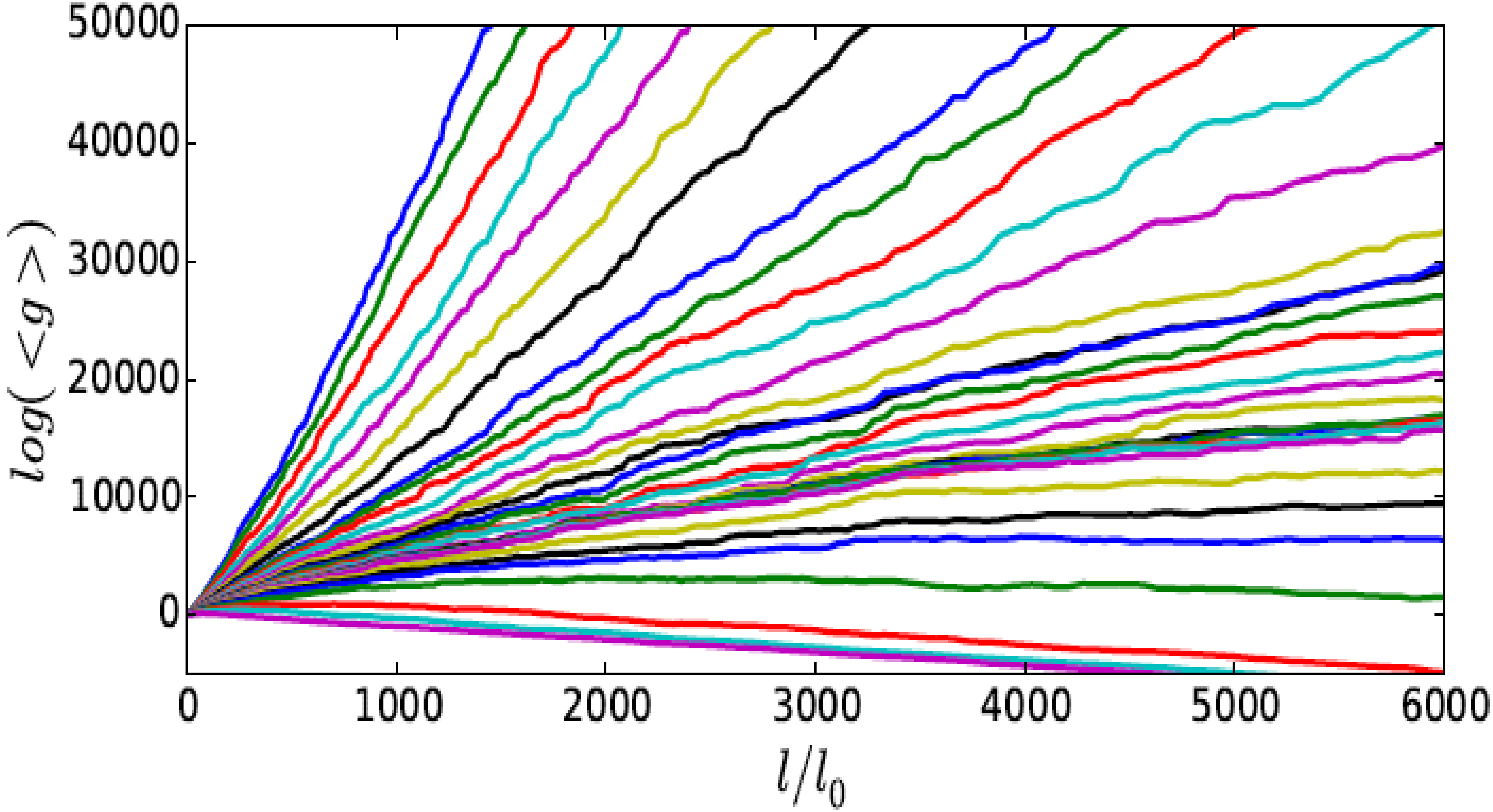
Outcome of the RG flow





Scaling and universal crossover in the MBL side?

Plot log<g> instead of <log g>:



