Local Integrals of Motion in MBL

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joint work with V.Ros (SISSA) and M.Mueller (ICTP & PSI)

Ros, Mueller, AS, *Nucl Phys B 891, (2015) p. 420*

Overview

- MBL in spin chains/interacting fermions
- Perturbation theory for the integrals of motion
- Convergence of perturbation theory

Are interactions relevant?

Theoretical discussions are based on non-interacting particles

$$H = -t \sum_i (c^\dagger_{i+1} c_i + c^\dagger_i c_{i+1}) + \sum_i \epsilon_i c^\dagger_i c_i$$

Would the introduction of interactions change the picture?

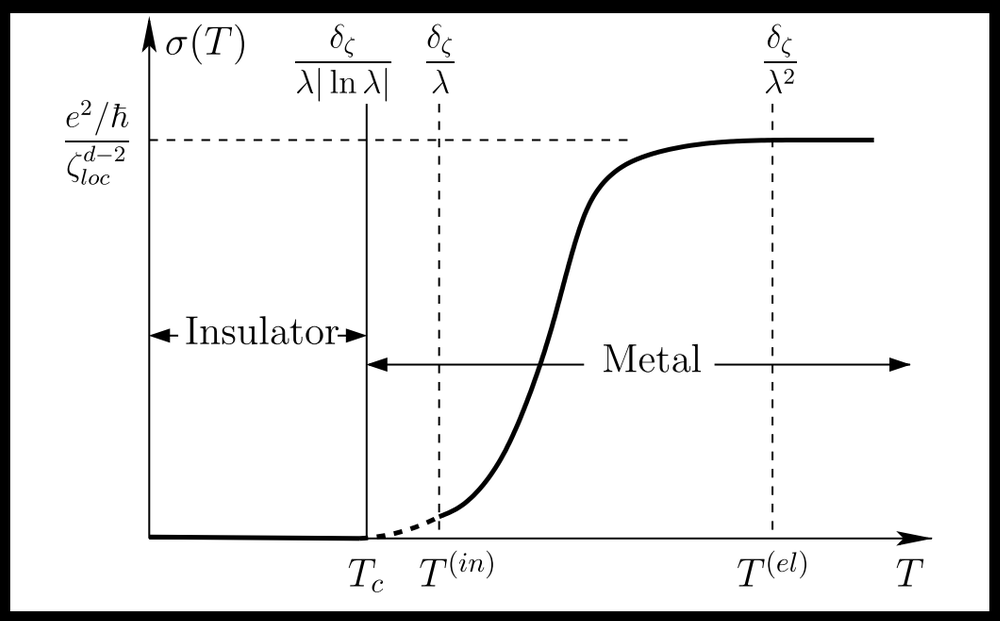
$$H = -t \sum_i c^\dagger_{i+1} c_i + c^\dagger_i c_{i+1} + \sum_i \epsilon_i n_i + \boxed{\lambda \sum_{i,j} v(|i-j|) n_i n_j}$$

Question asked already in Anderson 1958 but recently finally we think we have the answer

Many-body localization

In 2006 Basko, Aleiner and Altshuler presented
a perturbation theory calculation supporting the idea that,
for sufficiently small λ (and short-range interactions), localization survives

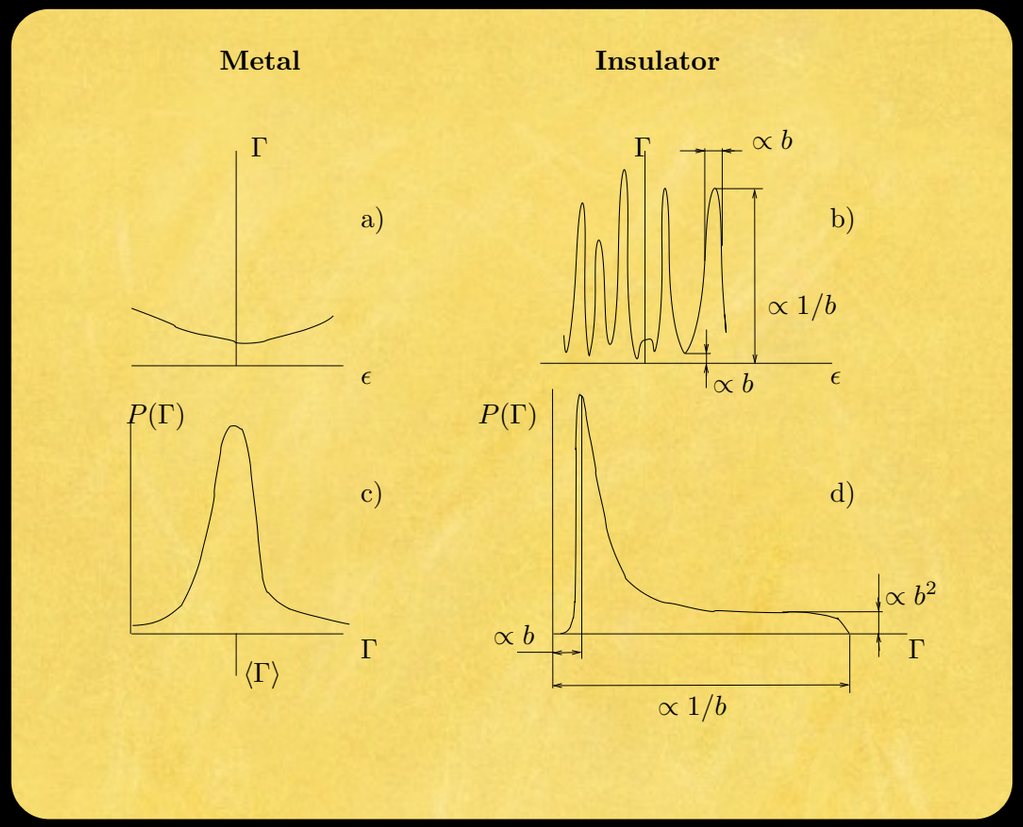
$$H = -t \sum_i c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1} + \sum_i \epsilon_i n_i + \lambda \sum_{i,j} v(|i-j|) n_i n_j$$



$$\lambda \lesssim \frac{\delta_\zeta}{T \ln(T/\delta_\zeta)}$$ Insulator

Many-body localization

In BAA this conclusion is based on an analysis of the distribution of $Im\Sigma$ in the spirit of Anderson 58



This was achieved by a diagrammatic analysis of the Green's function (Keldysh)

MBL in spin chains

Soon after BAA some authors (Huse, Oganesyan, Pal, Abanin and others) have proposed that the same phenomenon should be observed in spin chains, even at **infinite** temperature

$$H = J \sum_i \vec{s}_i \cdot \vec{s}_{i+1} - \sum_i h_i s_i^z$$

long wavelength $$M = \sum_{j=1}^{N} S_j^z e^{2\pi i j/L}$$

initial state $$\rho = 1 + \epsilon M^\dagger$$

MBL in spin chains

$$M(t) = \mathrm{Tr}(\rho(t)M)$$

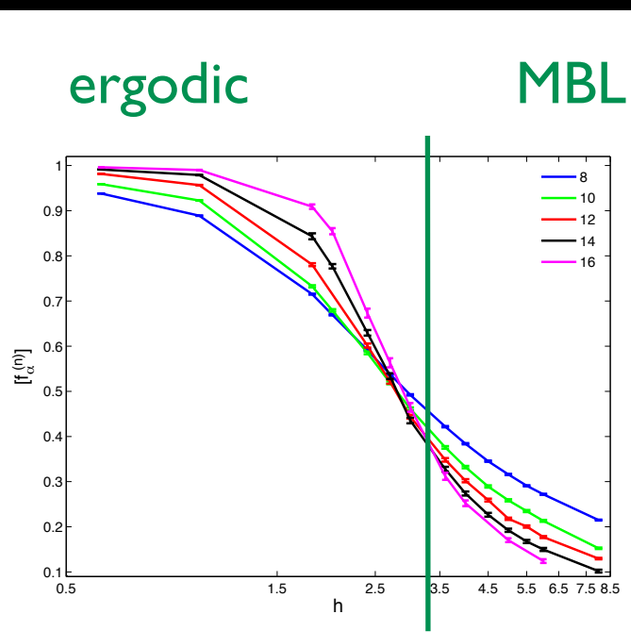


FIG. 2: (Color online) The fraction of the initial spin polarization that is dynamic (see text). The sample size L is indicated in the legend. In the ergodic phase (small h) the polarization decays substantially under the dynamics, while in the localized phase (large h) the decay is small, and this distinction gets sharper as L increases.

