Classical Nonlinear Lattice Waves – any MBL ?

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Based on collaborations with:
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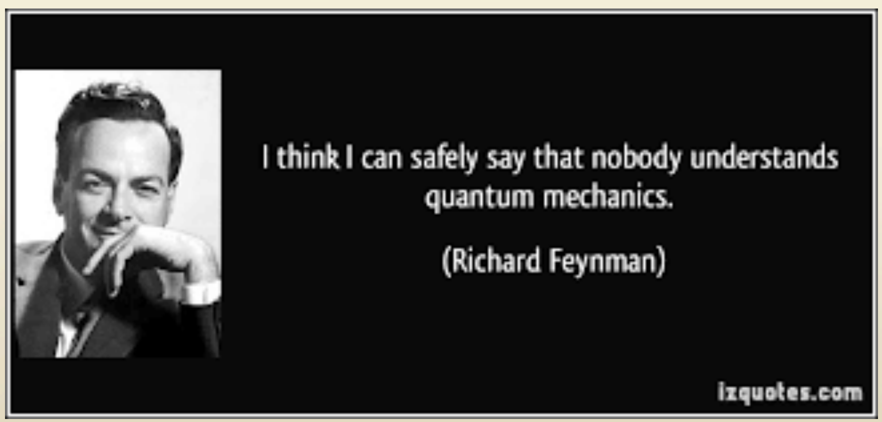


Messenger Lectures on ‘The Character of Physical Law’ Cornell U 1964

lecture 6 : Probability and Uncertainty — the Quantum Mechanical View of Nature

Nonlinear Hamiltonian lattice waves ?

- all about systems which are close to integrability

$$H = H_0(\vec{J}) + \epsilon H_1(\vec{J}, \vec{\theta})$$

- e.g. countable set of linear modes
  integrable

$$H_0 = \sum \omega_l J_l$$

- add nonlinear interaction between these modes
  typically the system becomes nonintegrable

$$H_1 = \sum_{l,m,p,q} I_{l,m,p,q} V((J,\theta)_l, (J,\theta)_m, (J,\theta)_p, (J,\theta)_q)$$

- follow the spreading of a localized (e.g. single mode) wave packet

- or compute properly defined conductivities

Examples: disorder (AL), quasiperiodics (AA), finite systems (FPU)
or simply weakly coupled anharmonic oscillators/rotors

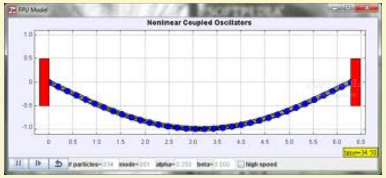
MBL? No? But perhaps outside MBL, in the bad metal nonergodic regime ?

Nonlinear Hamiltonian lattice waves ?

- MBL – quantum, nonergodic bad metal – classical ?
- What about KAM, Arnold diffusion, stochastic web ?
- Is QM simply coarse-graining over fine classical phase space structures?
- Or is classical dynamics a brutal projection from high-d Hilbert into low(er)-d phase space ?

Example: FPU Paradox : selective but long range coupling

Origin of equipartition and ergodicity?
Wave interactions ?

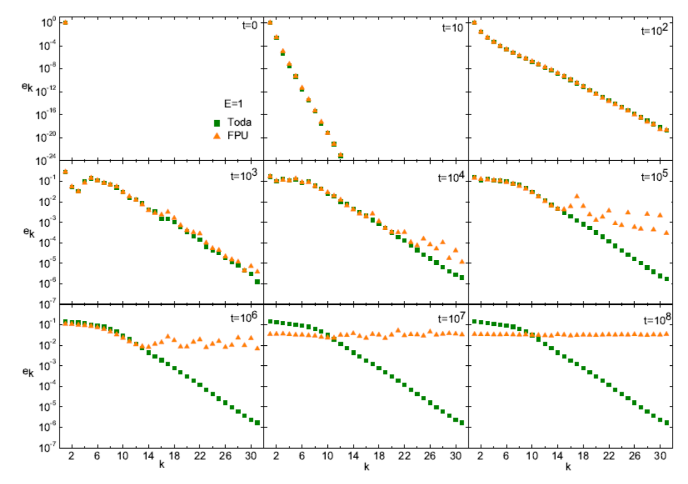


FPU problem:
excited mode q=1
did not observe equipartition
energy stays localized in few modes
recurrences after more integrations
thresholds in energy, system size etc

$$\ddot{x}_n = (x_{n+1} - 2x_n + x_{n-1}) + \alpha[(x_{n+1} - x_n)^2 - (x_n - x_{n-1})^2]$$

$$\ddot{Q}_q + \omega_q^2 Q_q = -\frac{\alpha}{\sqrt{2(N+1)}} \sum_{l,m=1}^{N} \omega_q \omega_l \omega_m B_{q,l,m} Q_l Q_m \qquad \omega_q = 2\sin(\frac{\pi q}{2(N+1)})$$

$$B_{q,l,m} = \sum_{\pm} (\delta_{q\pm l\pm m,0} - \delta_{q\pm l\pm m,2(N+1)})$$



two time scales
T1: formation of exponentially localized packets
T2: gradual destruction and equipartition

Computing periodic orbits, obtain boundary of pert. theory: **T2 ~ T1:** $\frac{E_{th}}{N} \sim \pi^4/(\alpha^2 N^4)$

FPU: Examples of open problems

- dependence of T2 on parameters
- where is KAM
- dynamical mechanisms of spreading
- Quantum case: Finite System MBL ?