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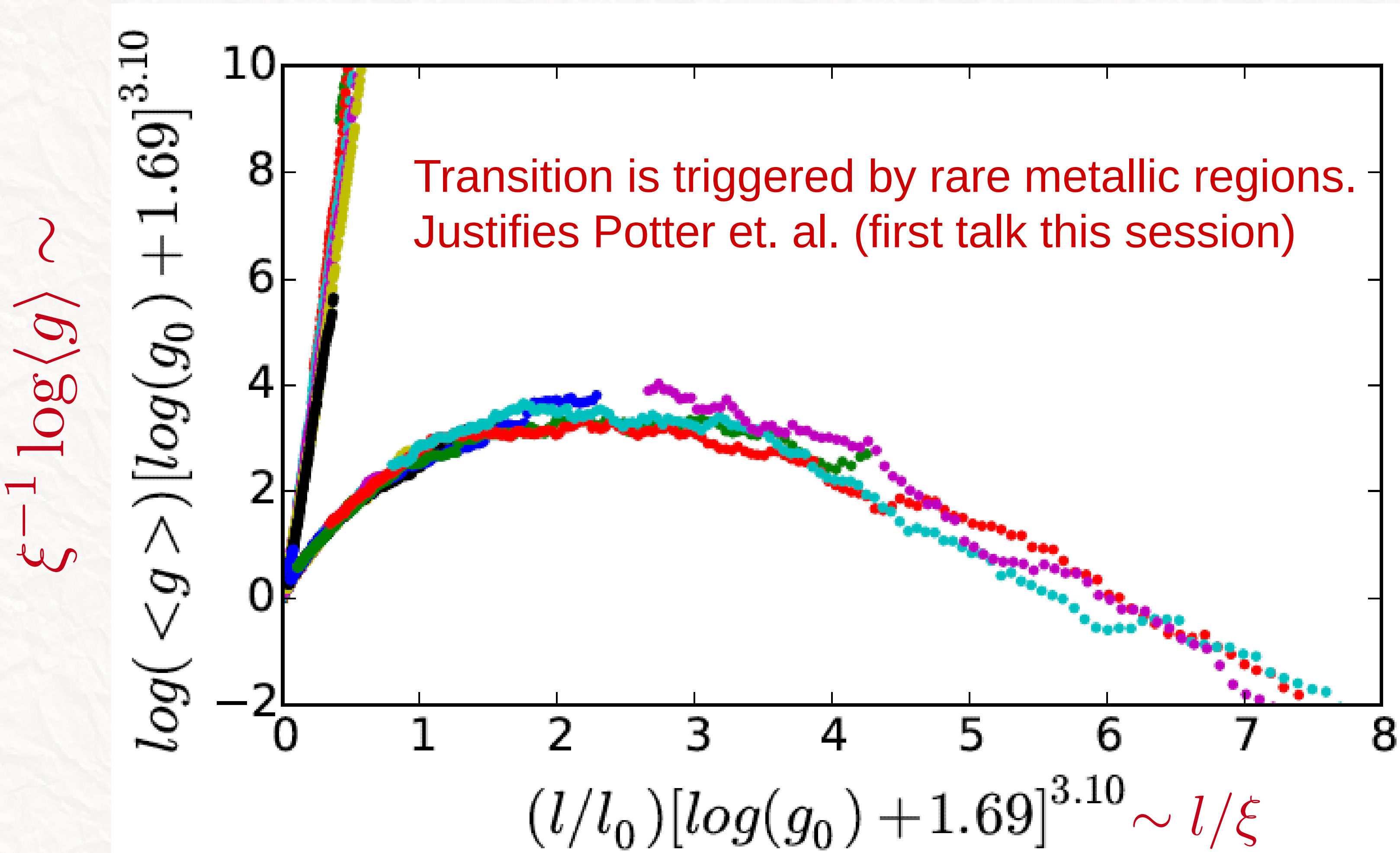
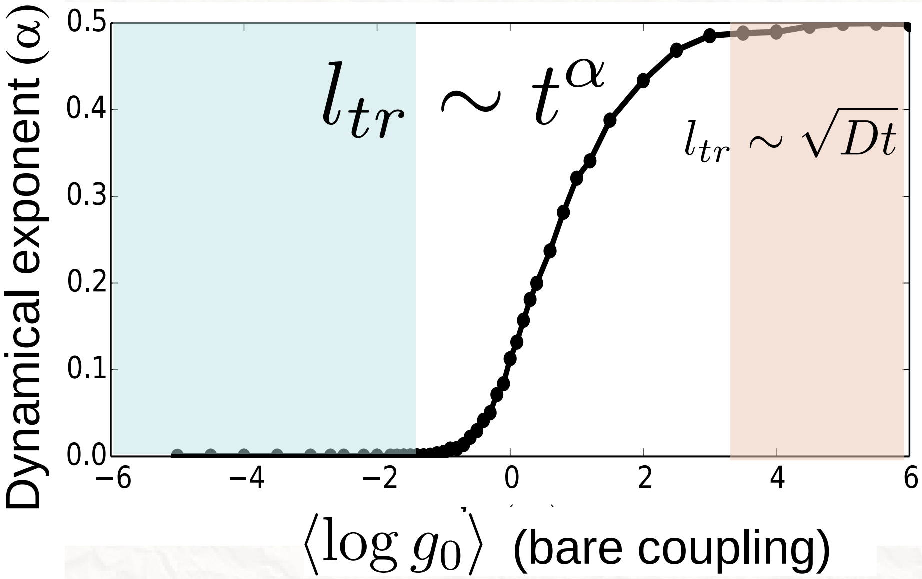
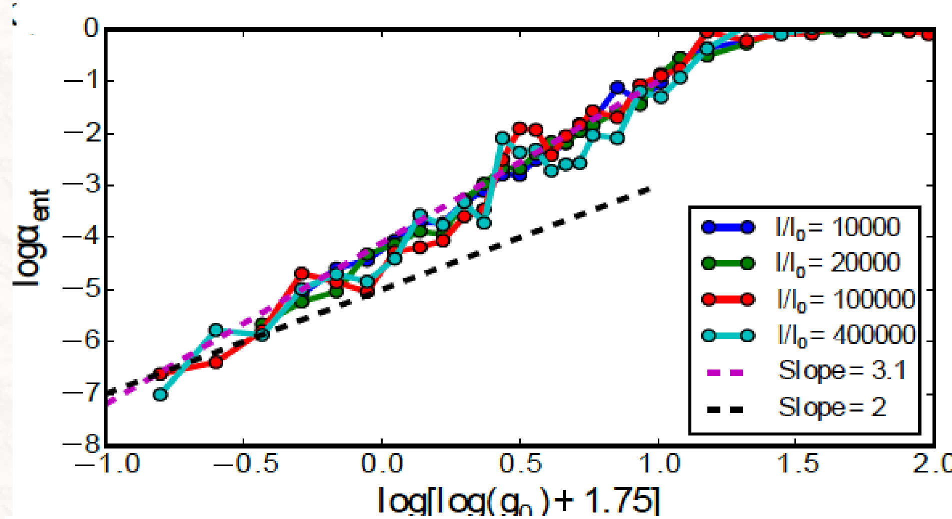


