# MBL, ETH, and entanglement; an "overview"

for KITP-MBL conference 11-16-15
—D. Huse

To be specific (and to avoid "controversial" cases) consider

<u>closed</u> <u>one-dimensional</u> system with dynamics:

$$\Psi(t+1) = U\Psi(t)$$

(spin chain, or lattice particles)

<u>local</u> unitary, can be $e^{-iH}$ ← <u>local</u> Hamiltonian

Many-body state, in <u>product Hilbert space</u> (product over <u>lattice</u> sites)

If there is an $H$, we are at $E = \langle H \rangle$ corresponding to $T \neq 0$:
(no ground states)

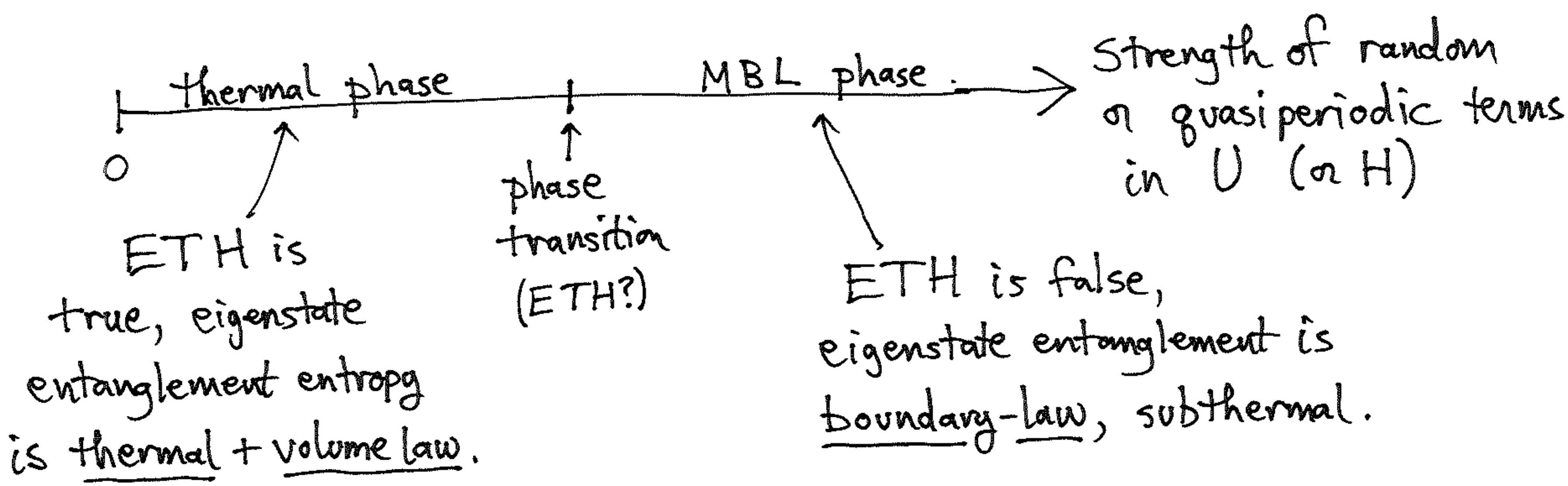
Thermal equilibrium entropy is "volume-law" (extensive).

Stay away from "traditional" integrable systems.