Short range mode-mode interactions: model inflation

$$\mathcal{H}_{\mathrm{D}} = \sum_{\mathbf{r}} \left[\epsilon_{\mathbf{r}} \left|\psi_{\mathbf{r}}\right|^2 + \frac{2\beta \left|\psi_{\mathbf{r}}\right|^{\sigma+2}}{\sigma+2} - \sum_{\mathbf{n}} \psi_{\mathbf{r}} \psi_{\mathbf{n}}^* \right]$$

$$i\dot{\psi}_{\mathbf{r}} = \epsilon_{\mathbf{r}} \psi_{\mathbf{r}} + \beta \left|\psi_{\mathbf{r}}\right|^{\sigma} \psi_{\mathbf{r}} - \sum_{\mathbf{n}} \psi_{\mathbf{n}}$$

$$\mathcal{H}_{\mathrm{K}} = \sum_{\mathbf{r}} \left[\frac{p_{\mathbf{r}}^2}{2} + \frac{\tilde{\epsilon}_{\mathbf{r}} u_{\mathbf{r}}^2}{2} + \frac{\left|u_{\mathbf{r}}\right|^{\sigma+2}}{\sigma+2} + \frac{1}{4W} \sum_{\mathbf{n}} \left(u_{\mathbf{n}} - u_{\mathbf{r}}\right)^2 \right]$$

$$\ddot{u}_{\mathbf{r}} = -\tilde{\epsilon}_{\mathbf{r}} u_{\mathbf{r}} - \left|u_{\mathbf{r}}\right|^{\sigma} u_{\mathbf{r}} + \frac{1}{W} \sum_{\mathbf{n}} \left(u_{\mathbf{n}} - u_{\mathbf{r}}\right)$$

$$\mathcal{H} = \sum_{n} \left[\frac{p_n^2}{2} + \frac{\varepsilon_n x_n^2}{2} + \sum_{m \in D(n)} \frac{(x_m - x_n)^{\gamma}}{2\gamma} \right],$$

Various classes of models can be defined and are available on the market:

- **Number of additional integrals of motion (e.g. norm = particle number)**
- **Power (exponent) of nonlinearity, not only restricted to two-body int.**
- **Space dimension**
- **Connectivity between normal modes (number, long vs short range)**

Anderson localization

Anderson (1958)

$$i\frac{\partial \psi_l}{\partial t} = \epsilon_l \psi_l - \psi_{l+1} - \psi_{l-1}$$

$\{\epsilon_l\}$ in $[-W/2, W/2]$

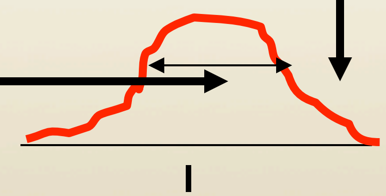
Eigenvalues: $\lambda_\nu \in \left[-2 - \frac{W}{2}, 2 + \frac{W}{2}\right]$

Width of EV spectrum: $\Delta = 4 + W$

Asymptotic decay: $A_{\nu,l} \sim \mathrm{e}^{-l/\xi(\lambda_\nu)}$

Localization length: $\xi(\lambda_\nu) \leq \xi(0) \approx 100/W^2$

Localization volume of NM: L



Anderson localization

Anderson (1958)

$$\lambda A_l = \epsilon_l A_l - A_{l-1} - A_{l+1}$$

$\{\epsilon_l\}$ in $[-W/2, W/2]$

Eigenvalues: $\lambda_\nu \in \left[-2 - \frac{W}{2}, 2 + \frac{W}{2}\right]$

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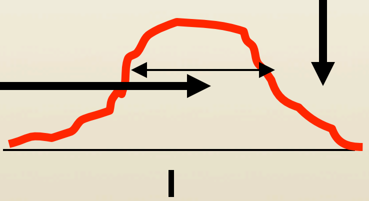

Topical Reviews:

- SF, Springer 2015
- T.V.Lapteva,M.V. Ivanchenko and SF, J Phys A Topical Review 47 (2014) 493001

Nonlinear waves: spreading!

- a disordered medium
- linear equations of motion: all eigenstates are localized with a finite upper bound on the localization length
- add short range nonlinearity (interactions)
- follow the spreading of an initially localized wave packet
- these models may serve as approximations to quantum many body systems in certain limits, e.g. of a large number of (bosonic) particles – photons or cold atoms

$$i\dot{\psi}_l = \epsilon_l \psi_l + \beta |\psi_l|^2 \psi_l - \psi_{l+1} - \psi_{l-1}$$

Two conserved quantities: energy and norm (aka number of particles)

Topical Reviews:
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Defining the problem

- a disordered medium
- linear equations of motion: all eigenstates are localized with a finite upper bound on the localization length
- add short range nonlinearity (interactions)
- follow the spreading of an initially localized wave packet

Will it delocalize?

Yes because of nonintegrability and ergodicity

No because of energy conservation – spreading leads to small energy/norm density, nonlinearity can be neglected, dynamics becomes integrable, and Anderson localization is restored

Equations in normal mode space:

$$i\dot{\phi}_\nu = \lambda_\nu \phi_\nu + \beta \sum_{\nu_1,\nu_2,\nu_3} I_{\nu,\nu_1,\nu_2,\nu_3} \phi^*_{\nu_1} \phi_{\nu_2} \phi_{\nu_3}$$

$$I_{\nu,\nu_1,\nu_2,\nu_3} = \sum_l A_{\nu,l} A_{\nu_1,l} A_{\nu_2,l} A_{\nu_3,l}$$

NM ordering in real space: $X_\nu = \sum_l l A^2_{\nu,l}$

Characterization of wavepackets in normal mode space:

$z_\nu \equiv |\phi_\nu|^2 / \sum_\mu |\phi_\mu|^2$ $\bar{\nu} = \sum_\nu \nu z_\nu$

Second moment: $m_2 = \sum_\nu (\nu - \bar{\nu})^2 z_\nu$ → location of tails

Participation number: $P = 1/\sum_\nu z_\nu^2$ → number of strongly excited modes