$$M(\infty) = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt M(t)$$

$$M(\infty) = 0 \qquad \text{if ergodic}$$

$$f = 1 - \frac{M(\infty)}{M(0)} \qquad \text{“ergodicity measure”}$$

MBL in spin chains

WD

↓

Poisson

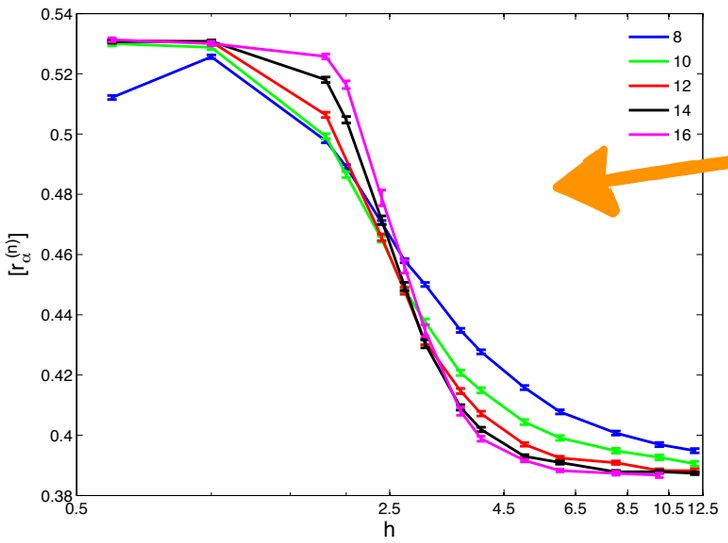
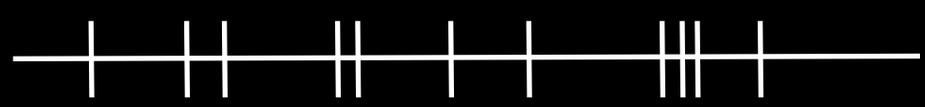


FIG. 3: (Color online) The ratio of adjacent energy gaps (defined in the text). The sample size L is indicated in the legend. In the ergodic phase, the system has GOE level statistics, while in the localized phase the level statistics are Poisson.

Spectral signatures

$$r = \left\langle \frac{\min(dE_i, dE_{i+1})}{\max(dE_i, dE_{i+1})} \right\rangle_i$$



Increasing disorder the level statistics goes from Wigner-Dyson to Poisson, and level repulsion disappears

MBL in spin chains

Ergodicity implies that $|\psi(a)|^2 \sim 1/V$ volume of the Hilbert space

More precisely: $x = V|\psi(a)|^2$

Then $\phi_N(x) \to \phi(x)$

Distribution of the wave function coefficients

(diagonal entropy)

Ergodic

Non-ergodic

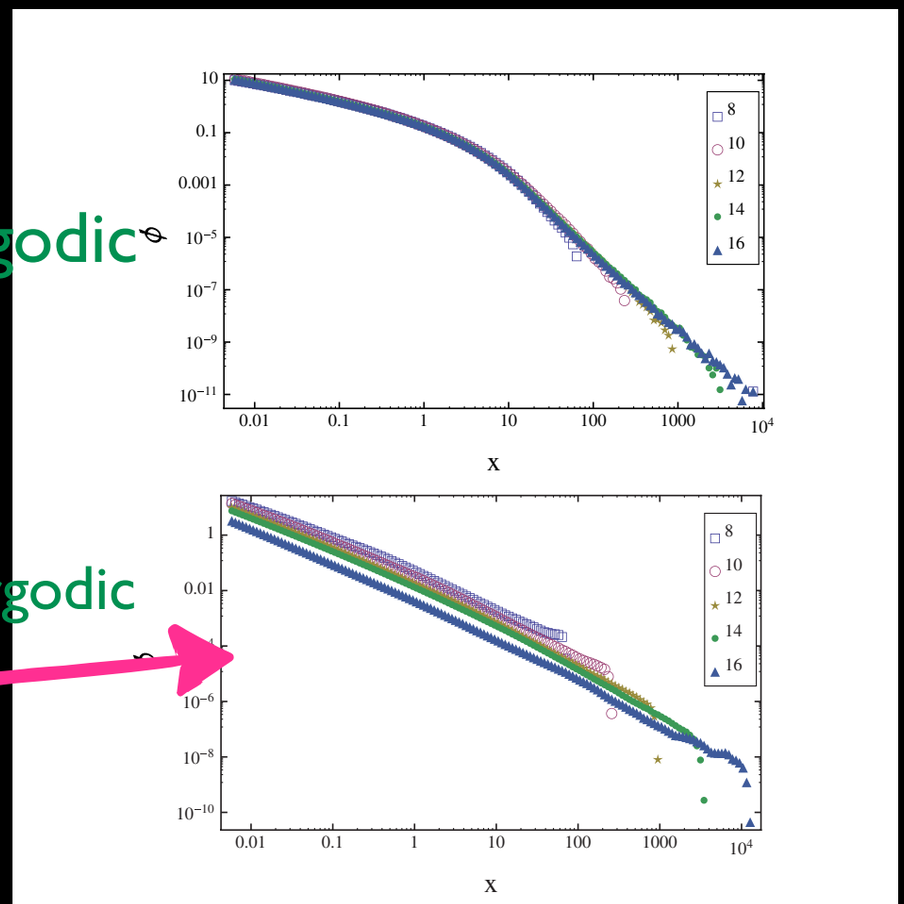
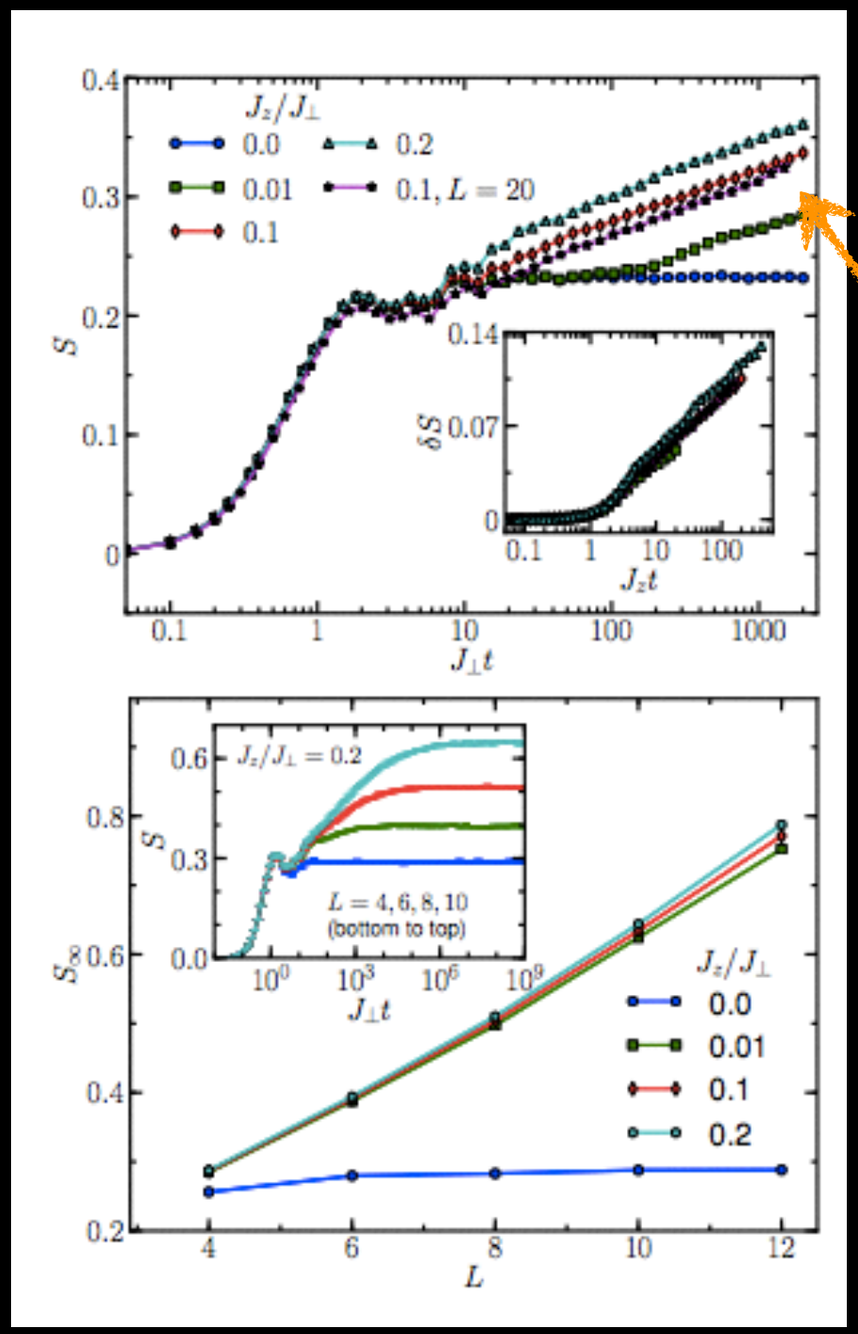


Fig. 3: (Colour on-line) The distribution ϕ of scaled wave function amplitudes $x = \mathcal{N}|\langle a|E\rangle|^2$ for different values of h. Upper panel: $h = 1.2$ in the middle of the ergodic phase where the scaling is perfectly verified; lower panel: $h = 4.2$ in the many-body localized phase. In each panel the different curves correspond to different values of N, from 8 to 16. Each curve is obtained by binning of not less than 3 10^6 squared amplitudes.