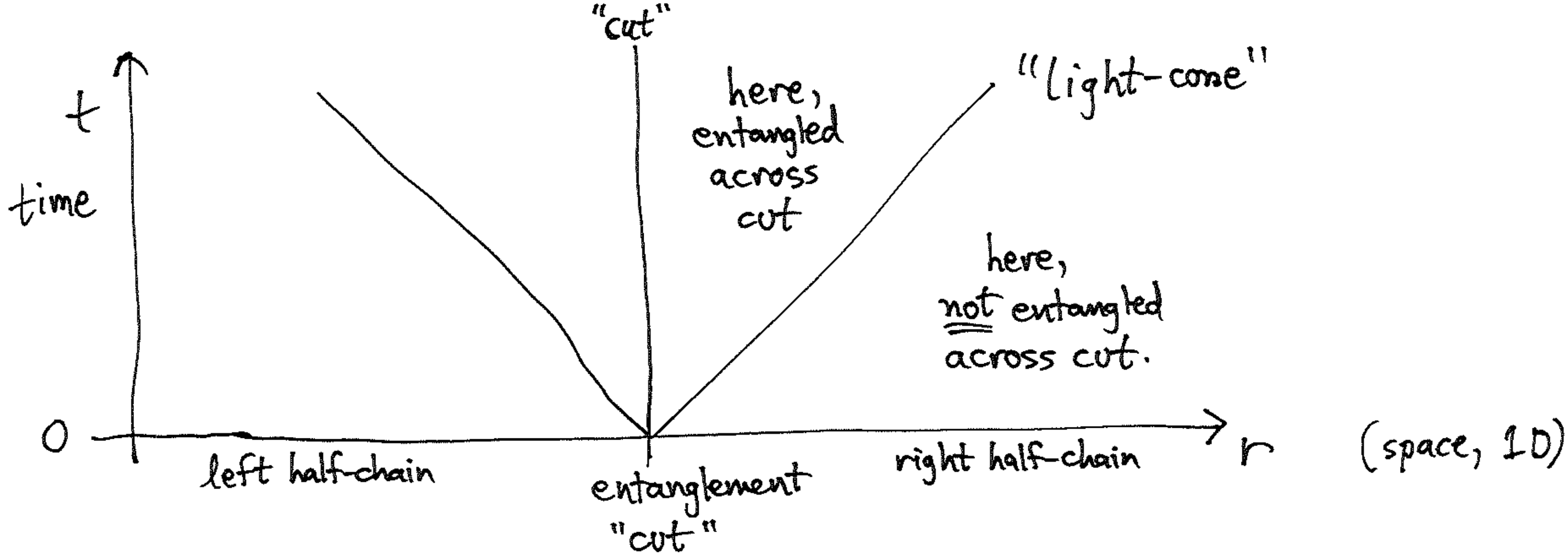
$\Psi(t+1) = U\Psi(t)$ $\qquad$ $U|n\rangle = \lambda_n|n\rangle$ are eigenstates.
$(|\lambda_n| = 1)$

We consider dynamics of entanglement, starting from product (zero initial entanglement) state $\Psi(t=0)$. And entanglement of eigenstates, $|n\rangle$.

Dynamic/Eigenstate phase diagram:




First, look "deep" in thermal phase, MBL phase, well away from phase transition.




<u>Deep</u> in thermal phase, entanglement spreads at "Lieb-Robinson" speed:

$$S_{ent.} \sim t$$

due to short-range interactions. Calabrese+Cardy 2005 (integrable)
Liu + Suh 2013 (holographic)
Kim + H. 2013 (nonintegrable Ising chain)

An "epidemic" spread by <u>contact</u>.
Quickly goes to local thermal equilibrium entanglement entropy.

<u>Closer</u> to MBL transition in <u>random</u> systems in 1D,
spread is slower: $S_{ent.} \sim t^{1/z}$ $z > 1$, due to Griffiths rare regions effects.
Agarwal, et al.; Vosk, H, Altman 2015

In the MBL phase, all eigenstates have only boundary-law entanglement.

Yet, the entanglement does spread: $S_{\text{ent.}} \sim \log t$, leaving behind <u>sub</u>thermal <u>volume-</u>law entanglement.

numerics $\rightarrow$ Znidaric, Prosen, Prelovsek 2008

$\rightarrow$ Bardarson, Pollman, Moore 2012

Why? Consider spin-$\frac{1}{2}$ chain, <u>bare</u> spins $\vec{\sigma}_i$. Only short-range interactions.

In MBL phase these can be "dressed" to make <u>localized</u> conserved pseudospins: $\vec{\tau}_i$ with $[U, \tau_{iz}] = 0$

"$\ell$-bits"

MBL phase is a new type of integrable system.