Compactness index: $\zeta = \frac{P^2}{m_2}$

→ K adjacent sites equally excited: $\zeta = 12$

→ K adjacent sites, every second empty or equipartition: $\zeta = 3$

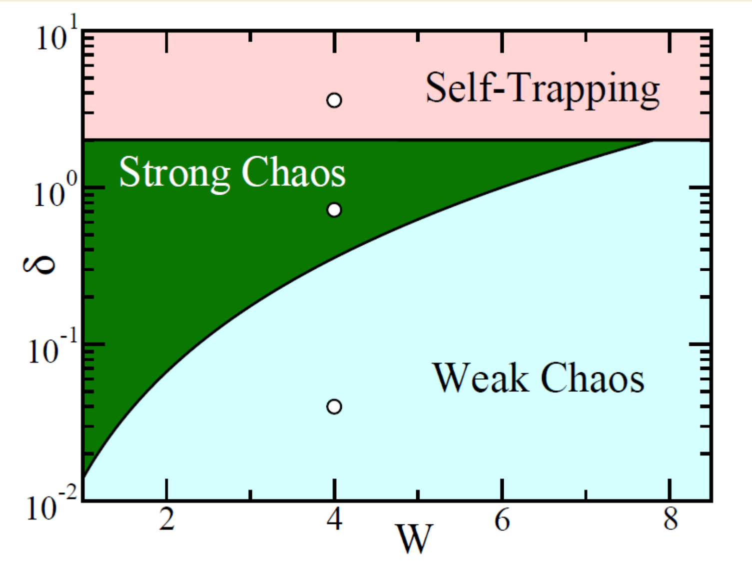
Scales

W=4 :

- Eigenvalue (frequency) spectrum width: $\Delta = W + 4$ — **8**
- Localization volume of eigenstate: $V \approx 360/W^2$ — **~18 (sites)**
- Average frequency spacing inside localization volume: $d = \Delta/V$ — **0.43**
- Nonlinearity induced frequency shift: $\delta_l = \beta|\psi_l|^2$

Three expected evolution regimes:
Weak chaos : $\delta < d$
Strong chaos : $d < \delta < 2$
(partial) self trapping : $2 < \delta$

SF Chem Phys 2010, TV Laptyeva et al EPL 2010



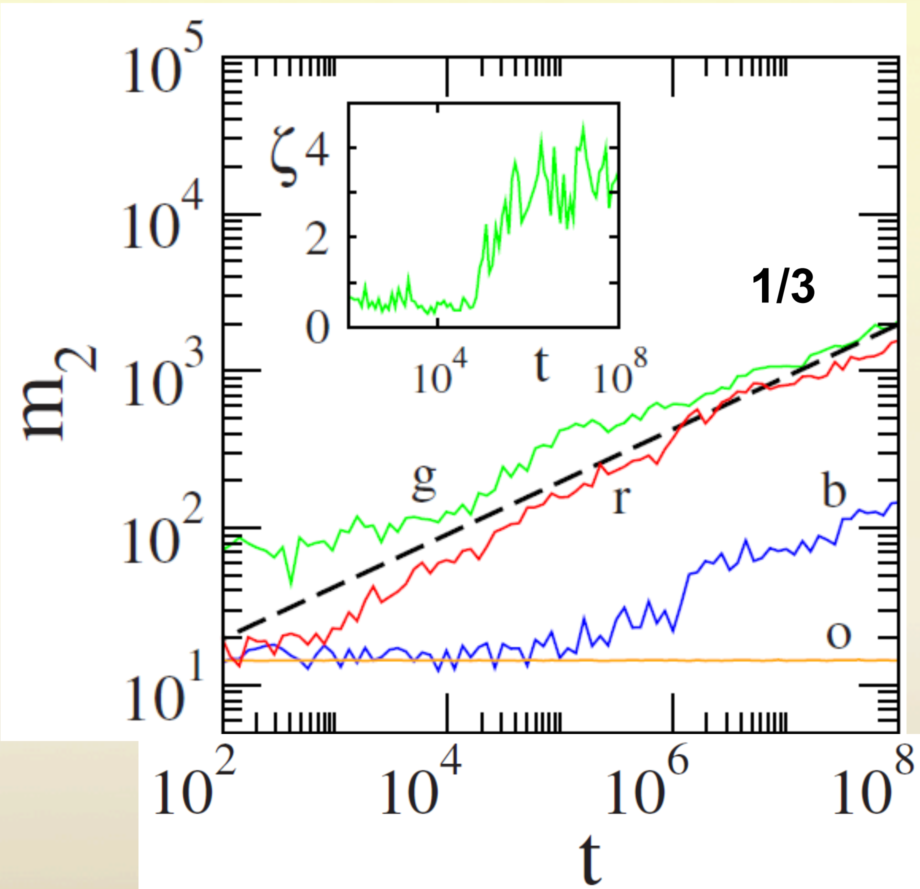
Disordered chains: subdiffusion!