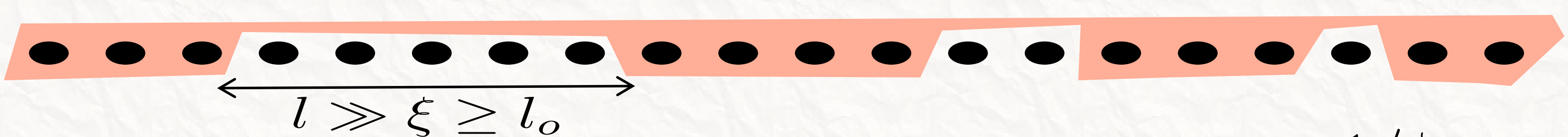
Prediction for dynamics: Sub-diffusive behavior in the ergodic phase





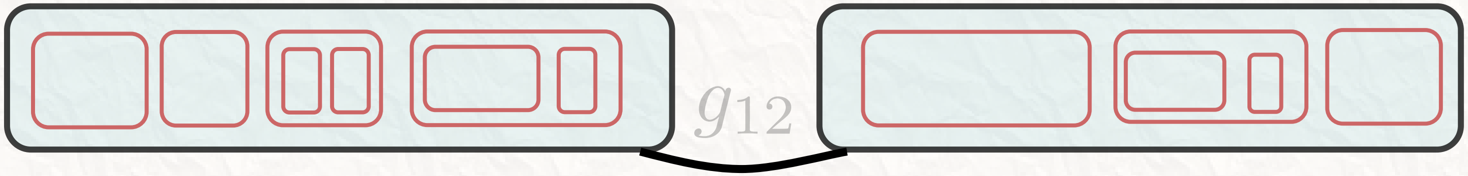
Seen also in ED studies: Bar-Lev et.al 2014; Agarwal et.al 2014

A result of Griffiths regions:
exponentially rare regions that cause exponential delay of transport.