Serbyn, Papic, Abanin 2013; Bauer, Nayak 2013,
H, Nandkishore, Oganesyan 2014

In the MBL phase, the system's Hamiltonian is an Ising model:
(when written in terms of the $\{\tau_{iz}\}$)

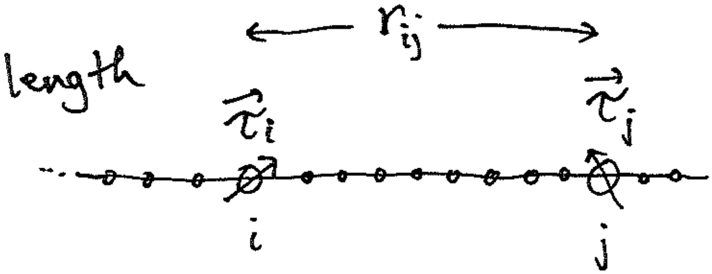
$$H = \sum_i h_i \tau_{iz} + \sum_{i,j} J_{ij} \tau_{iz} \tau_{jz} + \sum_{i,j,k} K_{ijk} \tau_{iz} \tau_{jz} \tau_{kz} + \cdots$$

with interactions $J_{ij}$, $K_{ijk}$, $\cdots$ falling off exponentially with distance. Thus if we fix all other $l$-bits, $l$-bits $i$ and $j$ interact as