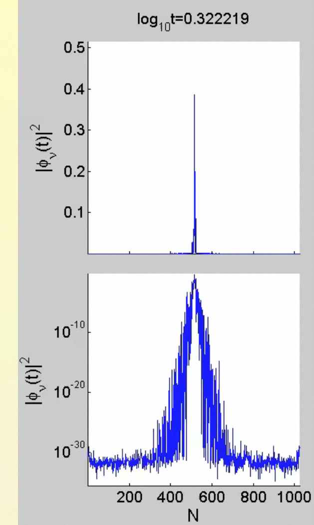
$$\psi_l = \delta_{l,l_0}$$

$$\epsilon_{l_0} = 0$$

SF+D.Krimer+H.Skokos 09

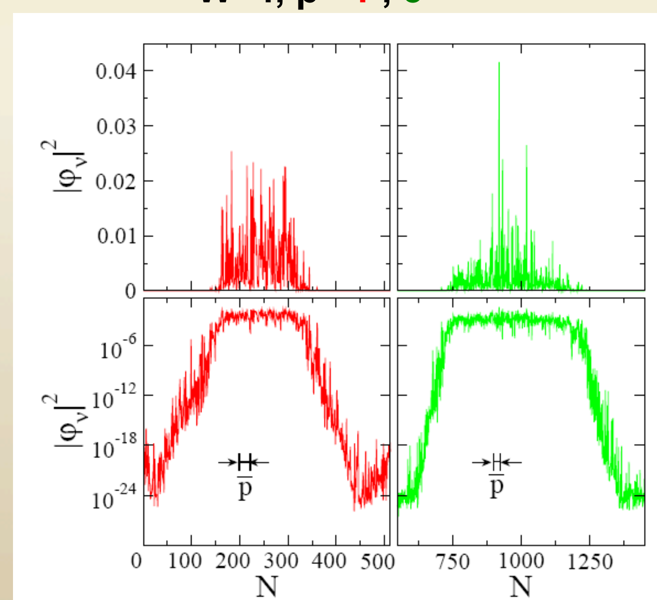
W=4, β= 0, 0.1, 1, 4.5





Wavepacket spreads way beyond localization volume.
DNLS at $t = 10^8$

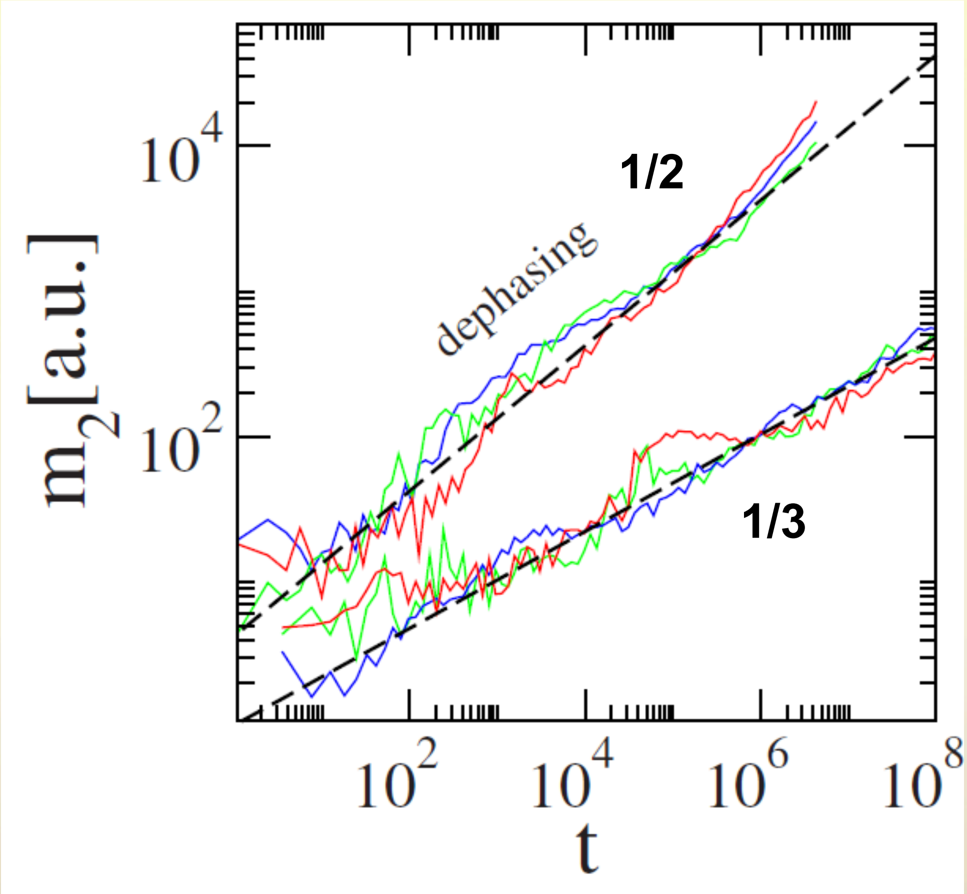
W=4, β= 1 , 5



M. Molina 98,
D. Shepelyansky+A. Pikovsky 08,
SF+D.Krimer+H.Skokos 09, + MANY MORE

Test: additional manual dephasing in normal mode space

W=4,7,10 β= 3,4,6



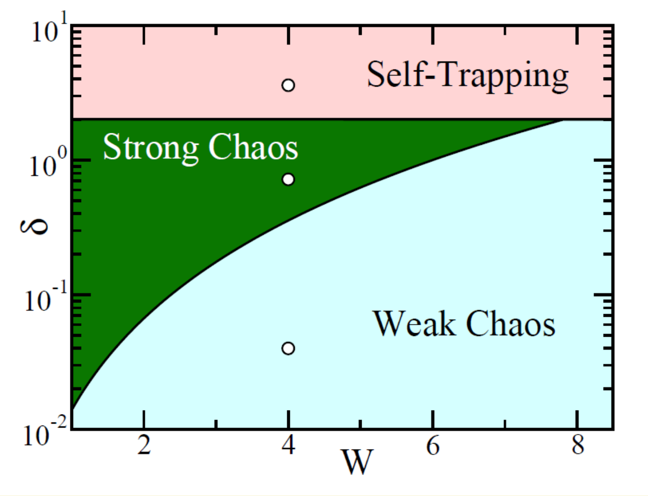
$$m_2 \sim t^{\alpha}$$

$$\alpha = ???$$

SF et al PRL 2009

W=4
Wave packet with 20 sites
Norm density = 1
Random initial phases
Averaging over 1000 realizations

$\alpha = 0.33 \pm 0.02$ (DNLS)
$\alpha = 0.33 \pm 0.05$ (KG)



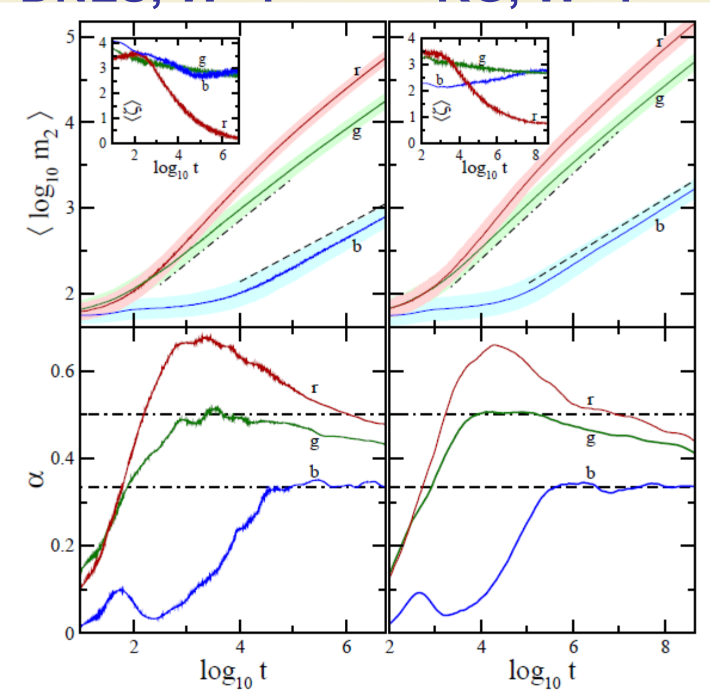
SF et al PRL 2009

Strong chaos and crossover to weak chaos

TV Laptyeva et al EPL 2010

KG

DNLS, W=4 KG, W=4



$$\alpha(\log_{10} t) = \frac{\mathrm{d}}{\log_{10} t} \langle \log_{10} m_2 \rangle$$