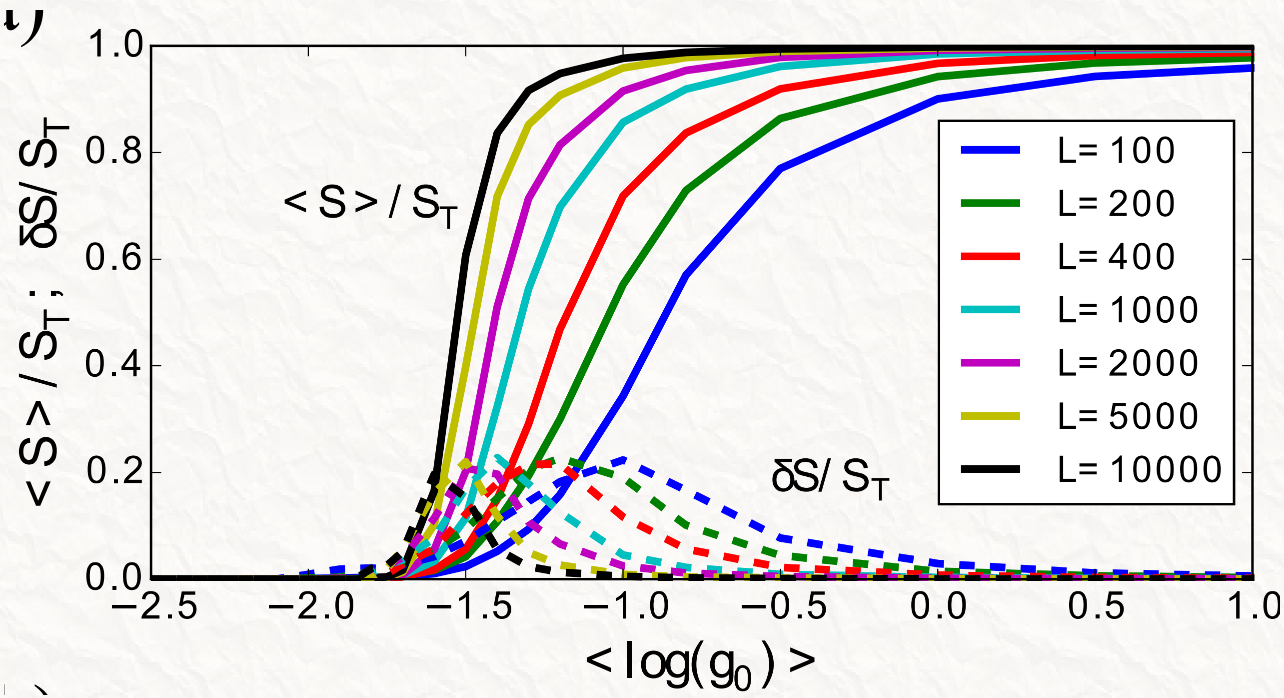


$\alpha \sim 1/\xi$

RG result II - eigenstate entropy



$$S_E(L/2) \sim \log_2 [g(L) + 1]$$



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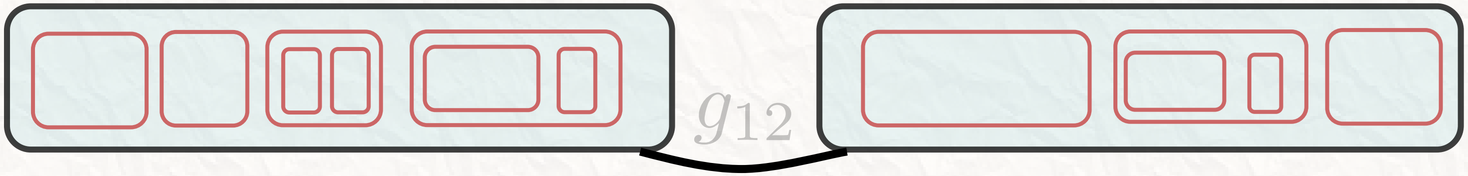

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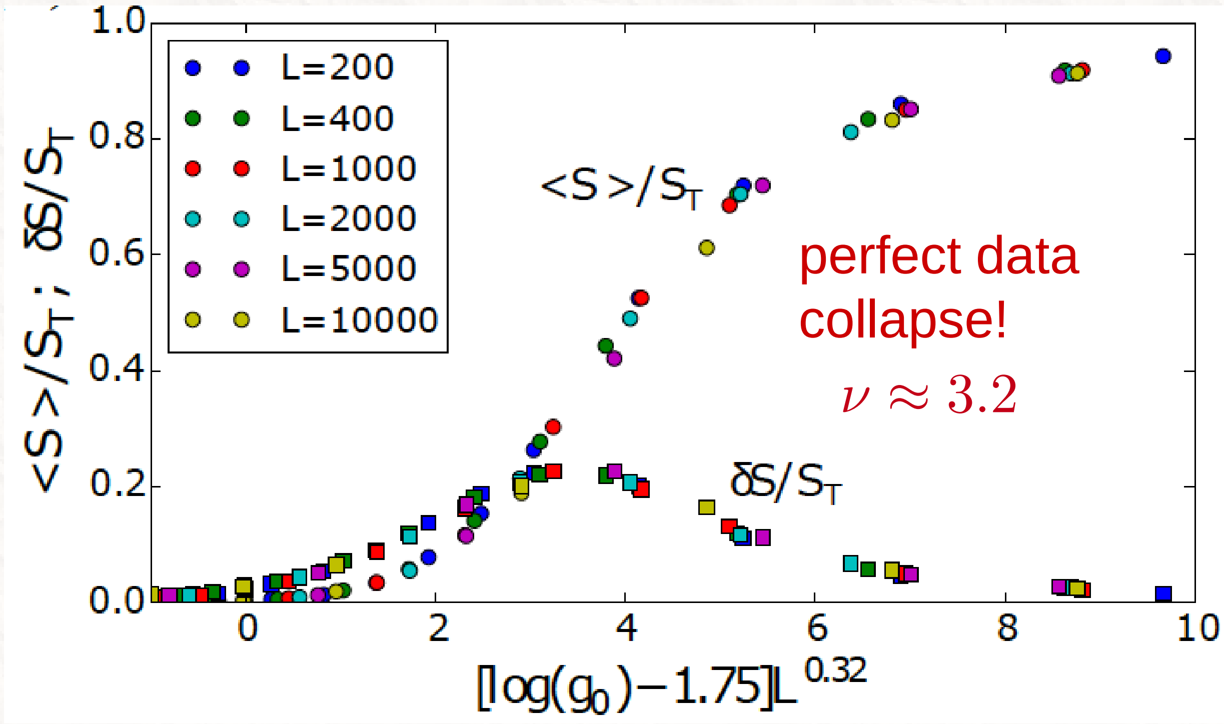
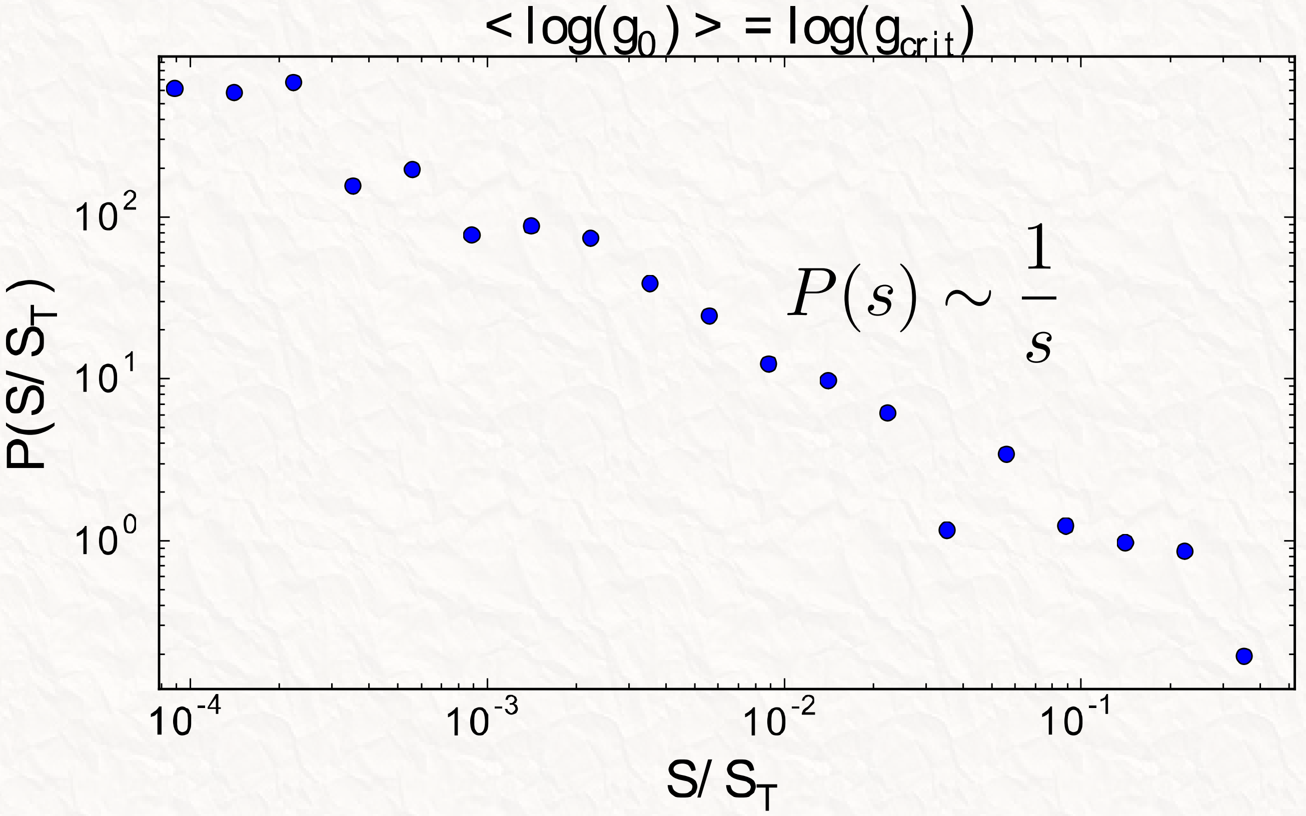