distance

decay length

$$H^{\text{eff}}_{ij} = J_0 e^{-r_{ij}/\xi} \tau_{iz} \tau_{jz}$$



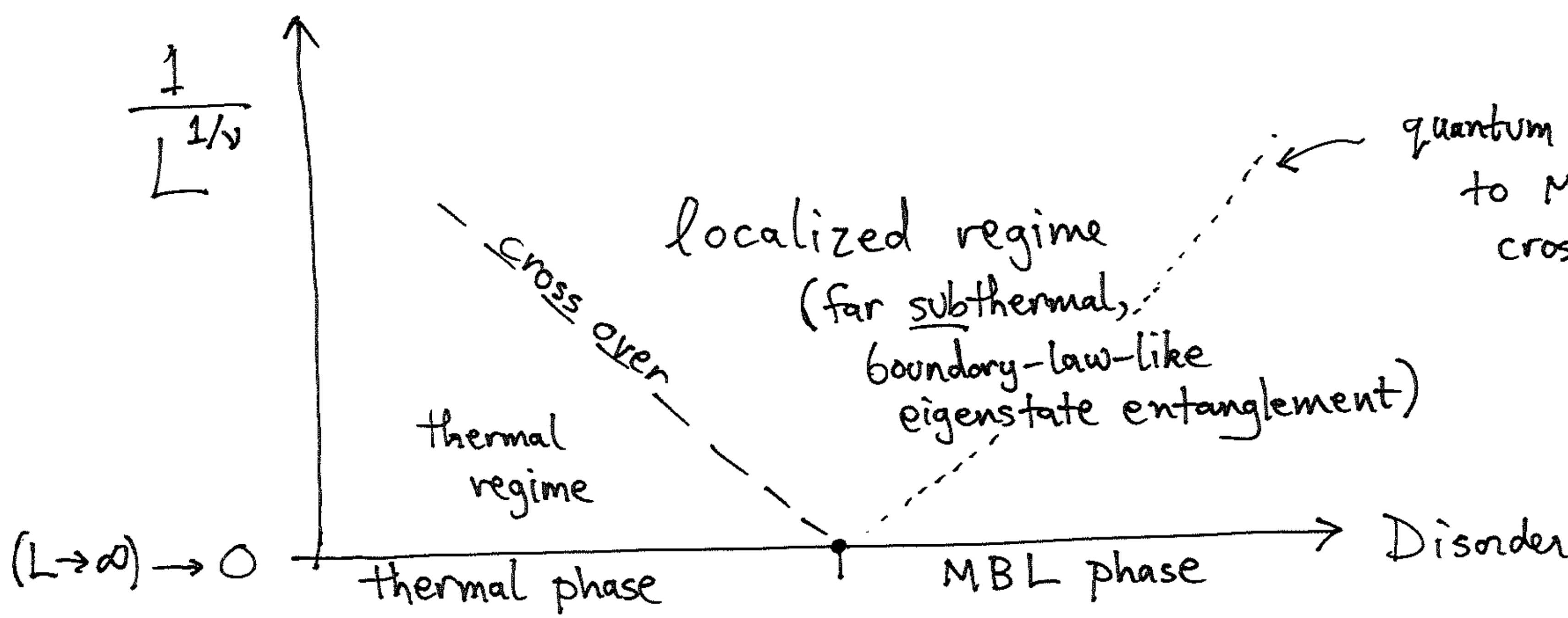

General initial state is not an eigenstate of either $\tau_{iz}$ or $\tau_{jz}$.
xy components of $\vec{\tau}$'s get entangled after time $t \sim \left(J_0 e^{-r_{ij}/\xi}\right)^{-1}$

So distance entanglement spreads $\sim \xi \log (J_0 t)$ due to these Ising interactions.

(any power-law interactions will speed this up)

# What do we <u>know</u> about the ETH–MBL phase transition?

Not much. Finite-size crossover: $L$ = length of system



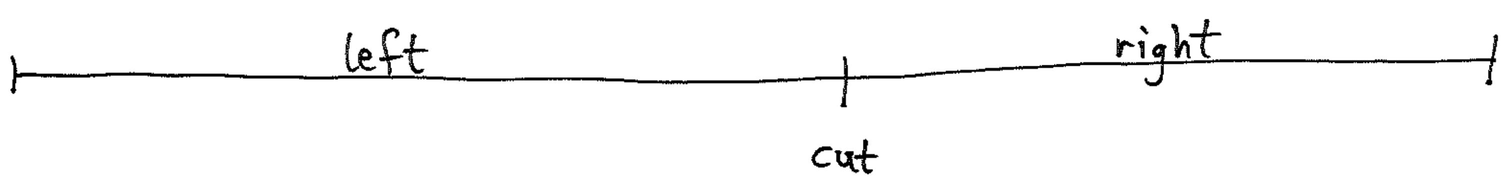

From numerics, and approximate RG (Vosk, H, Altman 2015) crossover is almost all on thermal side of transition.

At transition, MBL state goes unstable to thermalization only in $L\to\infty$ limit. May be <u>discontinuous</u> in some respects.

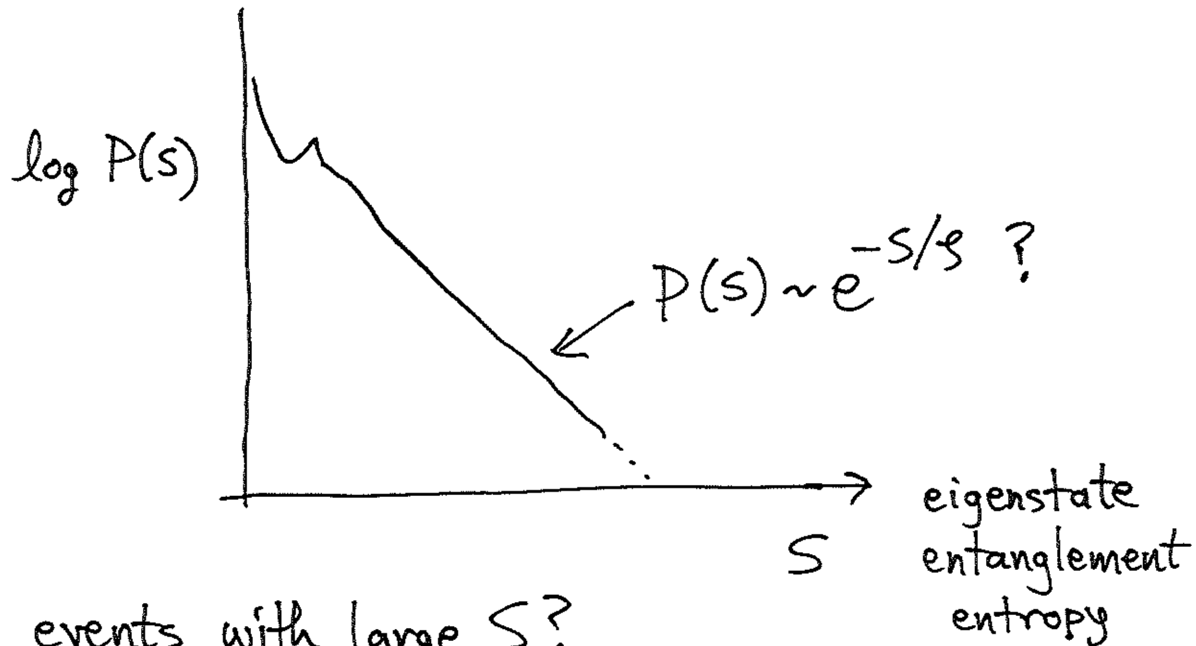
For systems with quenched randomness, eigenstate entanglement $S$ is a random quantity that depends on sample, eigenstate, and where you put the cut. It has a probability distribution $P_L(S)$ for systems of length $L$.