Integrability or chaos?

$$i\dot{\phi}_\nu = \lambda_\nu \phi_\nu + \beta \sum_{\nu_1,\nu_2,\nu_3} I_{\nu,\nu_1,\nu_2,\nu_3} \phi^*_{\nu_1} \phi_{\nu_2} \phi_{\nu_3}$$

$$I_{\nu,\nu_1,\nu_2,\nu_3} = \sum_l A_{\nu,l} A_{\nu_1,l} A_{\nu_2,l} A_{\nu_3,l}$$

Integral approximation (Nr. 1):
NO SPREADING!

$$\mathscr{H}_{int} = \sum_\nu \lambda_\nu J_\nu + \beta \sum_{\nu_1,\nu_2,\nu_3,\nu_4} I_{\nu_1,\nu_2,\nu_3,\nu_4} \sqrt{J_{\nu_1} J_{\nu_2} J_{\nu_3} J_{\nu_4}}$$

$$i\dot{\chi}_\nu = \beta \sum_{\nu_1,\nu_2,\nu_3} I_{\nu,\nu_1,\nu_2,\nu_3} \chi^*_{\nu_1} \chi_{\nu_2} \chi_{\nu_3} \mathrm{e}^{i(\lambda_\nu + \lambda_{\nu_1} - \lambda_{\nu_2} - \lambda_{\nu_3})t}$$

$$\phi_\nu = \mathrm{e}^{-i\lambda_\nu t} \chi_\nu$$

Average over time, obtain secular normal form = integral approximation Nr 2:
NO SPREADING!

$$i\dot{\chi}_\nu = \beta \sum_{\nu_1} I_{\nu,\nu,\nu_1,\nu_1} |\chi_{\nu_1}|^2 \chi_\nu$$

$$\chi_\nu(t) = \eta_\nu \mathrm{e}^{-i\Omega_\nu t} \;, \; \Omega_\nu = \beta \sum_{\nu_1} I_{\nu,\nu,\nu_1,\nu_1} |\eta_{\nu_1}|^2$$

One of the neglected terms: perturbation approach

$$\dot{\chi}_\nu = \beta I_{\nu,\mathbf{n}} \chi^*_{\nu_1} \chi_{\nu_2} \chi_{\nu_3} \mathrm{e}^{i\lambda_{\nu,\mathbf{n}} t} \qquad \lambda_{\nu,\mathbf{n}} \equiv \lambda_\nu + \lambda_{\nu_1} - \lambda_{\nu_2} - \lambda_{\nu_3}\ ,\ \mathbf{n} \equiv (\nu_1, \nu_2, \nu_3)$$

$$|\chi_\nu^{(1)}| = |\beta \eta_{\nu_1} \eta_{\nu_2} \eta_{\nu_3}| R^{-1}_{\nu,\mathbf{n}}\ ,\ R_{\nu,\mathbf{n}} \sim \left| \frac{\lambda_{\nu,\mathbf{n}}}{I_{\nu,\mathbf{n}}} \right|$$

$$\mathscr{W}_\lambda(\lambda_{\nu,\mathbf{n}}) \qquad \mathscr{W}_\lambda(x) \approx \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{x^2}{2\sigma^2}}\ ,\ \sigma^2 = \frac{\Delta^2}{12}$$

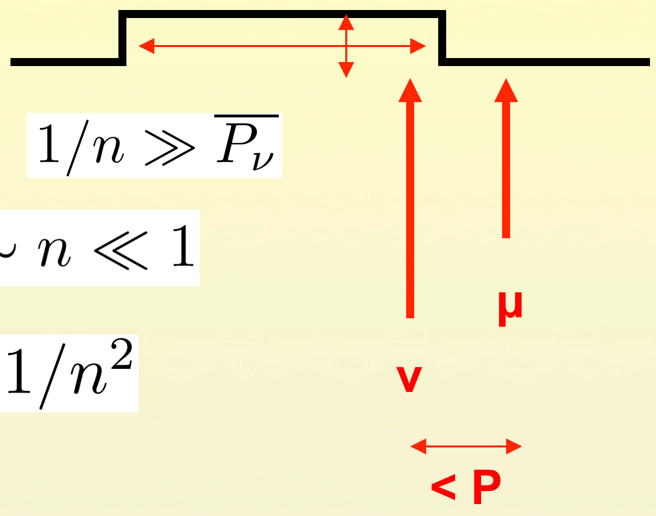
$$\mathscr{W}_R(x) = \langle I \rangle \mathscr{W}_\lambda(\langle I \rangle x)\ ,\ \mathscr{W}_R(0) = \frac{\sqrt{3}\langle I \rangle}{\sqrt{2\pi}\Delta}$$

Breakdown of perturbation approach = resonance

$$\beta n < R_{\nu,\mathbf{n}}$$

$$\mathscr{P}_\nu = 1 - \left(1 - \int_0^{\beta n} \mathscr{W}_R(x)dx\right)^{V^3} \ , \ \mathscr{P}_\nu|_{\beta n \to 0} \to \frac{\sqrt{3}V^3\langle I \rangle}{\sqrt{2\pi}\Delta}\beta n$$

Effective noise theory



- at some time t packet contains 1/n modes: $1/n \gg \overline{P_\nu}$
- each mode on average has norm $|\phi_\nu|^2 \sim n \ll 1$
- the second moment amounts to $m_2 \sim 1/n^2$

Simplest assumption:

