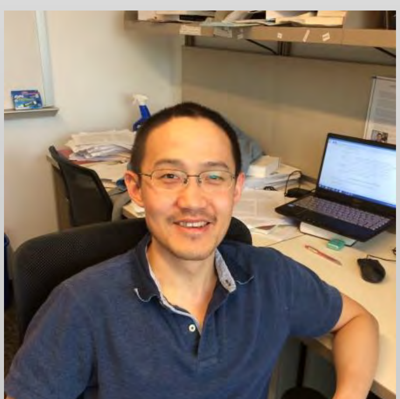
Topological protection, disorder, and interactions: Transport and delocalization at the surface of 3D topological superconductors

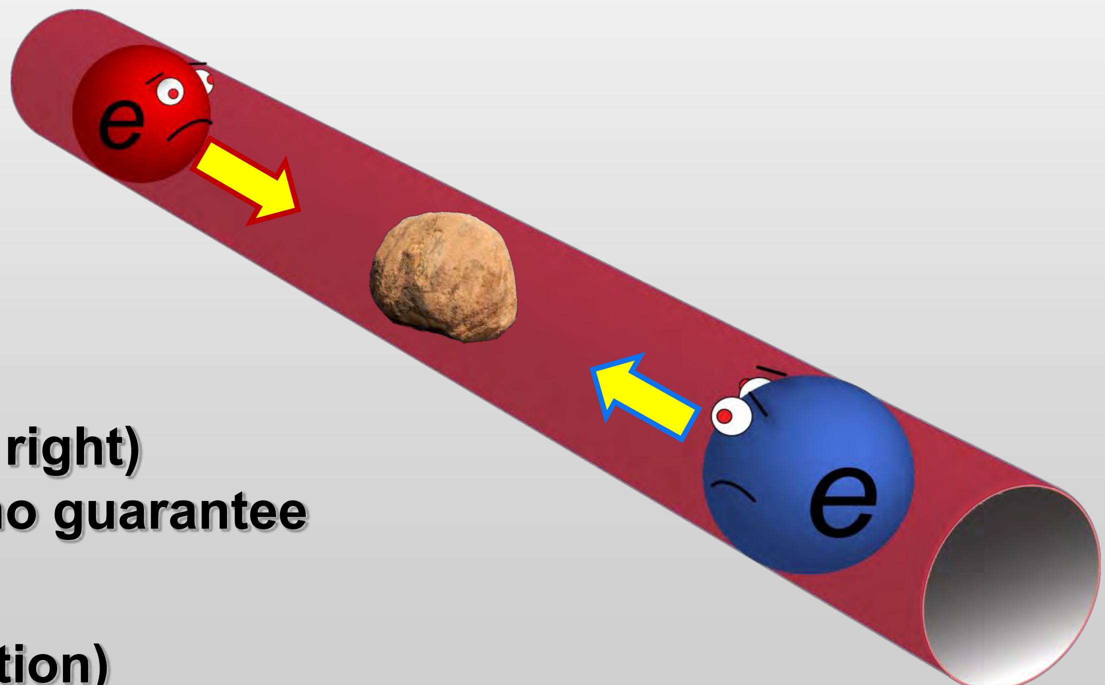
Hong-Yi Xie, Yang-Zhi Chou, Emil Yuzbashyan, and M. S. Foster



MSF & EY (2012)
MSF, H-YX, Y-ZC (2014)
Y-ZC & MSF (2014)
H-YX, Y-ZC, & MSF (2015)

Normal 1D quantum wire (e.g., carbon nanotube)

- **Left, right moving electrons**
- **Scattering (e-e, e-impurity) can change left-mover into right-mover, vice-versa**
- **Left and right movers are not separately conserved**