Thermal phase: $P_L(S) \xrightarrow[L\to\infty]{} \delta\left(\frac{S}{L} - \left(\frac{S}{L}\right)_{eq.}\right)$ (ETH)

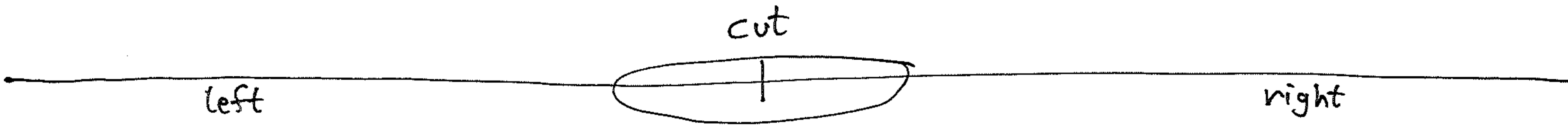
MBL phase: $P_L(S) \xrightarrow[L\to\infty]{} P(S)$, a non-trivial distribution of boundary-law entanglement. (e.g., Lim + Sheng 1510.....)



In MBL phase:




What are the rare events with large $S$?




rare region or rare eigenstate with high local entanglement. "Griffiths regions"

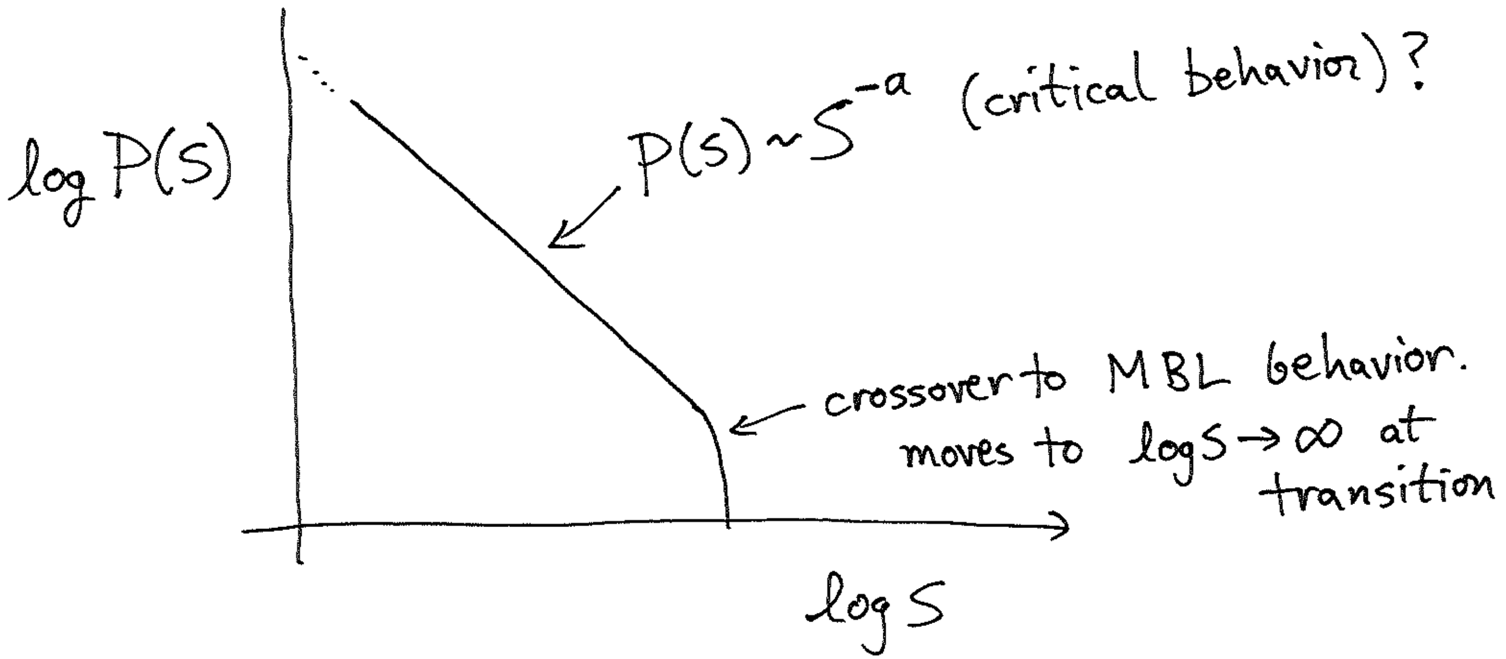
Here the system is "attempting" to thermalize itself, (but failing)