- some modes in packet interact resonantly and therefore evolve chaotic
- Probability of resonance: P(βn)
- all phases decohere on some time scale

exterior mode:

$$i\dot{\phi}_\mu \approx \lambda_\mu \phi_\mu + \beta n^{3/2} \mathcal{P}(\beta n) f(t)$$

$$\langle f(t) f(t') \rangle = \delta(t - t')$$

confirmed by:
Michaely et al
PRE 2012
Skokos et al 2013

momentary diffusion rate:

$$D = 1/T \sim \beta^2 n^2 (\mathcal{P}(\beta n))^2$$

Main findings (sweeping a lot of stuff under the carpet)

- **d – mean level spacing in localization volume**

$$m_2 \sim \begin{cases} \beta t^{1/2}, & \beta n/d > 1 \,(\text{strong chaos}) \\ d^{-2/3}\beta^{4/3}t^{1/3}, & \beta n/d < 1 \,(\text{weak chaos}) \end{cases}$$

$$m_2 \sim (\beta^2 t)^{\frac{2}{2+\sigma D}}, \quad \text{strong chaos},$$
$$m_2 \sim (\beta^4 t)^{\frac{1}{1+\sigma D}}, \quad \text{weak chaos}.$$

- **Strong chaos: intermediate but potentially long lasting regime**
- **Weak chaos: so far asymptotic**
- **Spread: up to 300ξ, time up to 10^{10}, averaging up to 1000 realizations**
- **Chaotic dynamics, positive Lyapunovs**
- **No signs of slowing down**
- **confirmed for quasiperiodics, disorder, nonlinear quantum kicker rotor, 1d, 2d**
- **Qualitatively similar for Wannier Stark ladder (but exponents nonuniversal)**

Restoring Anderson localization? A matter of probability and KAM

Fix the size of a wave packet:

- the probability P to hit a regular trajectory tends to unity in the limit of zero nonlinearity (Aubry/Johansson, Fishman/Pikovsky, Basko, Ivanchenko/Laptyeva/SF)

Fix the norm/energy but vary the size of a wave packet:

- The probability to hit a regular trajectory tends to unity in the limit of infinite size (Fishman/Pikovsky)
- This depends on the particular model, and nonlinearity exponent (Ivanchenko/Laptyeva/SF)

NOTE: spreading wave packets are observed to penetrate this KAM regime !

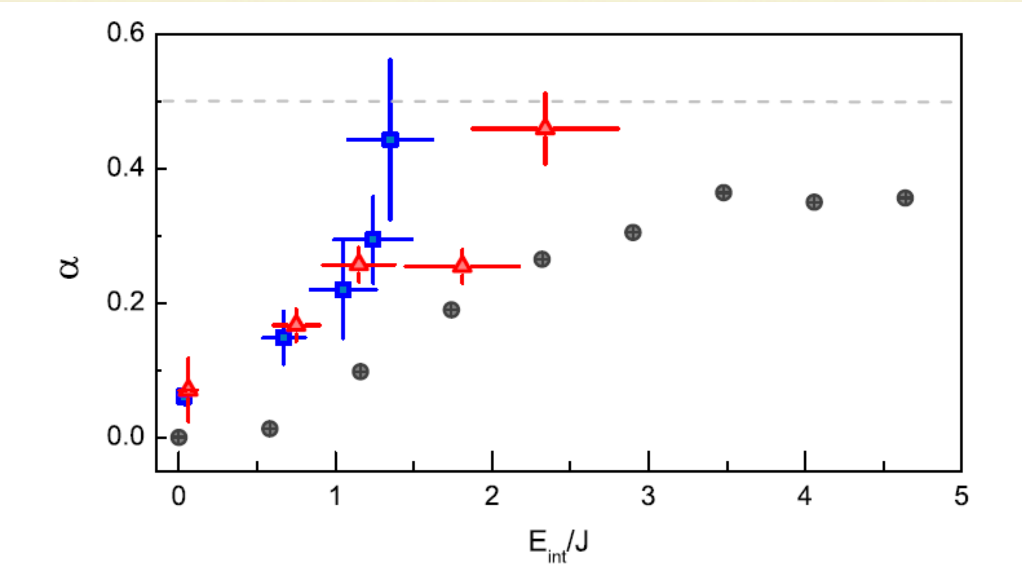
Interacting BEC with quasiperiodic Aubry-Andre potential

PRL 106, 230403 (2011) PHYSICAL REVIEW LETTERS wee
10 J

Observation of Subdiffusion in a Disordered Interacting System

E. Lucioni,[1,*] B. Deissler,[1] L. Tanzi,[1] G. Roati,[1] M. Zaccanti,[1,†] M. Modugno,[2,3] M. Larcher,[4]
F. Dalfovo,[4] M. Inguscio,[1] and G. Modugno[1,‡]

Bose-Einstein condensate of ^{39}K atoms



Finite temperature conductivities

$$\mathcal{H}_K = \sum_l \frac{p_l^2}{2} + \frac{\tilde{\epsilon}_l}{2}u_l^2 + \frac{1}{4}u_l^4 + \frac{1}{2W}(u_{l+1} - u_l)^2$$

$$\kappa = jN/(T_N - T_1)$$

$$\ddot{u}_{1,N} = -\partial H/\partial u_{1,N} + \xi_{1,N} - \lambda \dot{u}_{1,N}$$

$$\langle \xi_{1,N} \rangle = 0 \text{ and } \langle \xi_{1,N}(t)\xi_{1,N}(0)\rangle = 2\lambda T_{1,N}\delta(t)$$