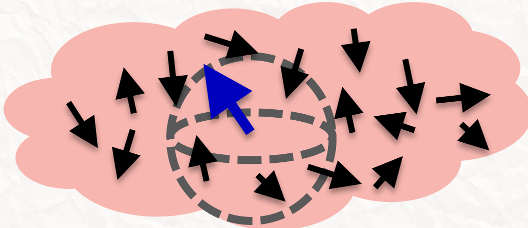
- Universal jump to full thermal entropy → Direct transition to thermal state

Broad entropy distribution at criticality



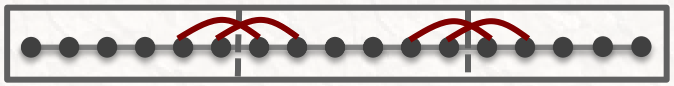
Summary

Many-body localized | **Thermalizing**



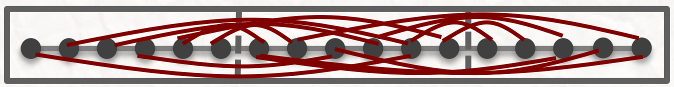
Quantum coherent dynamics

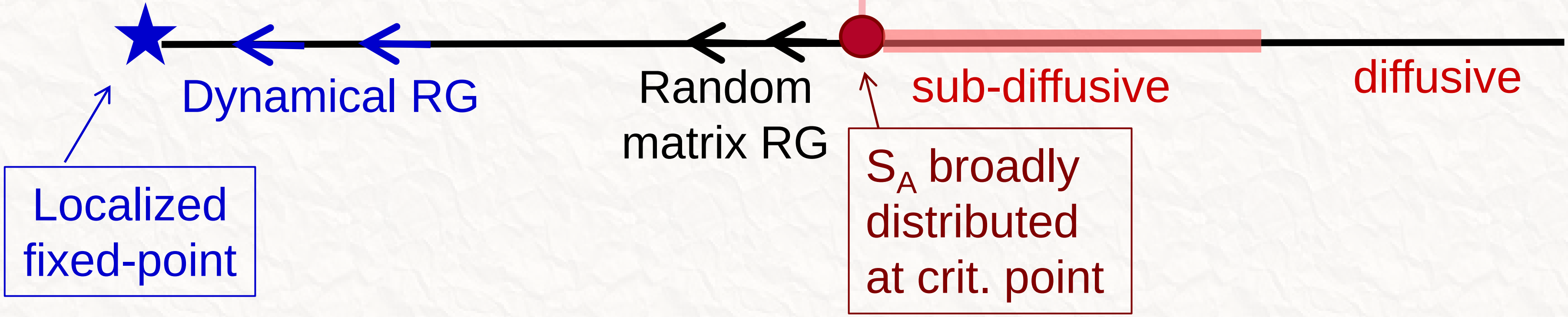
Area law entanglement



"Classical" dynamics

Volume law entanglement





- More microscopic foundations for the RG ? Controlled numerics ?
- Generalization to higher dimensions ?

Part II:
Many-body delocalization via marginally localized phonons ?