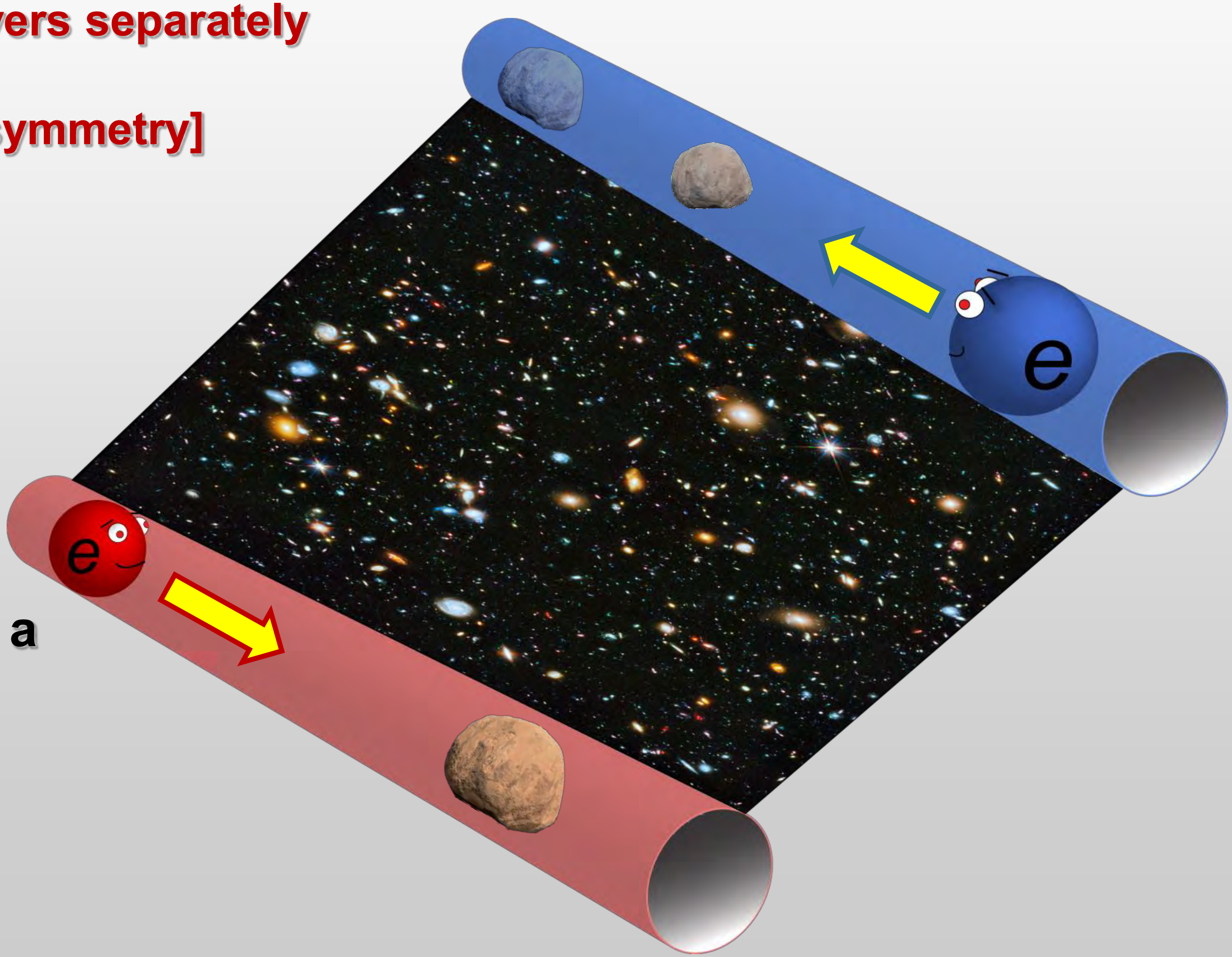
Total charge (left + right) is conserved, but no guarantee it will flow

(Anderson localization)

Chiral edge state: Half of a normal quantum wire

Chiral edge states in the quantum Hall effect

- Left, right moving electrons separated by macroscopic bulk
- Scattering ineffective: left mover cannot be scattered into right
- Left and right movers separately conserved
  [anomalous U(1) symmetry]