Entanglement growth



Entanglement between two half-chains

A | B

$$S = -\mathrm{Tr}_A(\rho_A \ln \rho_A)$$

logarithmic growth

Far away regions of our system cannot exchange energy or spin but they can get entangled

(Bardarson et al. 2014)

Ergodicity breaking

What is the phenomenology of the MBL phase?

- Energy and charge transport is suppressed
- Some memory of the initial state is conserved forever in **local** quantities
- Eigenstates close in energy have different "footprints" of local observables
- Eigenstates have area law entanglement (even at high *T*)*
- Entanglement of an initial product state grows slowly, but to an extensive value

Local IOM

A unifying phenomenology

$$H = J \sum_i \vec{s}_i \cdot \vec{s}_{i+1} - \sum_i h_i s_i^z$$

in the MBL phase can be rewritten as

$$H = -\sum_i h' \tau_i^z - \sum_{ij} J_{ij} \tau_i^z \tau_j^z - \sum_{ijk} J_{ijk} \tau_i^z \tau_j^z \tau_k^z + ...$$

The operators τ_i^z are conserved quantities

$$[H, \tau_i^z] = 0$$ called *l-bits*

Local IOM

Which are local

$$\tau_1^z = \frac{1}{Z}(\sum_i A_i^{(1)} s_i^z + \sum_{i,j} A_{ij}^{(2)} s_i^+ s_j^- + \sum_{i,j,k} A_{ijk}^{(3)} s_i^+ s_j^z s_k^- + ...)$$

$$|A_i^{(1)}| < e^{-|i-1|/\xi_1} \qquad |A_{ij}^{(2)}| < e^{-(|i-1|+|j-1|)/\xi_2}$$

This LIOMs constrain the dynamics of the system in such a way that ergodicity cannot be achieved

This conjecture was proven for one particular spin chain by Imbrie (arXiv:1403.7837)

Local IOM

We constructed explicitly the LIOM for

$$H = -t \sum_i c^\dagger_{i+1} c_i + c^\dagger_i c_{i+1} + \sum_i \epsilon_i n_i + \lambda \sum_{i,j} v(|i-j|) n_i n_j$$

connecting the formulation in terms of LIOM to the perturbation theory of BAA

The LIOMs are number operators dressed with strings of excitations

$$I_1 \simeq n_1 + A_2 c_2^+ n_1 c_0 + A_3 c_3^+ n_2 n_1 c_0 + ...$$

Ros, Mueller, AS, Nucl Phys B 2015

Local IOM

First of all we diagonalize the quadratic part

$$H = \sum_{\alpha} \epsilon_\alpha c^\dagger_\alpha c_\alpha + \lambda \sum_{\alpha,\beta,\gamma,\delta} u_{\alpha,\beta,\gamma,\delta} c^\dagger_\alpha c^\dagger_\beta c_\gamma c_\delta$$

$|\alpha\rangle$ single particle localized states

we coarse grain the system into "quantum dots" of size ξ and then we consider only matrix elements between the same or n.n. quantum dots

$$[H_0 + \lambda V, I_\alpha] = 0$$

We need to solve $[H_0, I^{(n)}_\alpha] + [V, I^{(n-1)}_\alpha] = 0$

Local IOM

$$I_1 = n_1 + A_{2,1} c_2^\dagger n_1 c_0 + A_{2,2} c_2^\dagger c_0 + \ldots$$
$$+ A_{3,1} c_3^\dagger c_2^\dagger c_1^\dagger c_2 c_1 c_0 + A_{3,2} c_3^\dagger c_2^\dagger c_2 c_0 + \ldots$$

the number of terms at distance r 4^r

the amplitudes are random numbers

$$|A_r| = \max_i |A_{r,i}|$$

In the localized regime we expect

$$\exists \xi > 0 \quad \lim_{r \to \infty} P(|A_r| < e^{-r/\xi}) = 1$$

so that the operators are *(quasi-)*local

Local IOM

One could use perturbation theory to construct the conserved operator, removing non-commuting terms order by order (*a la Poincaré*) (Chandran et al, Rademaker)

Instead, to make analytic progress, we focused on the tail of the operators and estimated

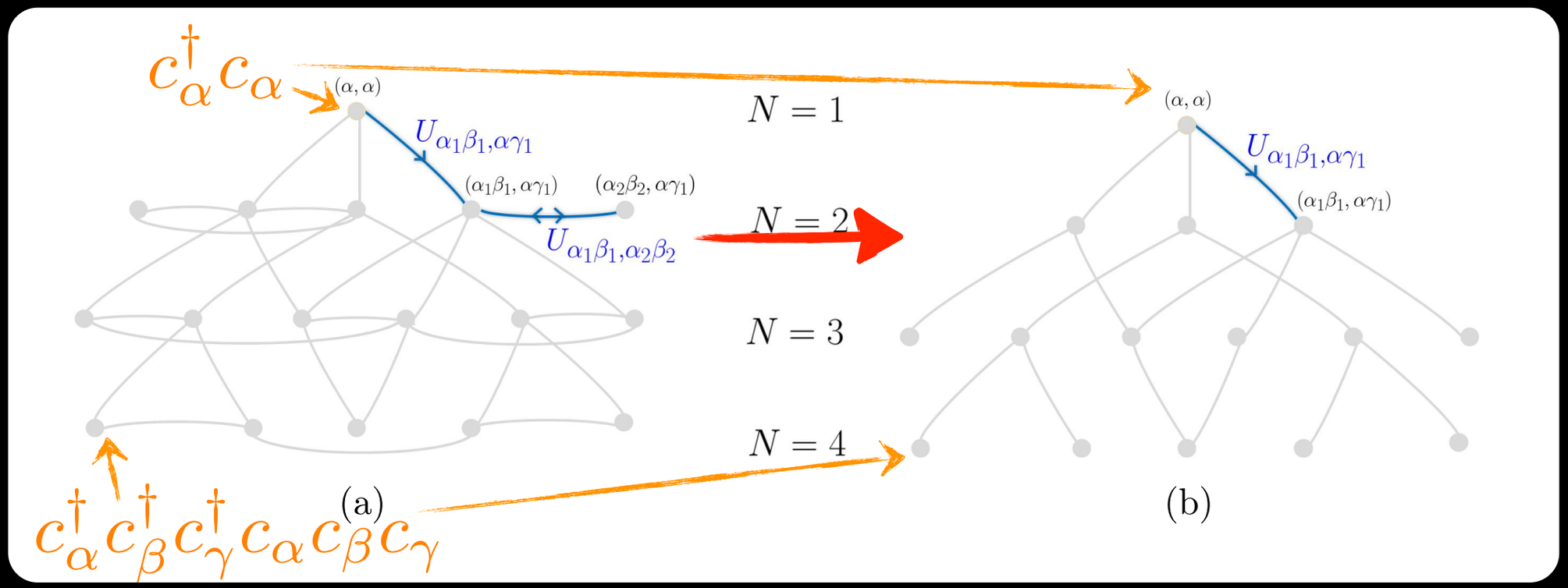
$$A_r = \lambda^r c_0(r) + \lambda^{r+1} c_1(r) + ...$$

to lowest order in perturbation theory

$$c_0(r) \simeq q^r$$

Perturbation theory

Hopping in operator space



Lowest order: shortest paths from a short to a long operator (forward approximation)

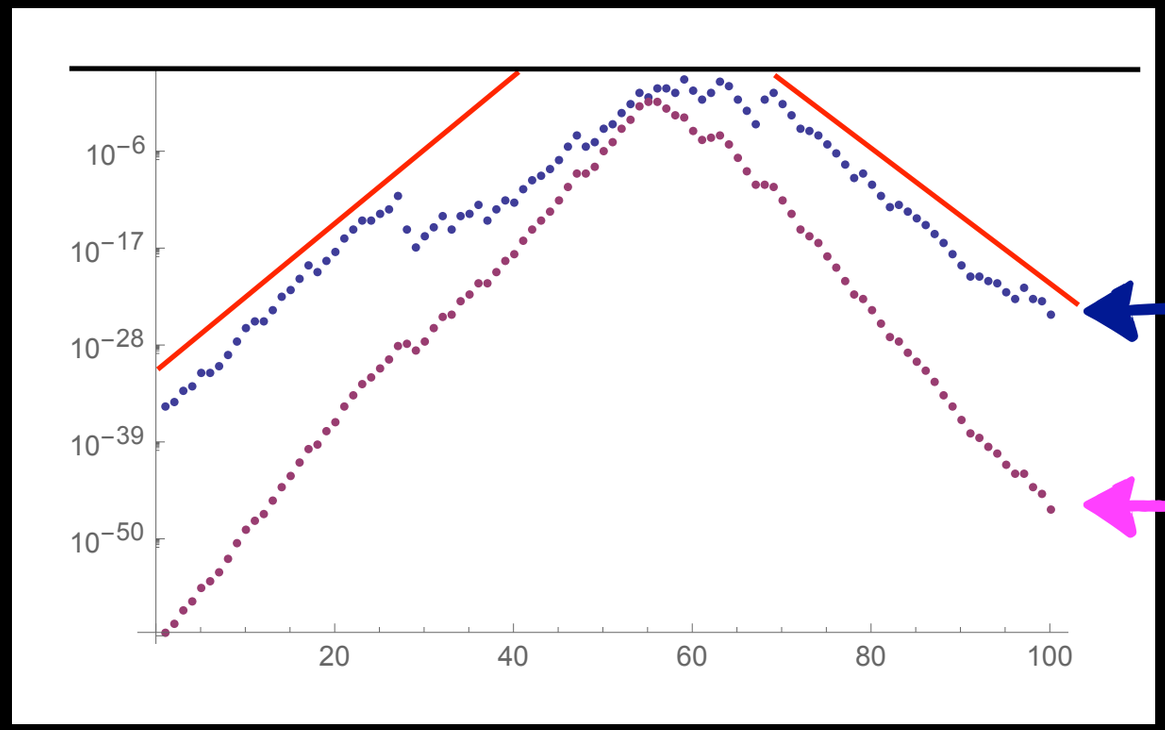
This should give a lower bound for the critical interaction