Sumilan Banerjee and EA arXiv:1511.03676

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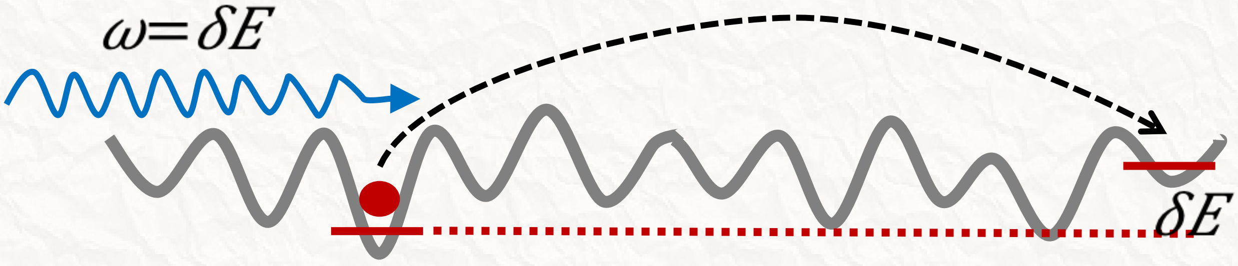
BSF United States – Israel Binational Science Foundation

Usual argument for absence of MBL with phonons

<u>Phonon assisted hopping (Mott):</u>

$$\sigma = \sigma_0 e^{-(T_0/T)^{1/(d+1)}}$$

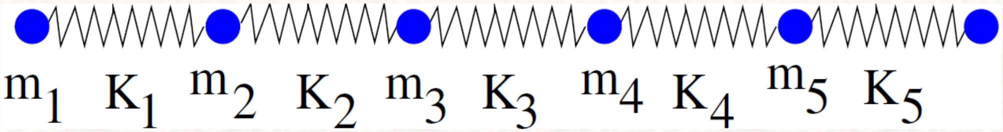
Phonons provide a bath with continuous spectrum.



But what if the phonons themselves are localized?

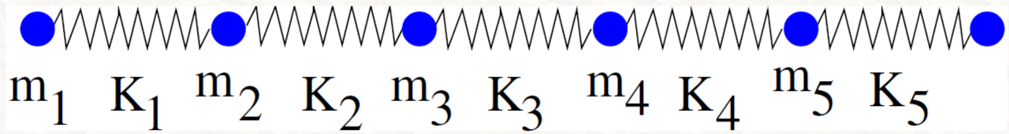
Examples:

1. Random harmonic chain



2. Phonons of a disordered superfluid

Phonon localization in 1d



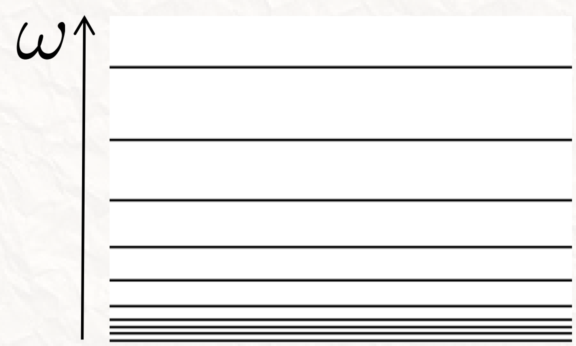
- Phonons localized at all non vanishing frequencies
- But the zero frequency mode is protected (Goldstone mode)
- Divergence of the localization length at low frequency:

$$l_\omega = l_0(\omega_0/\omega)^\alpha \qquad 1 < \alpha \leq 2$$

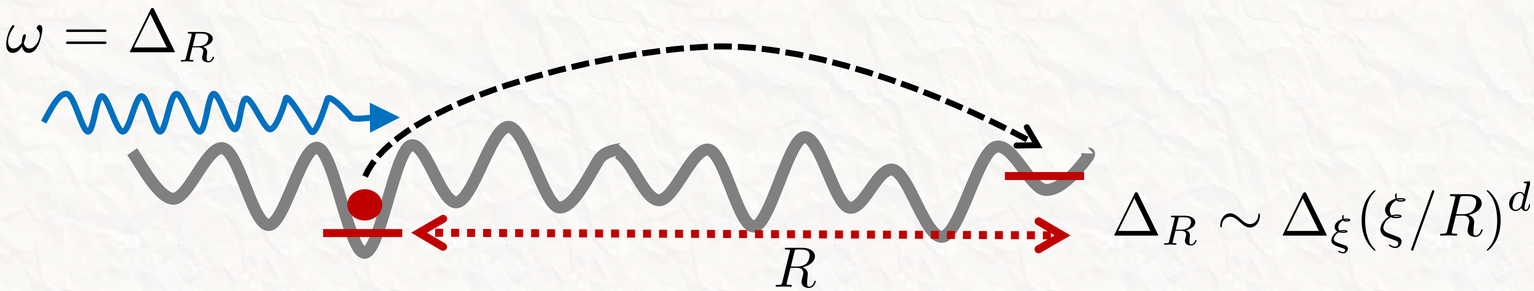
Gurarie, Refael and Chalker, PRL (2008)

- Implies discrete local spectrum:

$$\delta_\omega \approx c/l_\omega \sim (\omega/\omega_0)^\alpha$$



standard VRH calculation



$$\frac{1}{\tau} \approx \sum_R g^2 \rho(\omega_R) e^{-R/\xi} \; e^{-\omega_R/T}$$

Saddle-point evaluation of the integral (d=1):