MBL phase: such entangled regions remain finite, do not "percolate".
(but dominate low-$\omega$ response, Gopalakrishnan, et al. 2015)

thermal phase: entanglement "percolates": makes a "reservoir".

Approach the MBL-ETH transition from the MBL side:

Expect (Vosk, H, Altman 2015):




If $a < 1$: $S \to \infty$ at transition for almost all cuts/states/samples

but

If $a > 1$: $P(S)$ remains nonzero, finite, boundary-law for almost all cuts/states/samples, thus $P(S)$ is discontinuous. (even while transition is continuous in other respects).

Slow entanglement spread in thermal phase due to Griffiths (rare regions) bottlenecks: Agarwal et al; Vosk, H, Altman 2015

phase transition

thermal phase | MBL phase | Disorder

$S_{ent} \sim t$ "ballistic"

$S_{ent} \sim t^{1/z}$

$z=1$ ← $z$ varies smoothly → $z \to \infty$

$S_{ent} \sim \log t$ (also, $\underline{\underline{\text{at}}}$ transition)

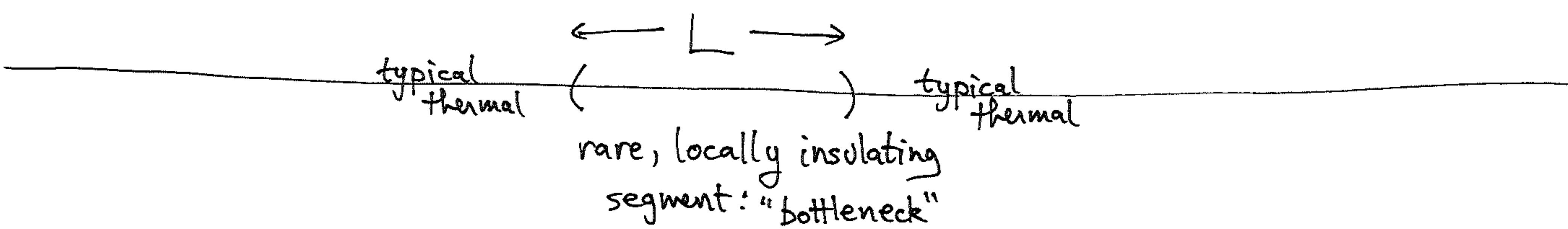
Griffiths regime: in thermal phase, but near phase transition.