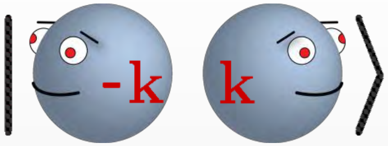
No scattering:
Left, right edges form a perfect quantum wire

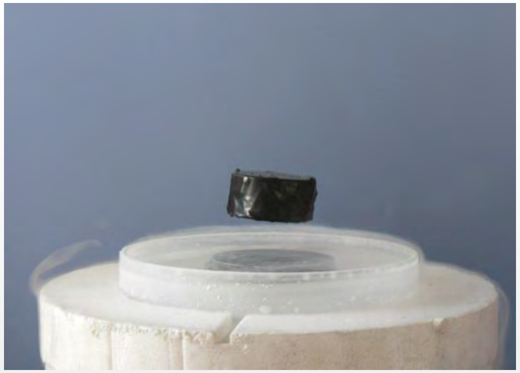
$$\sigma_{xy} = \frac{e^2}{h}$$

Topological Superconductor: Gapped bulk, Majorana fluid boundary

Superconductivity

Collective motion of loosely bound electron pairs at low temperatures





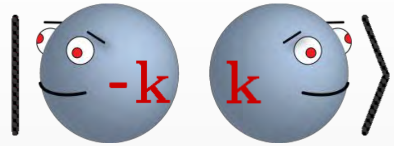
Mai-Linh Doan, Wikipedia

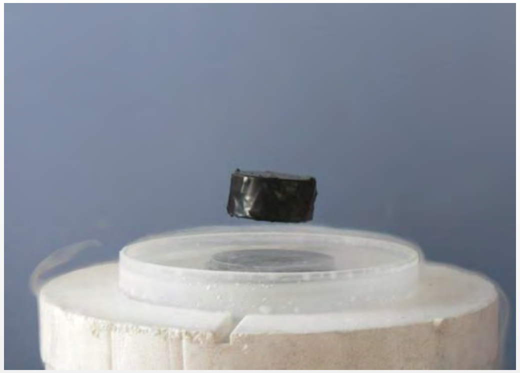
- **Superfluidity: Electrical resistance is zero**
- **No heat or spin transport in the superfluid**
- **Topological superconductor:**
  Theorized to possess a charge neutral surface fluid of unpaired "Majorana" fermions

Topological Superconductor: Gapped bulk, Majorana fluid boundary

Superconductivity

Collective motion of loosely bound electron pairs at low temperatures





Mai-Linh Doan, Wikipedia

- Superfluidity: Electrical resistance is zero
- No heat or spin transport in the superfluid
- **Topological superconductor:**
  Theorized to possess a charge neutral surface fluid of unpaired "Majorana" fermions

Spaceballs the Movie

"New" idea: 3D Bulk topological superconductivity

Integer-valued winding number $\nu \in \mathbb{Z}$

2D Majorana surface fluid, envelopes bulk

Schnyder, Ryu, Furusaki, Ludwig 2008; Kitaev 2009

- Transport properties?
- Stability? (Exposed crystal surface: disorder)

Experimental realizations

- Helium 3B (neutral topological superfluid) — Volovik 1988
- $Cu_xBi_2Se_3$?, Cd_3As_2 ?, LuPdBi ?

Topological superconductor surface state Majorana fluid

For a 3D topological superconductor with bulk winding number ν, what do the Majorana surface states "look like?"

Spaceballs the Movie

Bulk models

- **CI [spin SU(2), spin singlet]**
  Schnyder, Ryu, Ludwig (2009)
  Schnyder, Brydon, Manske, Timm (2010)
- **AIII [spin U(1), e.g. p-wave]**
  Xie, Chou, Foster (2015)
- **DIII ["Rashba" S.O.C.]**
  Fu and Berg ($Cu_xBi_2Se_3$, 2010)

⋮

Topological superconductor surface state Majorana fluid

A lot like graphene!