Typical hopping range: $\bar{R} = \xi \sqrt{\Delta_\xi / T}$

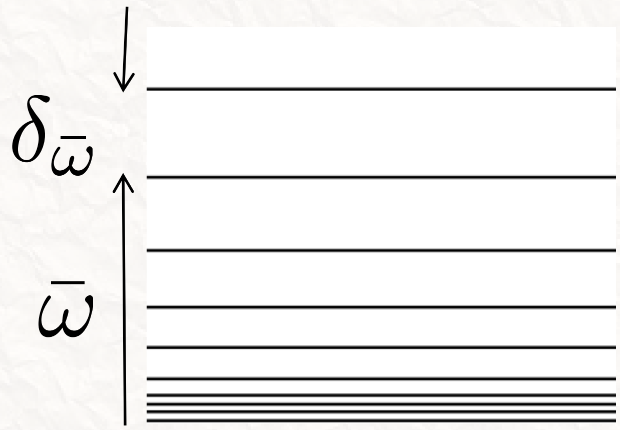
Typical energy taken from phonon: $\bar{\omega} = \Delta(\bar{R}) = \Delta_\xi \sqrt{T/\Delta_\xi}$

$$\Rightarrow \quad \tau^{-1} \approx \frac{\bar{g}^2}{c} \tilde{T} \exp\left(-2\tilde{T}^{-1/2}\right)$$

Is this a valid Fermi golden rule result?

How does the rate compare to the level spacing of the bath at energy $\bar{\omega}$?

Failure of VRH in a 1d random chain ?



At low T:

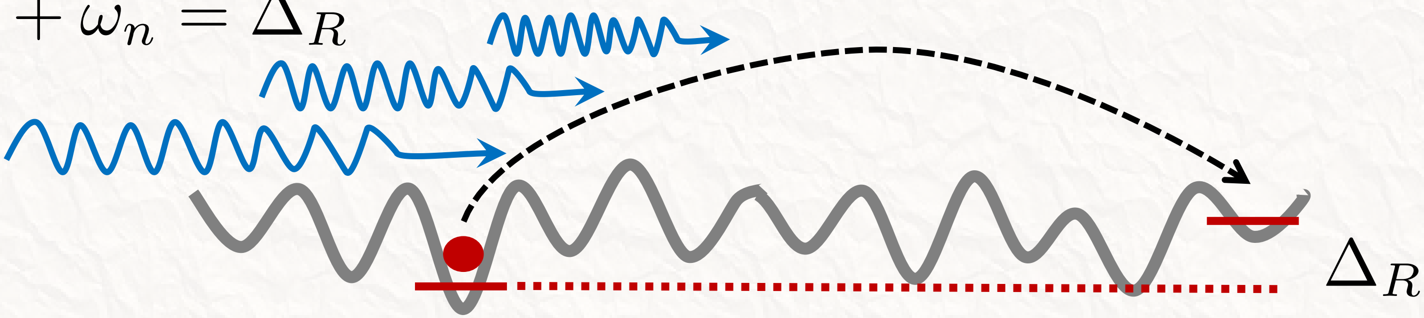
$$\tau^{-1} \approx \bar{g} e^{-2\sqrt{\Delta_\xi / T}} \quad << \quad \delta_{\bar{\omega}} \sim \bar{\omega}^{\alpha} \sim T^{\alpha/2}$$

Does the failure of the Fermi golden rule hopping rate at low T imply many-body localization ?

Not necessarily!

Hopping assisted by a multiple-phonon process ?

$$\omega_1 + \omega_2 + \ldots + \omega_n = \Delta_R$$

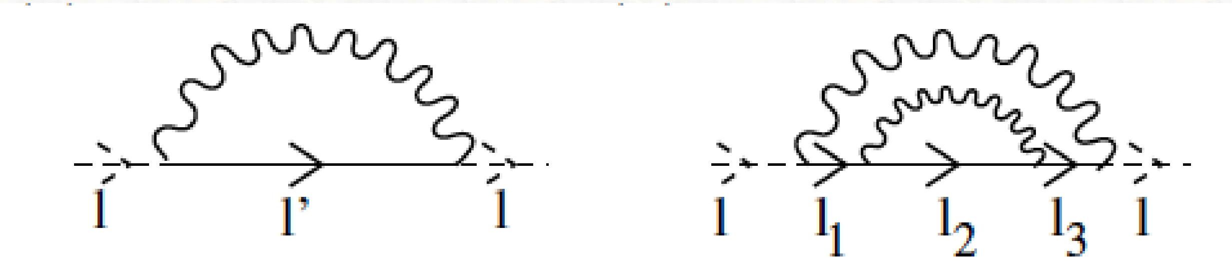


The many-phonon level spacing collapses very rapidly with the number of phonons n:

$$\delta_\omega^{(n)} \sim \omega^{\alpha^n}$$

Multi-phonon assisted hopping rate

Perturbative approach for two-phonon assisted hopping:



$$\tau_{(2)}^{-1} \sim |\bar{g}|^4 \, \tilde{T}^{3/2} \, \exp\left(-2\tilde{T}^{-1/2}\right) \quad << \quad \delta_{\bar{\omega}}^{(2)} \sim \tilde{T}^{\alpha^2/2}$$

- 2-phonon rate still smaller than level spacing at low T.
- May need to go to very high order process!

Better to use a non-perturbative approach !