Perturbation theory

Forward approximation

$$\psi_\alpha(n) \simeq \sum_{p \in \mathrm{paths}(0,n)} \prod_{i \in p} \frac{t}{\epsilon_0 - \epsilon_i}$$

Resonances are less important in the exact solution than in the fwd approx



$e^{-r/\xi}$

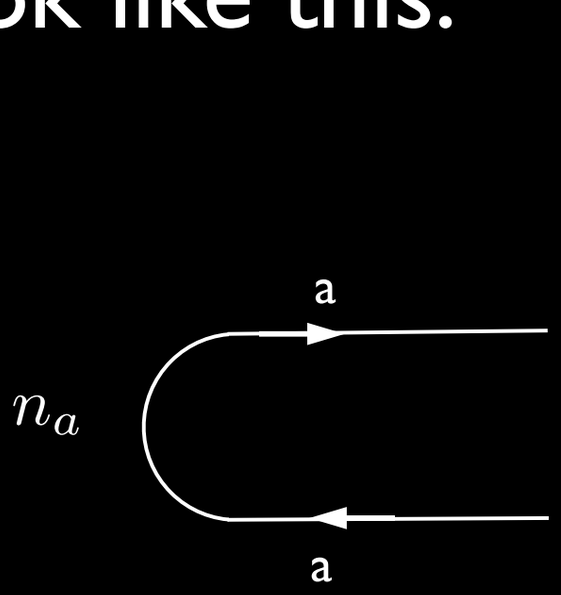
fwd approximation

exact eigenstate

This is equivalent to the ImSCBA in BAA's perturbation theory (see also Abou-Chacra, Anderson, Thouless 1973)

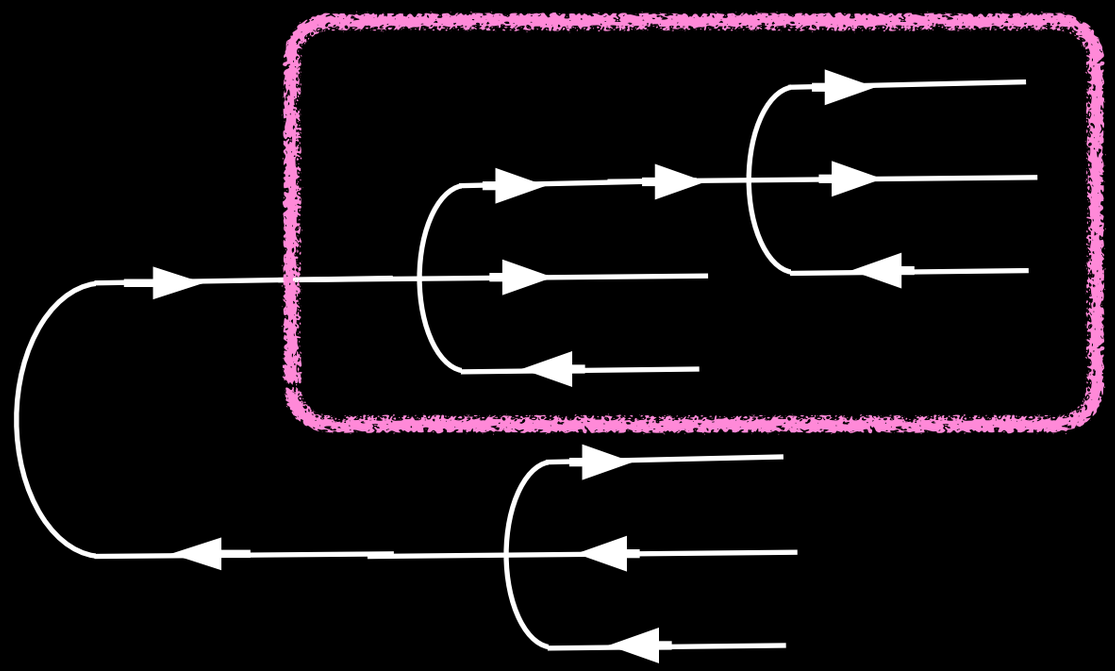
Perturbation theory

We only include terms in the *n*-th order operator which look like this:



λ^0

or

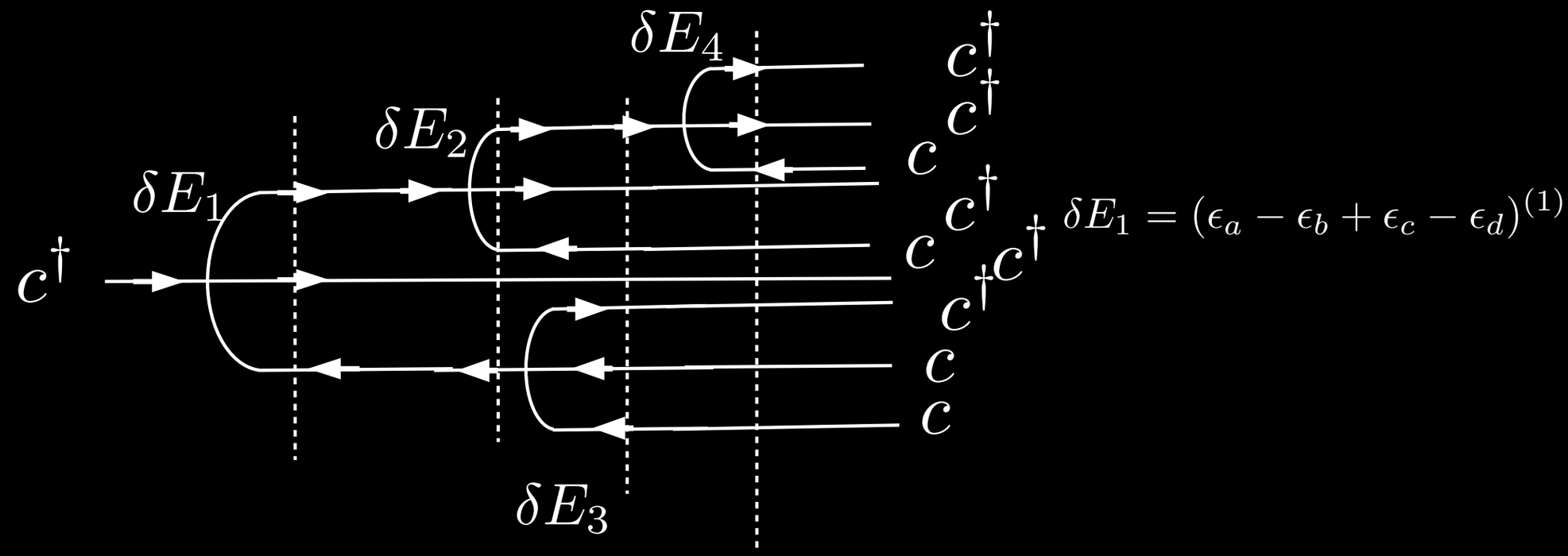


λ^3 $4\ c^\dagger,\ 4\ c$

Consider the two branching trees

Perturbation theory

One sub-tree generates an amplitude:



$$A = \frac{\lambda\delta_\xi}{E + \delta E_1} \frac{\lambda\delta_\xi}{E + \delta E_1 + \delta E_2} \frac{\lambda\delta_\xi}{E + \delta E_1 + \delta E_2 + \delta E_3} \frac{\lambda\delta_\xi}{E + \delta E_1 + \delta E_2 + \delta E_3 + \delta E_4}$$

Correlated denominators

The many-body amplitude

$$A = \prod_{a=1,\ldots,n} \frac{\lambda \delta_\xi}{E + \sum_{i=1}^{a} \delta E_i}$$