← L →

locally thermalizing (typical)

( locally insulating (rare segment) ) ← these become more common near phase transition.

locally thermalizing (typical)




Time for entanglement to spread across bottleneck $\sim \exp(L/\xi)$

order one.

At time $t$, the bottlenecks are of length $\sim \xi \log(t)$

Probability of having a bottleneck of length $L$ $\sim \exp(-rL)$

*Large-deviations "rate function"
$r \to 0$ at transition.

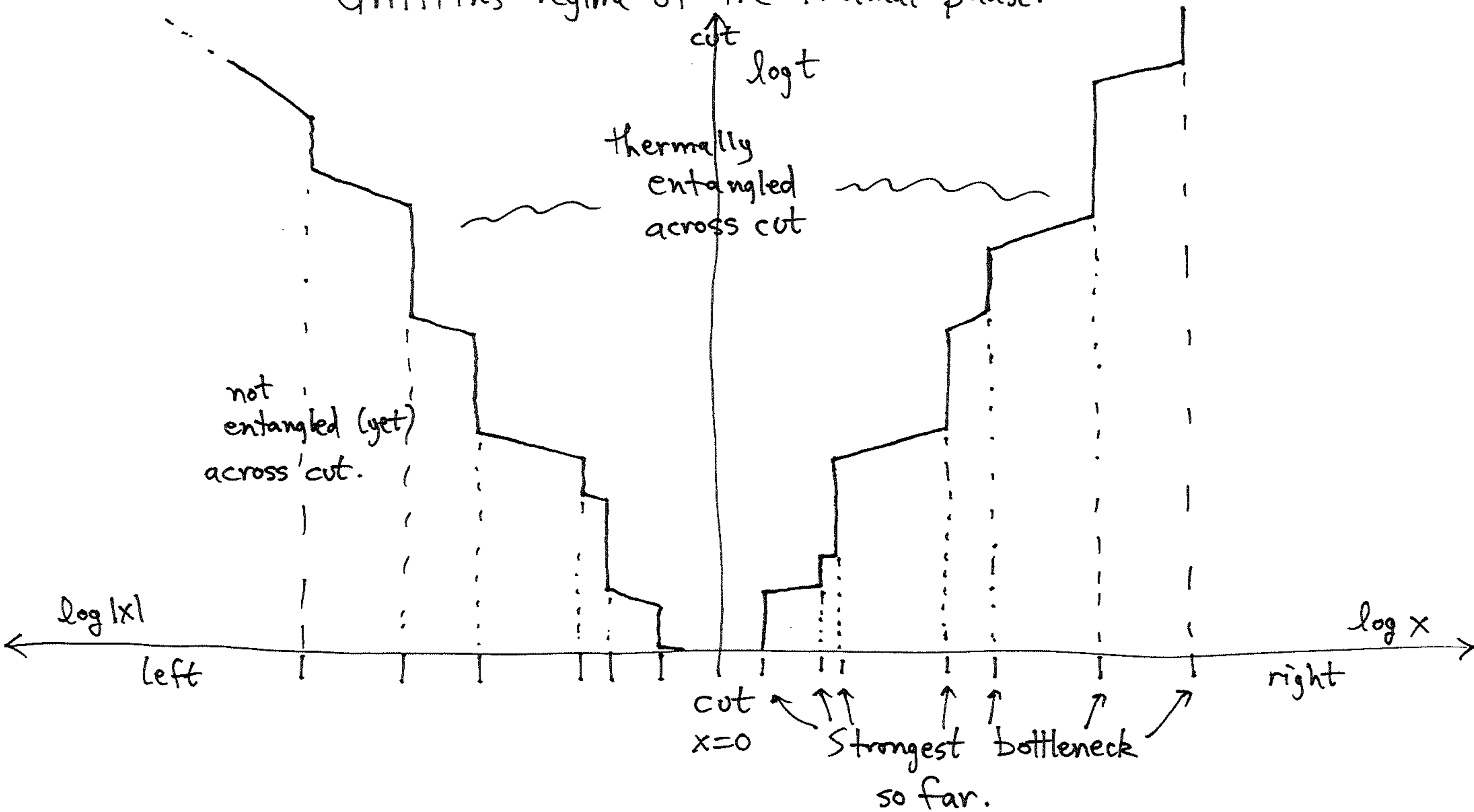
Typical spacing between bottlenecks:

$$\sim \exp(rL) \sim \exp(r\xi \log(t)) \sim t^{(r\xi)} = t^{1/z}$$ is how far entanglement can spread.