Polaron transformation

Electron-phonon Hamiltonian:

$$H_{el} = \sum_l \epsilon_l \tilde{c}_l^\dagger \tilde{c}_l \qquad H_{ph} = \sum_\mu \omega_\mu a_\mu^\dagger a_\mu \qquad H_{el-ph} = \sum_{i,\mu} g_{i,\mu} n_i (a_\mu^\dagger + a_\mu)$$

Eliminate coupling g by a unitary transformation:

$$H' = e^S H e^{-S} \qquad S = \sum_{\mu,i} \left(\frac{g_{\mu,i}}{\omega_\mu} \right) n_i (a_\mu^\dagger - a_\mu)$$

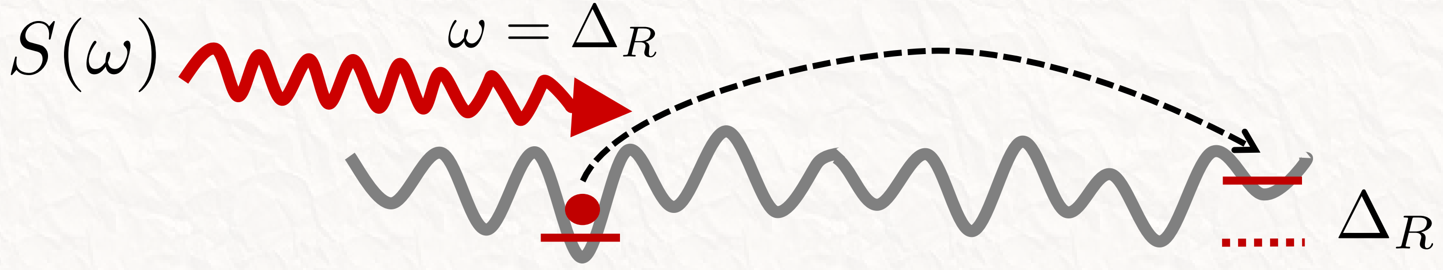
Small residual el-ph coupling (order of t):

$$\hat{T} = \sum_i \hat{B}_i \hat{B}_{i+1} c_i^\dagger c_{i+1} = \sum_{i,l,l'} \phi^*(R_l - R_i) \phi(R_{l'} - R_i) \hat{B}_i \hat{B}_{i+1} \tilde{c}_l^\dagger \tilde{c}_{l'}$$

Coupling contains infinite number of phonons !

$$\hat{B}_i = \exp \left[-\sum_\mu \frac{g_{\mu,i}}{\omega_\mu} (a_\mu + a_\mu^\dagger) \right]$$

Polaron variable range hopping



Now can use lowest order Fermi golden rule:

$$\frac{1}{\tau} \approx t^2 \sum_R e^{-R/\xi} S(\Delta_R)$$

But with “polaron” rather than phonon spectral function (bath DOS):

$$S(\omega) = \int_{-\infty}^{\infty} dt e^{-i\omega t} \langle \hat{B}_i(t) \hat{B}_{i+1}(t) \hat{B}_{i+1}(0) \hat{B}_i(0) \rangle$$

To assess validity of the Fermi golden rule rate:

compare $1/\tau$ with level spacing of $S(\omega)$

Rate for n-phonon process

Expand spectral function to orders of phonon numbers n:

$$\frac{1}{\tau} \approx t^2 \sum_R e^{-R/\xi} S(\Delta_R) = t^2 \sum_R e^{-R/\xi} \sum_n \frac{\bar{g}^{2n}}{n!} I_n(\Delta_R)$$

n-phonon rate for n>>1: $\tau_{(n)}^{-1} = C_n \, \tilde{T}^{(n\alpha-1)/2} e^{-2\tilde{T}^{-1/2}}$

n-phonon level spacing: $\delta_{\Delta_R}^{(n)} \sim \tilde{T}^{\alpha^n/2}$

Minimal n-phonon process: $n = -\ln \tilde{T} / \ln \alpha$

Modified VRH rate $\tau^{-1} \sim e^{-\frac{\alpha}{4 \ln \alpha} \ln^2 \tilde{T}} e^{-2/\sqrt{\tilde{T}}}$

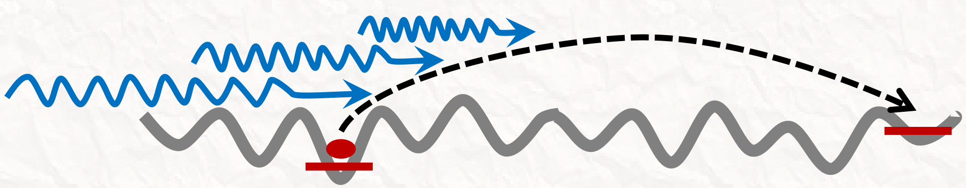
Highly singular and strongly suppressed prefactor

Summary of part II

- Standard (Mott) variable range hopping fail in a random harmonic chain

- Multi-phonon processes allow hopping $n = -\ln \tilde{T} / \ln \alpha$

$$\omega_1 + \omega_2 + \ldots + \omega_n = \Delta_R$$



- Modified variable range hopping rate Strongly suppressed prefactor.

$$\tau^{-1} \sim e^{-\frac{\alpha}{4 \ln \alpha} \ln^2 \tilde{T}} e^{-2/\sqrt{\tilde{T}}}$$

Ref: S. Banerjee and EA arXiv:1511.03676