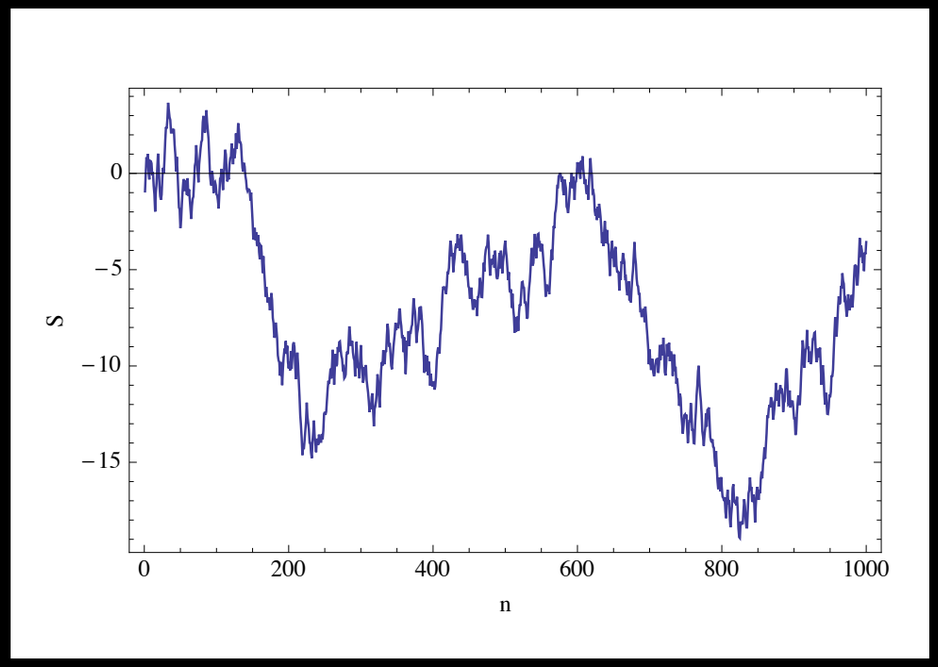
is different from the single-particle one

$$A = \prod_{a=1,\ldots,n} \frac{\lambda \delta_\xi}{E + \delta E_a}$$

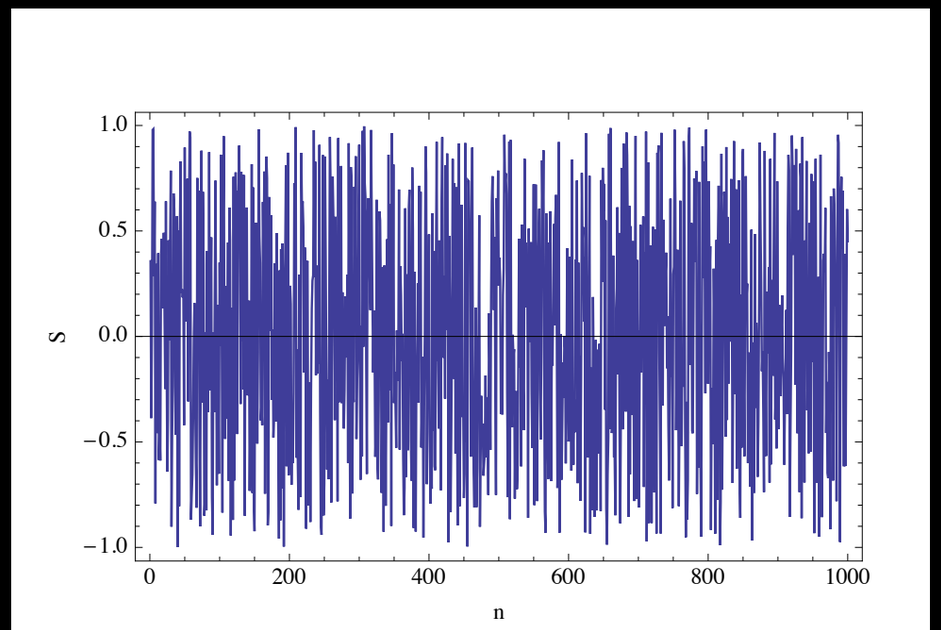
We need to find $P(A)$ and we cannot use the techniques used for single particle AL

Correlated denominators

Very different probability distributions...



many body



single particle

Consider $Y = -\ln|A|$

Correlated denominators

We can compute the Laplace transform

$$G_N(k) = \mathbb{E}[e^{-kY}]$$

and eventually invert it to get

$$P(Y) = \int_B \frac{dk}{2\pi i} e^{kY} G_N(k)$$

we anticipate that we are going to do a saddle point calculation with

$$Y \sim N$$

Correlated denominators

We cast the Laplace transform in a transfer matrix calculation

$$G_N(k) = \left(\frac{\delta_\xi^{2k}}{2\pi}\right)^{N/2} \qquad \langle\psi'|\mathcal{H}^N|\psi\rangle.$$

$$\mathcal{H}_{n,m} = \frac{\Gamma(\frac{1+k}{2}+m+n)}{\sqrt{\Gamma(1+2m)}\sqrt{\Gamma(1+2n)}}$$

So we need to find the largest eigenvalue of *H*

Correlated denominators

This can be done (for large Y) and we find that

$$P(Y) = \int \frac{dk}{2\pi i} e^{N(y+\log(\mu))} \simeq \left(\frac{Y}{y_0 N}\right)^N e^{-Y/y_0(1-\gamma/(2(Y/N)^2)}$$

We now need to find *how many terms are there in the sum*

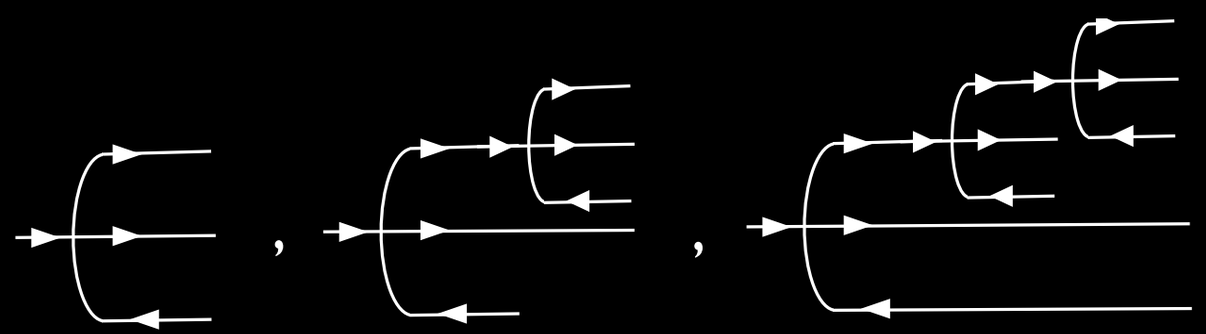
$$I_\alpha^{(n)} = \sum_{|\mathcal{I}|=|\mathcal{J}|=n} A_{\mathcal{I},\mathcal{J}} O_{\mathcal{I},\mathcal{J}}$$

of the order *n* correction to *I*

Counting diagrams

Topology + assignment of indices $\alpha, \beta, \gamma, \delta$

Topology:



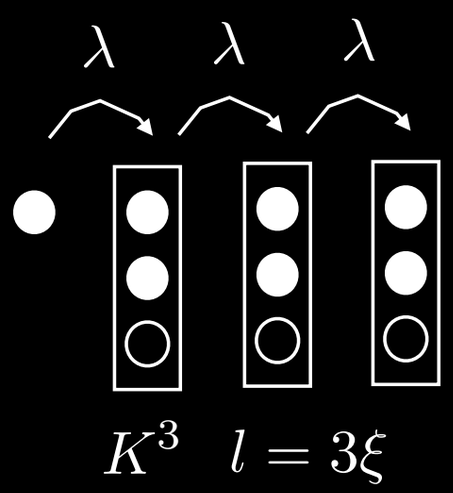
Classic combinatorics problem (generalized Catalan numbers)