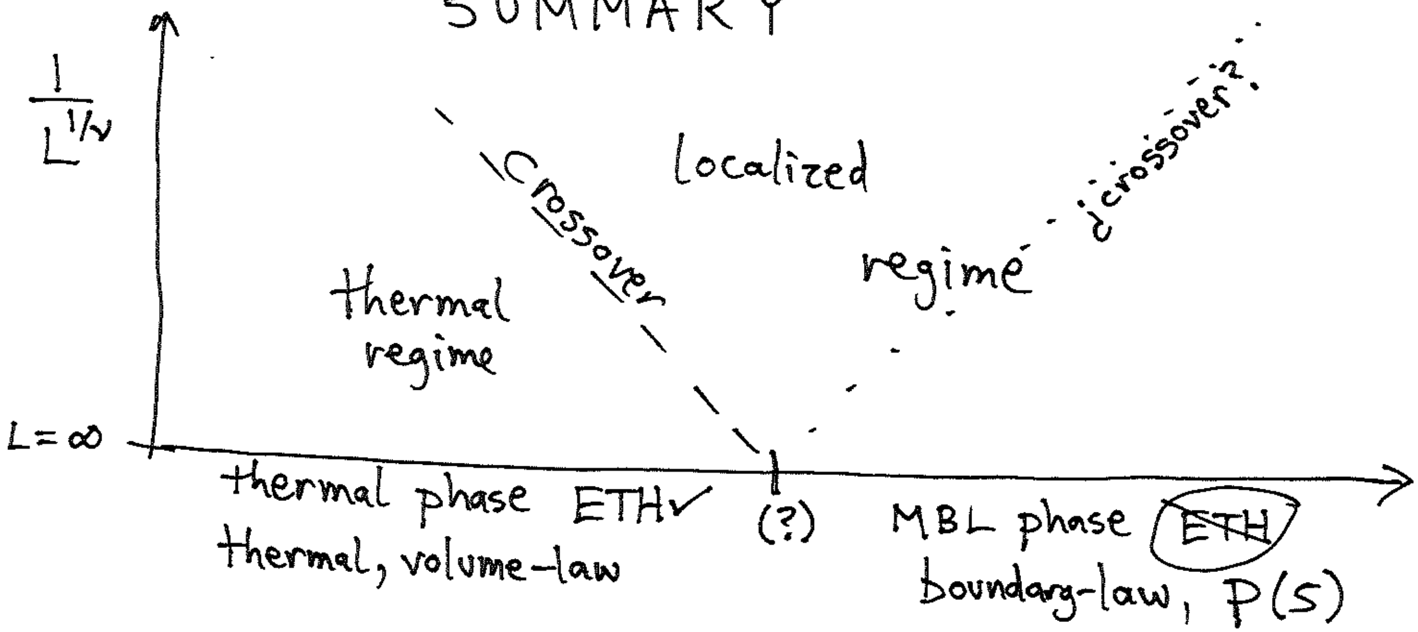
When $z > 1$ these slow the spread to "subballistic".

Power-law long-range interactions should remove this Griffiths regime in $S_{ent}(t)$.

"Sub-ballistic light-cone" for entanglement spreading in this Griffiths regime of the thermal phase:



# SUMMARY




Eigenstate Entanglement

---

Entanglement Spreading Dynamics:

thermal phase | Griffiths regime | MBL phase

$S_{ent} \sim t$ $\quad\uparrow_{z=1}\quad$ $S_{ent} \sim t^{1/z}$ $\quad\uparrow_{z=\infty}\quad$ $S_{ent} \sim \log t$

---

What about: $d>1$?, critical points in MBL phase in $d=1$?

power-law interactions? quasiperiodic systems?