- **Unpaired surface Majorana fermion quasiparticles**
- **$|\nu| = 2k$ "colors," $k = (1,2,3,\ldots)$ (class CI)**

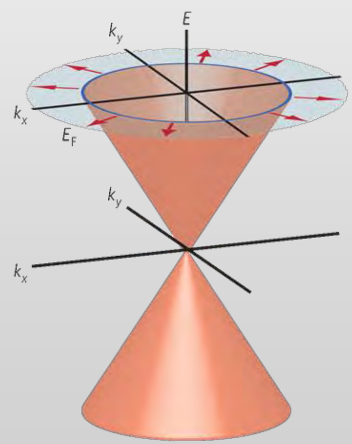
Low energy surface Andreev state Hamiltonian:

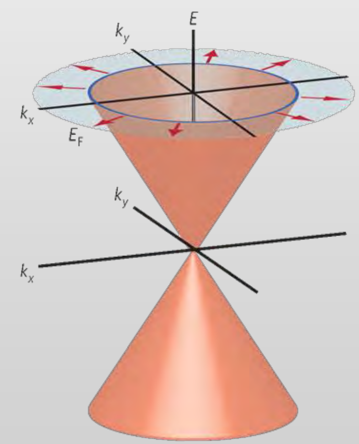
$$H = \int d^2\mathbf{r}\, \Psi^\dagger \left(-i\hat{\sigma}^1 \partial_x - i\hat{\sigma}^2 \partial_y\right) \Psi \equiv \Psi^\dagger \hat{h} \Psi$$

Majorana fermion carries Pseudospin (σ) and Color (κ) indices

$$\Psi_\kappa = \begin{bmatrix} c_{\uparrow,\kappa} \\ c^\dagger_{\downarrow,\bar{\kappa}} \end{bmatrix}$$

$$1 \leq \kappa \leq |\nu|$$





- **"Anomalous" chiral symmetry *(= physical time-reversal)*:** $-\hat{\sigma}^3 \hat{h} \hat{\sigma}^3 = \hat{h}$

Schnyder, Ryu, Furusaki, Ludwig 08; Bernard, LeClair 02

Topological superconductor surface state Majorana fluid

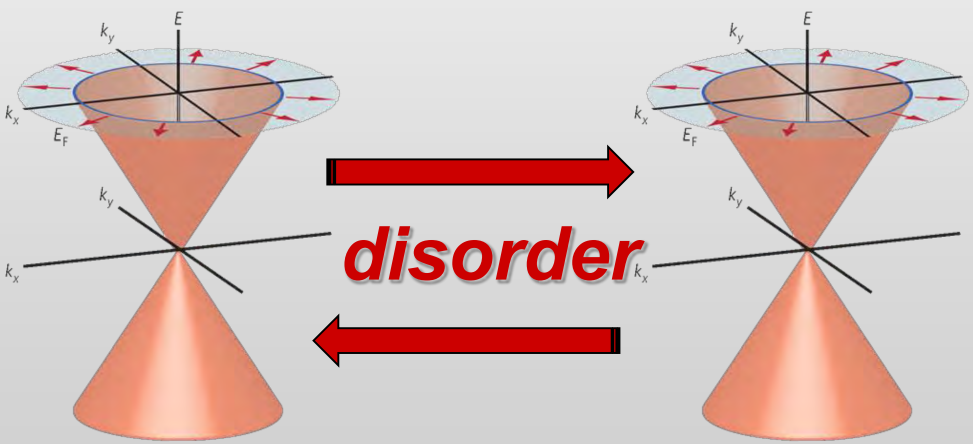
Effects of disorder

- **Junk is unavoidable at the surface!**
- ***Any* non-magnetic (time-reversal preserving) surface perturbation: intercolor vector potential $\hat{t}^i_\kappa \mathbf{A}_i(\mathbf{r})$!**

$$H = \int d^2\mathbf{r}\, \Psi^\dagger \left(-i\hat{\boldsymbol{\sigma}} \cdot \boldsymbol{\nabla} + \mathbf{A}_i \cdot \boldsymbol{\sigma}\, \hat{t}^i_\kappa\right) \Psi = H_0 + \int d^2\mathbf{r}\, \left(J^i_\kappa\, \bar{A}_i + \bar{J}^i_\kappa\, A_i\right)$$