$$T_n = \sum_{m_1, m_2, m_3, \sum_i m_i = n} T_{m_1} T_{m_2} T_{m_3}$$

$$T_n = \frac{1}{2n+1} \binom{3n}{n} \sim \left(\frac{27}{4}\right)^n$$

Counting diagrams

Assignment of indices defines the spatial structure of the excitations



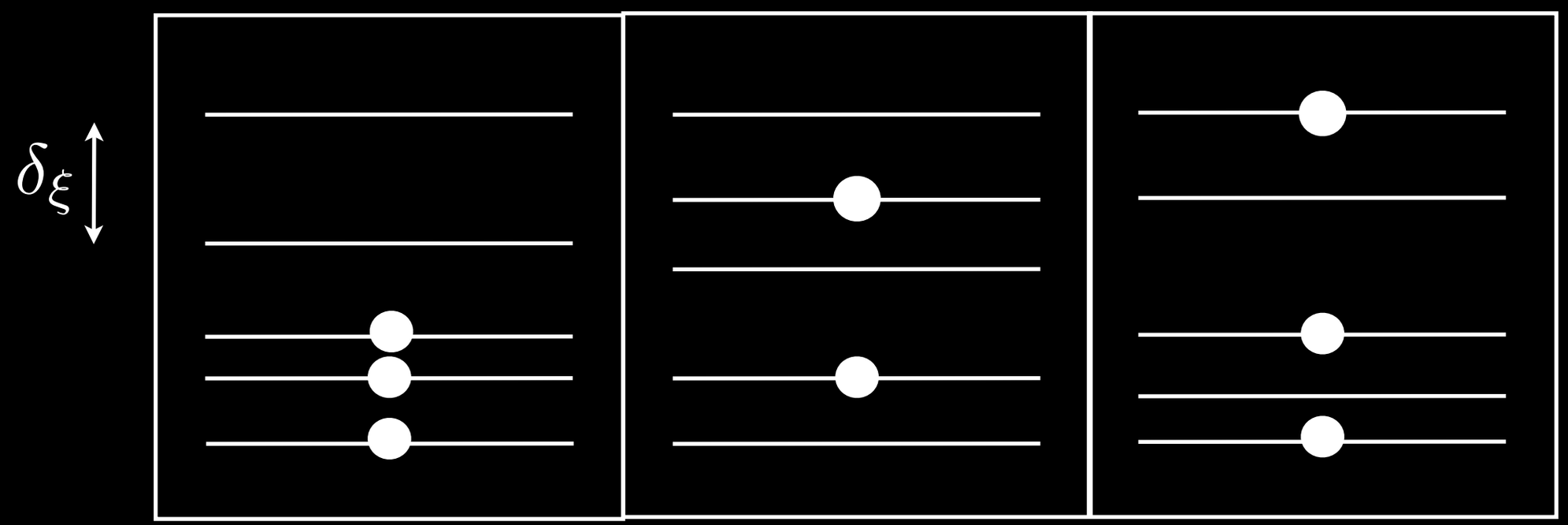
e.g. ballistic excitation

$$K = \xi/a$$

It is convenient to consider the picture of a particle which hops from volume to volume leaving a trail of excitations

Local IOM

The important processes are those in which an excitation can travel staying (almost) in resonance



$$\Delta E = (E_1 - E_2) + (E_3 - E_4) + (E_5 - E_6) \lesssim \delta_\xi$$

$$T \ll W \ : \ K \sim T/\delta_\xi \qquad\qquad T \gg W \ : \ K \sim W/\delta_\xi$$

We need to estimate the probability that a resonance occurs far away

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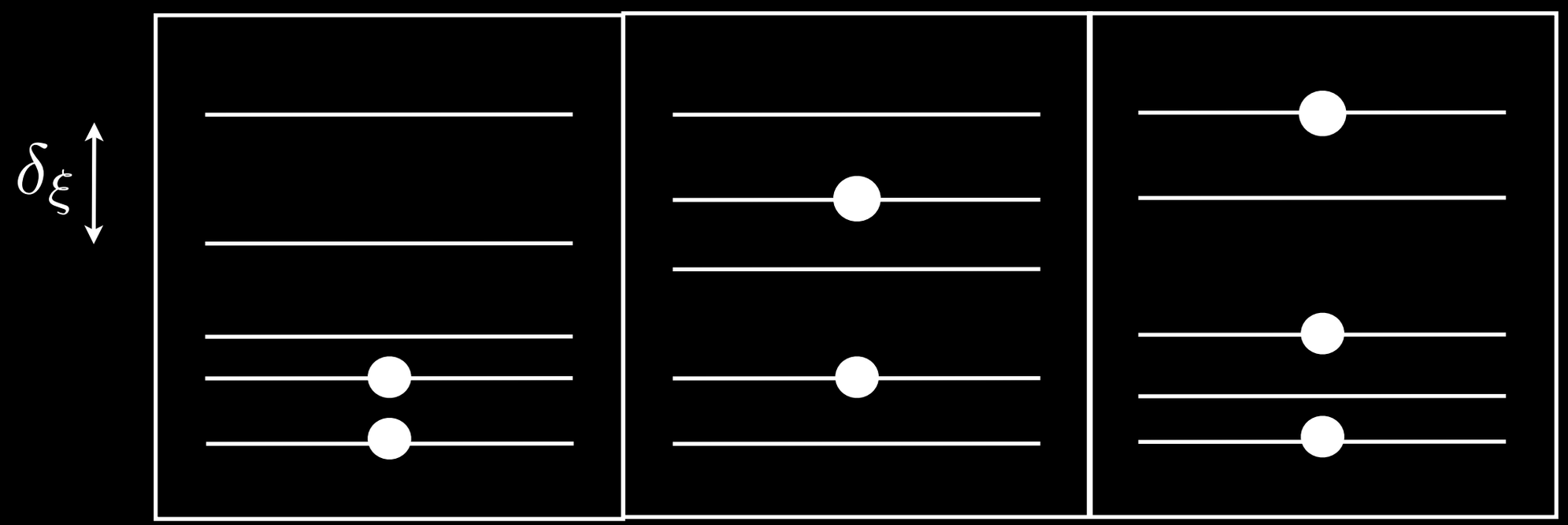

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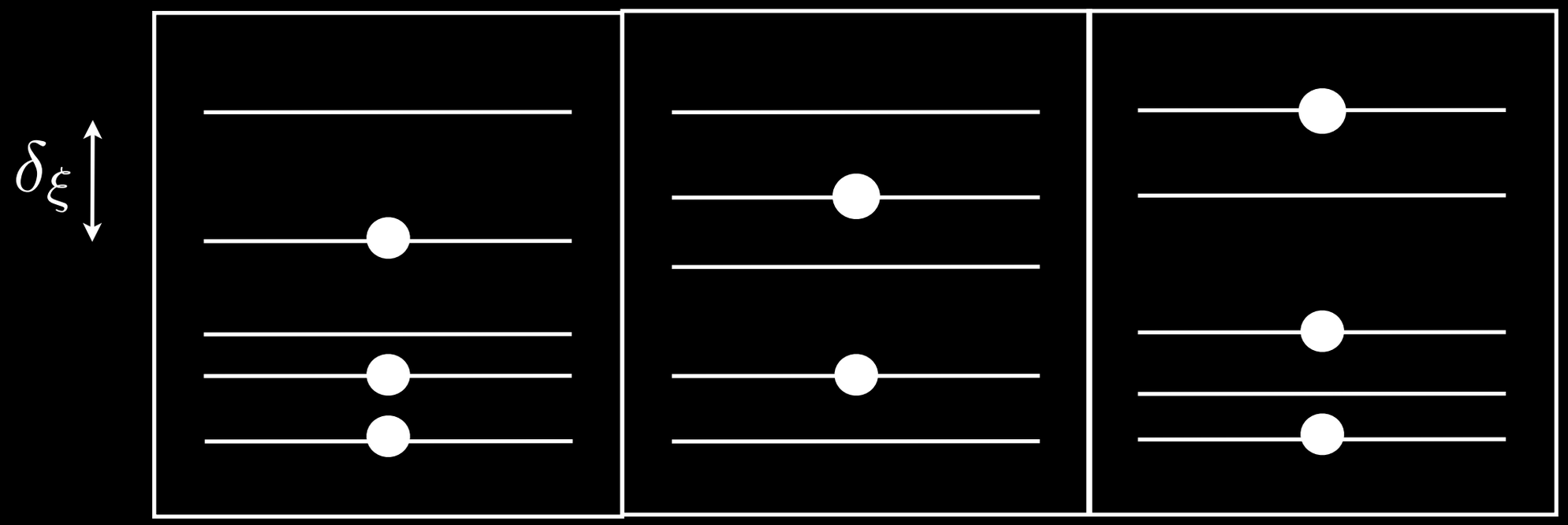

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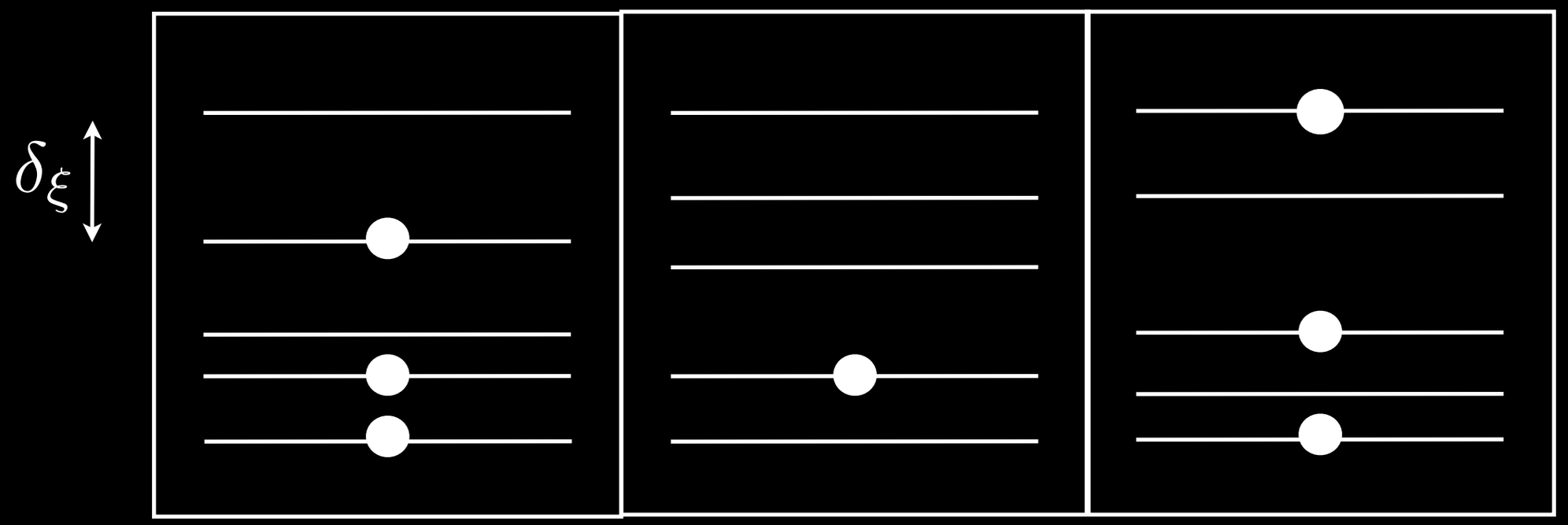