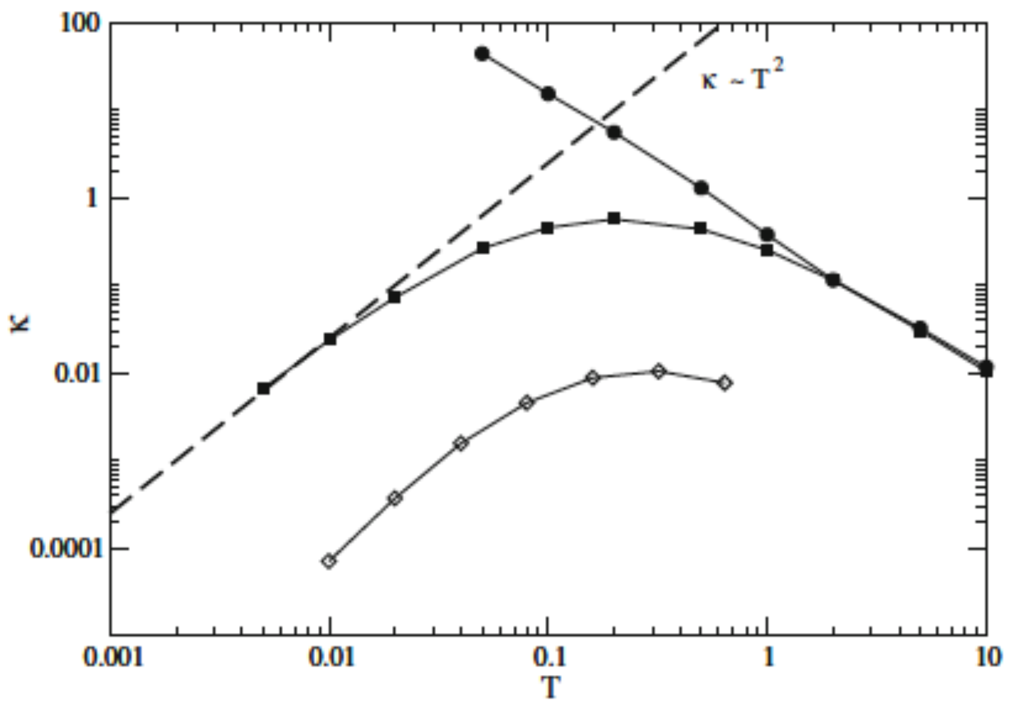


Figure 2. KG chain: $\kappa(T)$ for $W = 2$ (filled squares). For comparison we also show the data for $\tilde{\epsilon}_l \equiv 1$ (filled circles). Thin solid lines guide the eye. The dashed line corresponds to the power law T^2. The stronger disorder case $W = 6$ corresponds to the open diamond data points.

Some questions

- Will spreading slow down log-like in a further regime of KAM?
- Weak chaos definitely persists into a KAM regime!
- Is the weak chaos regime a nonergodic bad metal, with $D \sim (bn)^4$?
- If yes, what is then the predicted log-like VeryWeakChaos regime – some kind of classical pseudo-MBL ?
- If there is VWC, what kind of not-so-bad metal is then weak chaos?
- Spreading wave packets allow to explore the Arnold web by simply penetrating into it deeper and deeper upon spreading!
- Or is everything only the physics of roundoff errors?
- All this is about penetrating a KAM + Arnold diffusion/web regime for short range mode-mode interactions.
  Naive quantizing implies a coarse-graining of classical phase space structures. Is that enough to get some kind of MBL ?

Some questions

- With the assumption of chaos the spreading characteristics could be related to equilibrium properties at corresponding densities
- KAM: for a sufficiently localized wave packet, AL is restored for weak nonlinearities in a probabilistic way (Basko, Ivanchenko et al, Fishman)
- Can one perform similar computations in MBL settings?
- Experiments (light, cold atoms) so far reach 10^{4-5} in our dimensionless time units, and do observe onset of spreading, but no reliable exponents, not even mentioning the issue of asymptotics, or quantum deviations
- Is this all about zero density and of no relevance for finite T? Or are the insights from 'thermalizing' wave packets connecting nonequilibrium dynamics with expected equilibrium properties?

Nonlinear diffusion equations and scaling

$$\partial_t \mathcal{P} = \partial_x \left(\kappa \mathcal{P}^a \partial_x \mathcal{P} \right)$$

$$\mathcal{P}(x,t) = \left[A - \frac{a x^2}{2(2+a) t^{2/(2+a)}} \right]^{1/a} t^{-1/(2+a)}$$

Zeldovich, Kompaneets, Barenblatt et al, 1950s-1960s

$$m_\eta = \left[\frac{2(2+a)}{a} \right]^{\frac{\eta+1}{2}} \mathcal{B}\left[\frac{a+1}{a}, \frac{\eta+1}{2} \right] A^{\frac{a(\eta+1)+2}{2a}} t^{\frac{\eta}{2+a}}$$

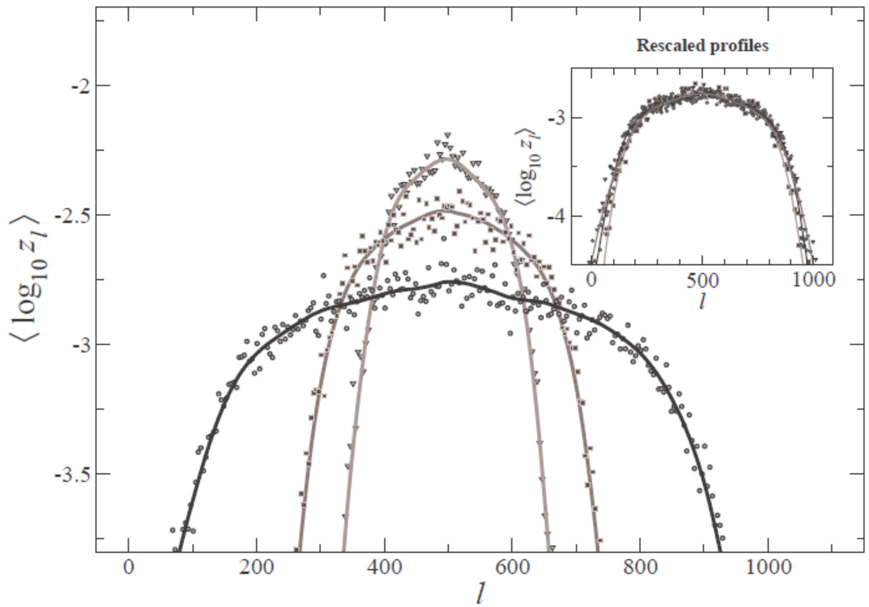


Figure 4: *Main:* The log of the normalized energy density distribution $\langle \log_{10} z_l \rangle$ at three different times (from top to bottom $t \approx 10^4$, $t \approx 10^7$, $t \approx 10^8$). The initial parameters are $E = 0.2$, $W = 4$ and $L = 21$. Symbols correspond