Sources of $\hat{t}^i_\kappa \mathbf{A}_i(\mathbf{r})$:

- **Impurities, vacancies**
- **External electric fields**
- **Edge, corner, dislocation potentials**



"Quenched" 2+1-D QCD: Dirac fermions in a sea of frozen gauge fluctuations

Schnyder, Ryu, Furusaki, Ludwig 08; Bernard, LeClair 02

Majorana fluid transport

Surface Majorana fluid can carry spin or heat

In 2D, wave interference dominates transport; quantum conductance corrections due to

1. **Multiple scattering off of impurities (weak localization)**

Majorana fluid transport

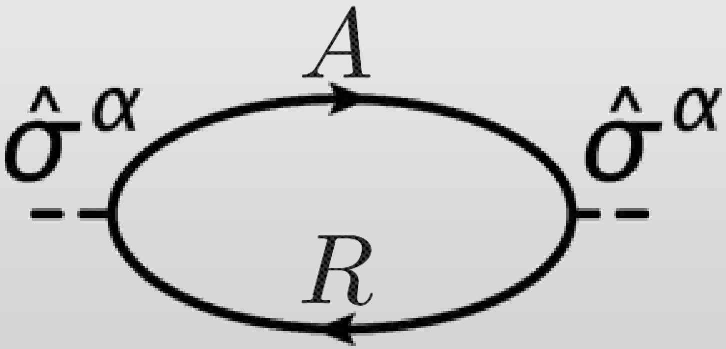
Surface Majorana fluid can carry spin or heat

In 2D, wave interference dominates transport; quantum conductance corrections due to

1. **Multiple scattering off of impurities (weak localization)**

Kubo formula for dc spin conductivity (class CI or AIII):

$$\sigma = \frac{1}{8\pi L^d} \int_{\mathbf{r_1},\mathbf{r_2}} \mathrm{Re}\left\{\mathrm{Tr}\left[\hat{\sigma}^\alpha \hat{G}^{(A)}(0;\mathbf{r_1},\mathbf{r_2})\hat{\sigma}^\alpha \hat{G}^{(R)}(0;\mathbf{r_2},\mathbf{r_1}) - \hat{\sigma}^\alpha \hat{G}^{(R)}(0;\mathbf{r_1},\mathbf{r_2})\hat{\sigma}^\alpha \hat{G}^{(R)}(0;\mathbf{r_2},\mathbf{r_1})\right]\right\}$$



Components:
Retarded, advanced Green's functions

$$\hat{G}^{(R,A)}(\varepsilon;\mathbf{r_1},\mathbf{r_2}) \equiv \langle \mathbf{r_1}| \frac{1}{\varepsilon \pm i\eta - \hat{h}} |\mathbf{r_2}\rangle$$

$$\hat{h} = -i\hat{\boldsymbol{\sigma}} \cdot \boldsymbol{\nabla} + \mathbf{A}_i(\mathbf{r}) \cdot \boldsymbol{\sigma}\, \hat{t}^i_\kappa$$

Majorana fluid transport

Surface Majorana fluid can carry spin or heat

In 2D, wave interference dominates transport; quantum conductance corrections due to

1. **Multiple scattering off of impurities (weak localization)**

Kubo formula for dc spin conductivity (class CI or AIII):

$$\sigma = \frac{1}{8\pi L^d} \int_{\mathbf{r}_1,\mathbf{r}_2} \mathrm{Re}\left\{\mathsf{Tr}\left[\hat{\sigma}^\alpha \hat{G}^{(A)}(0;\mathbf{r}_1,\mathbf{r}_2)\hat{\sigma}^\alpha \hat{G}^{(R)}(0;\mathbf{r}_2,\mathbf{r}_1) - \hat{\sigma}^\alpha \hat{G}^{(R)}(0;\mathbf{r}_1,\mathbf{r}_2)\hat{\sigma}^\alpha \hat{G}^{(R)}(0;\mathbf{r}_2,\mathbf{r}_1)\right]\right\}$$