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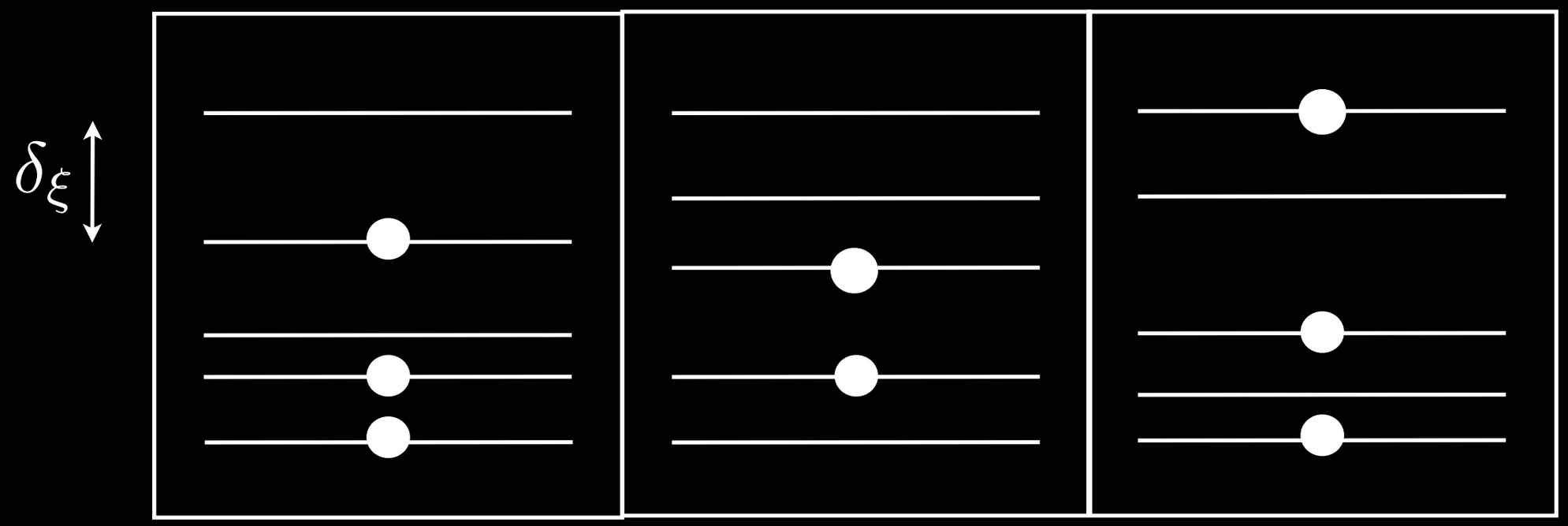

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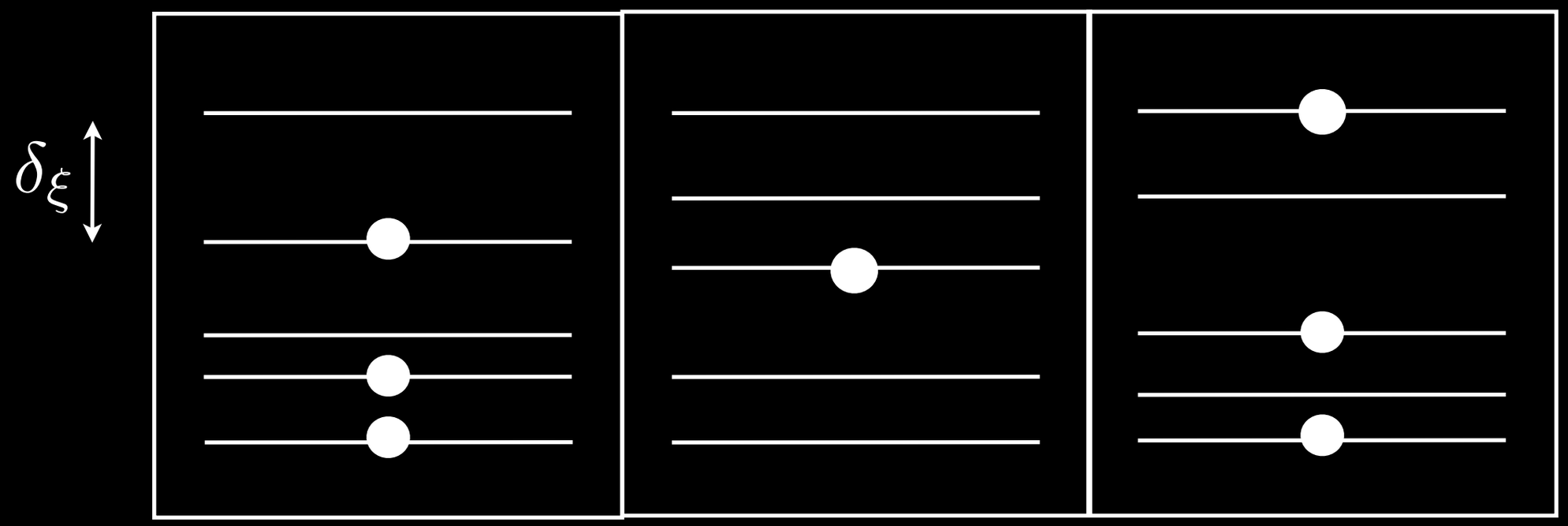

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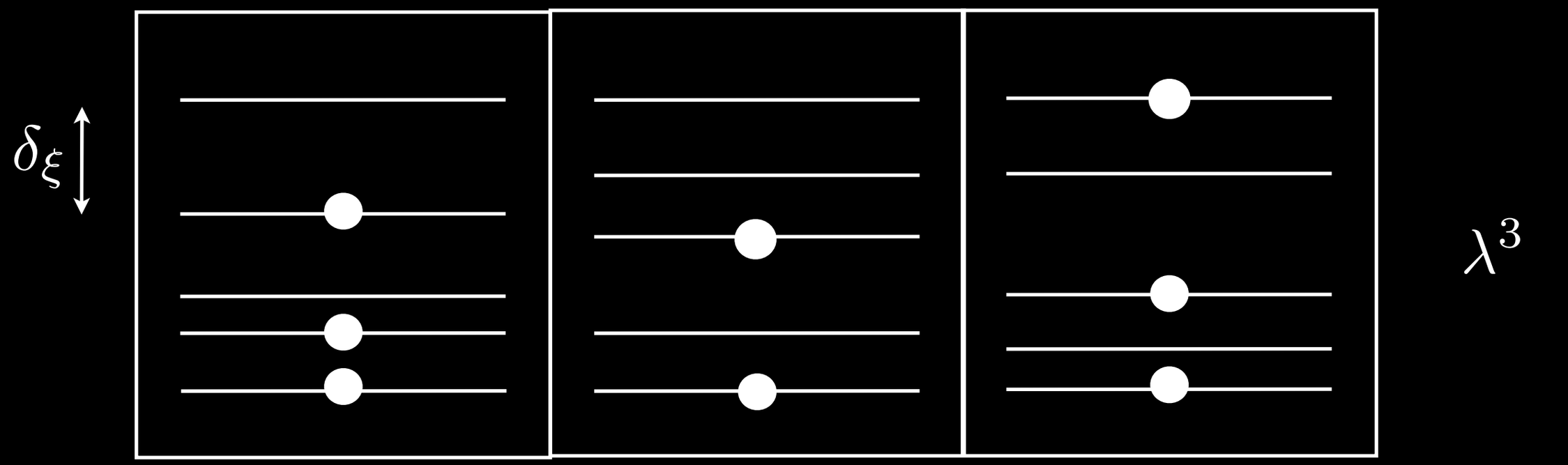