TV Laptyeva et al, Physica D 2013

Density resolved spreading

$$H = \sum_{l=1}^{N} \epsilon_l |\Psi_l|^2 - J(\Psi_l \Psi_{l+1}^* + cc) + \frac{1}{2}\beta |\Psi_l|^4$$

$$S = \sum_{l=1}^{N} |\Psi_l|^2$$

$$\text{scaled densities : } y = \beta H/N \ , \ x = \beta S/N$$

$$\text{partition function : } \mathcal{Z} = \int d\Gamma \mathrm{e}^{-\frac{1}{T}(H+\mu S)}$$

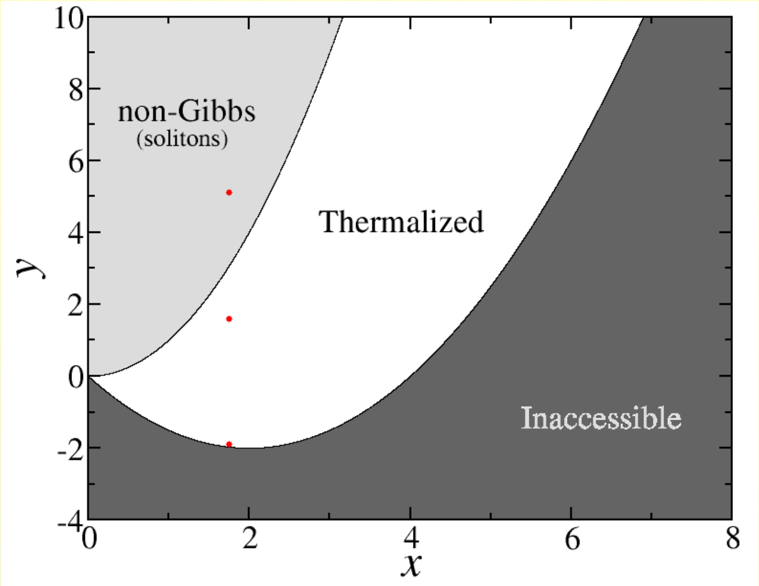
$$T = 0 \text{ line} \ , \ J = 1 \ , \ W \leq 10 \ :$$

$$y_0 \approx -(2 + W/2)x - x^2 \ln(x)/2 \ , \ x \leq d$$

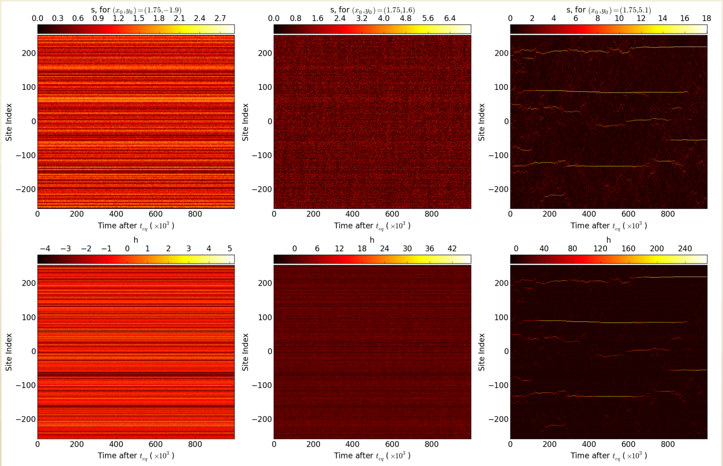
$$y_0 \approx -2x + x^2/2 \ , \ x \gg d$$

$$T = \infty \text{ line} \ , \ \text{any } J, W, \ \text{lattice dimension :}$$

$$y_\infty = x^2$$

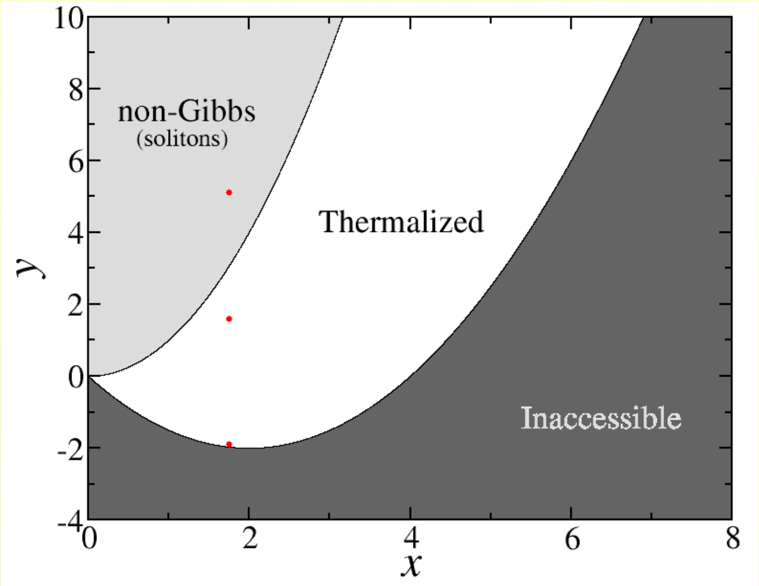


Rasmussen et al, PRL 2000 (W=0)



Courtesy J Bodyfelt

Density resolved spreading



Rasmussen et al, PRL 2000 (W=0)

wave packets spread along straight lines, straight to zero

x < d : weak chaos, observed down to x = d/100 !

crossing the Gibbs-nonGibbs line shows no impact on spreading

Ivanchenko et al, Basko:
x << d is KAM regime for wave packets $P_{KAM} \sim (1 - x/d)^{\xi}$

spreading wave packets launched outside the KAM regime with x > d simply spread into it once x < d

x << d might be a nonergodic regime with nonzero conductivity

work in progress …