Anomalous form of time-reversal symmetry

Retarded, advanced interchangable: $-\hat{\sigma}^3 \hat{G}^{(A)}(\varepsilon;\mathbf{r}_1,\mathbf{r}_2)\hat{\sigma}^3 = \hat{G}^{(R)}(-\varepsilon;\mathbf{r}_2,\mathbf{r}_1)$

$$\sigma = -\frac{1}{8\pi L^d} \sum_\alpha \int_{\mathbf{r}_1,\mathbf{r}_2} \mathrm{Re}\left\{\mathsf{Tr}\left[\hat{\sigma}^\alpha \hat{G}^{(R)}(0;\mathbf{r}_1,\mathbf{r}_2)\hat{\sigma}^\alpha \hat{G}^{(R)}(0;\mathbf{r}_2,\mathbf{r}_1)\right]\right\}$$

U(1) Ward identity (conserved z-component of spin):

$$\int_{\mathbf{r}} \hat{G}^{(R)}(\varepsilon;\mathbf{x}_1,\mathbf{r})\hat{\sigma}^\alpha \hat{G}^{(R)}(\varepsilon;\mathbf{r},\mathbf{x}_2) = -i\left(\mathbf{x}_1 - \mathbf{x}_2\right)^\alpha \hat{G}^{(R)}(\varepsilon;\mathbf{x}_1,\mathbf{x}_2)$$

Majorana fluid transport

Surface Majorana fluid can carry spin or heat

In 2D, wave interference dominates transport; quantum conductance corrections due to

1. **Multiple scattering off of impurities (weak localization)**

Kubo formula for dc spin conductivity:

$$\sigma = -\frac{1}{8\pi L^d} \sum_{\alpha} \int_{\mathbf{r_1},\mathbf{r_2}} \text{Re}\left\{ \text{Tr}\left[\hat{\sigma}^{\alpha} \hat{G}^{(R)}(0;\mathbf{r_1},\mathbf{r_2}) \hat{\sigma}^{\alpha} \hat{G}^{(R)}(0;\mathbf{r_2},\mathbf{r_1}) \right] \right\}$$

Anomalous form of time-reversal symmetry

Retarded, advanced interchangable, U(1) Ward identity

$$\Rightarrow \quad \sigma = -\frac{1}{4\pi} \lim_{\mathbf{r}\to\mathbf{r}'} \text{Im}\left\{ \text{Tr}\left[(\mathbf{r}-\mathbf{r}') \cdot \hat{\boldsymbol{\sigma}}\, \hat{G}^{(R)}(0;\mathbf{r},\mathbf{r}') \right] \right\} = \frac{|\nu|}{8\pi^2}$$

2+0-D Chiral anomaly

Universal spin, thermal conductivity (ala W-F), neglecting interactions

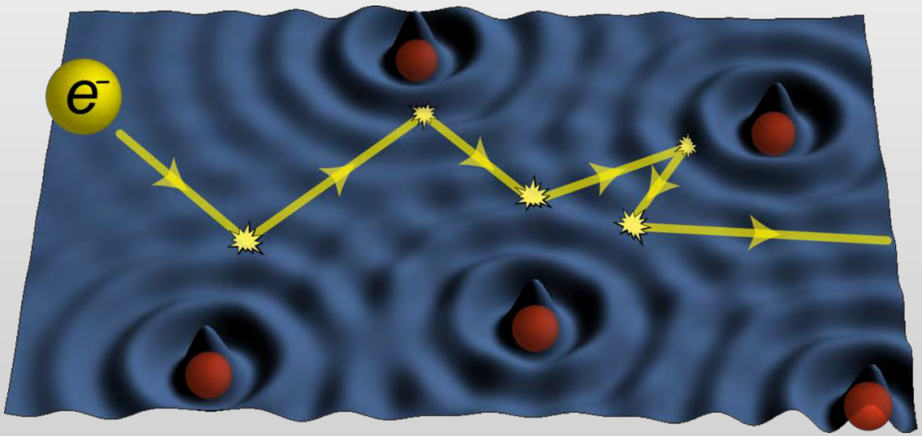
$$\sigma_s = \frac{|\nu|}{\pi h}\left(\frac{\hbar}{2}\right)^2, \qquad \kappa = \frac{|\nu|}{\pi h} \frac{\pi^2 k_B^2 T}{3}$$