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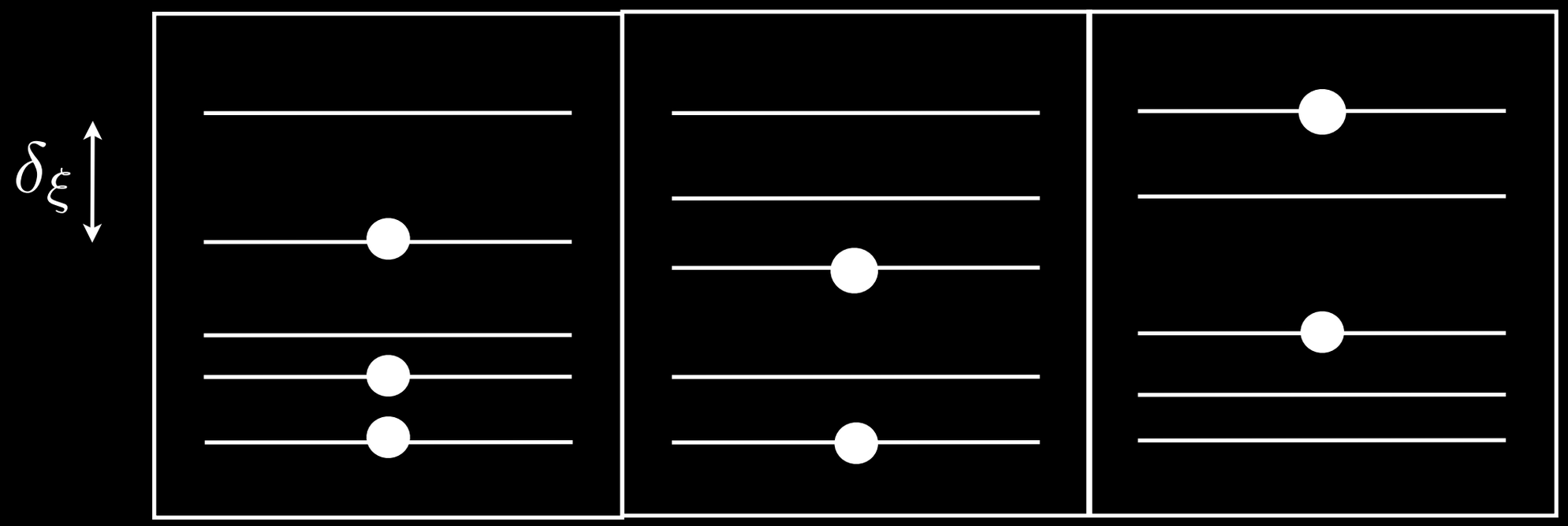

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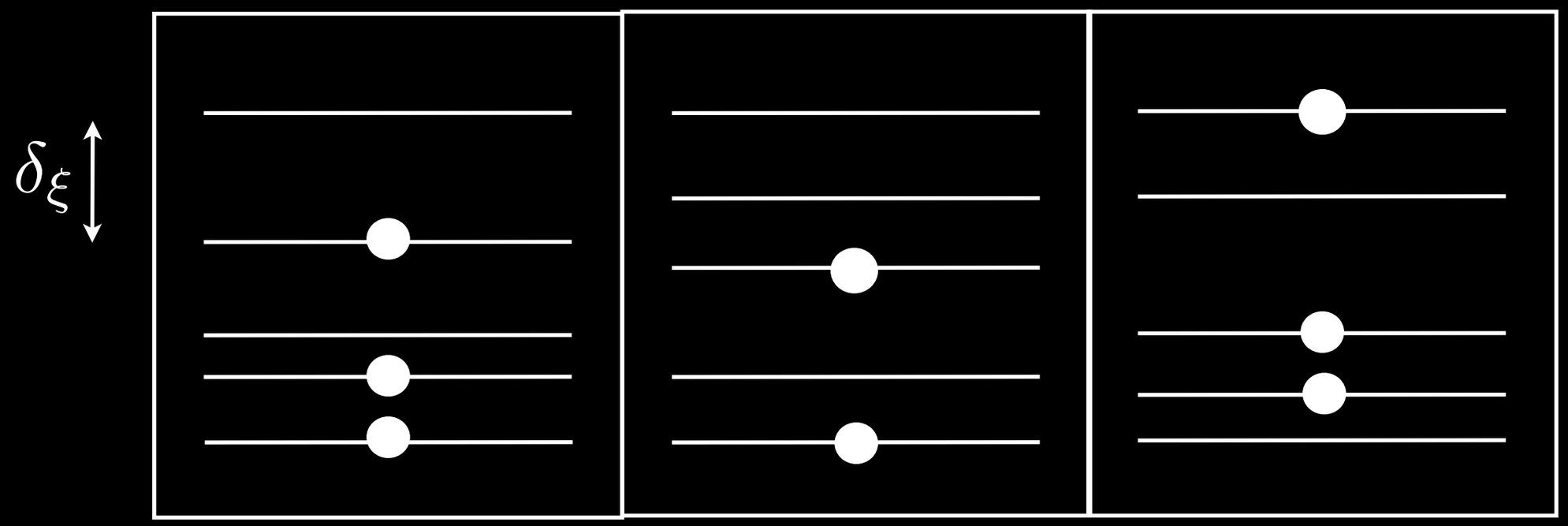

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We need to estimate the probability that a resonance occurs far away

Counting diagrams

Assigning indices: describing the trail of excitations

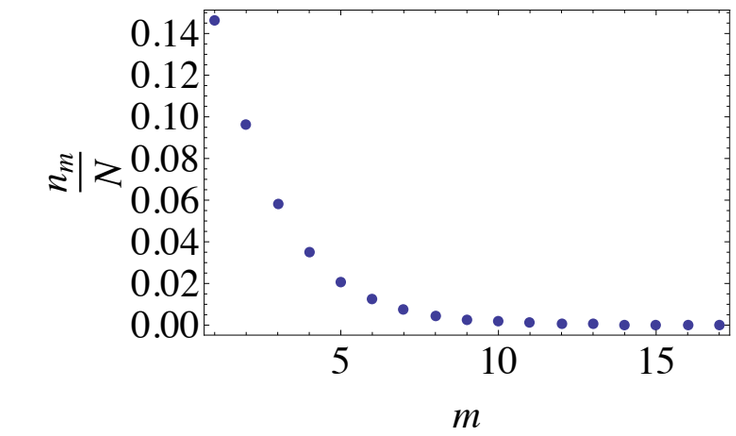
In every box *i* there is *m excitations*

$$\mathcal{N}_N \approx \overline{\mathcal{P}(d)} \sum_{\{m_i\}|\sum_i m_i = N} \frac{1}{2} \prod_{i=1}^{n} \left[2\mathcal{K}^{m_i} m_i \mathcal{T}_{m_i}\right],$$

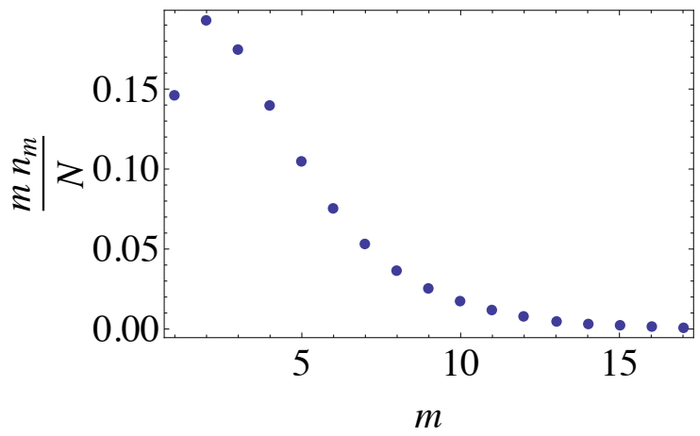
We need to maximize this number over the *m's*

$$\mathcal{N}_N \approx (\mathcal{K} e^{\mu})^N \approx (10.6\,\mathcal{K})^N$$

Counting diagrams



(a)



(b)

Figure 6: The plot (a) shows the distribution of the number n_m/N of groups of m particle-hole pairs in necklace diagrams dominating $\mathcal{N}_N$. The plot (b) shows the probability mn_m/N that a given pair belongs to a group containing m pairs.

Recapitulate

- 1) Distribution of amplitudes of a single diagram giving a long operator
- 2) Count the number of diagrams
- 3) Count the number of spatial processes pertaining to a given assignment of indices

Result

for $\lambda < \lambda_c = \dfrac{\sqrt{2\pi}}{\nu(1-\nu)2eC}\dfrac{1}{K\ln K}$

$18.97 < C < 36.25$

$K \sim T/\delta_\xi$

$K \sim W/\delta_\xi$

we can find operators I_α (one per site) $I_\alpha = 0, 1$

$$[H, I_\alpha] = 0$$

Then the eigenstates can be written as bit strings

$$|E_m\rangle = |0, 1, 0, 0, 0..., 1\rangle$$

each bit is the eigenvalue of a local operator

$$\mathrm{Tr}(I_\alpha c_r^\dagger c_r) \sim e^{-d(\alpha,r)/\ell}$$

Convergence of p.t. for LIOMs

We found that the IOM are local for

$$\lambda < \lambda_c = \frac{\sqrt{2\pi}}{\nu(1-\nu)2eC} \frac{1}{K \ln K}$$

For $\lambda > \lambda_c$ there are several different scenarios

a) All LIOMs die, becoming non-local
b) Some LIOMs "die," some don't (*a la* KAM)

This problem is open

Conclusions

- MBL phenomenology can be recovered by conjecturing the existence of local IOMs
- We can show, under the same approximations of BAA, that LIOMs exist for weak interactions/strong disorder
- We can find the radius of convergence of the perturbation theory