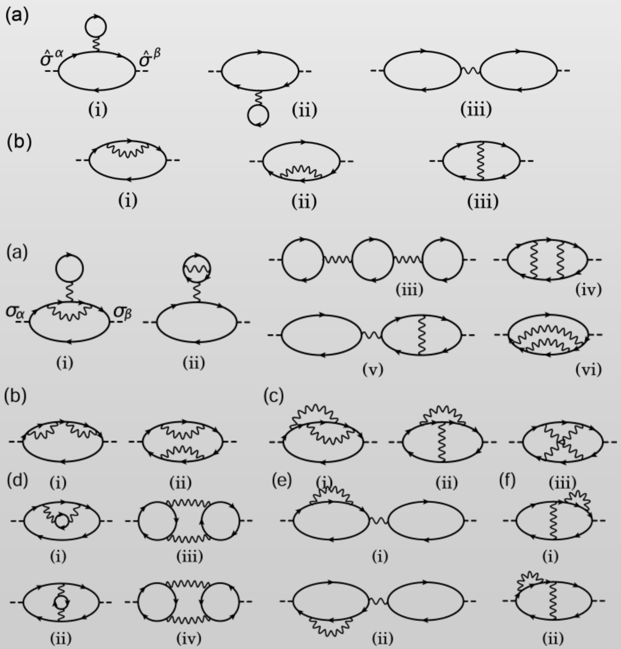
Ludwig, Fisher, Shankar, Grinstein (1994)
Tsvelik (1995)
Ostrovsky, Gornyi, Mirlin (2006)

Surface Majorana fluid can carry spin or heat

In 2D, wave interference dominates transport; quantum conductance corrections due to

1. Multiple scattering off of impurities (weak localization)
2. Scattering off of impurity-induced density Friedel oscilations (Altshuler-Aronov corrections, short-ranged interactions)





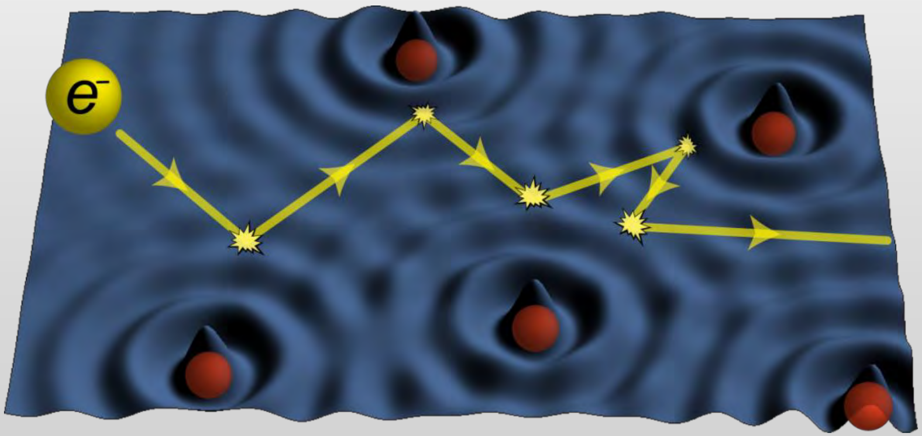
Altshuler and Aronov 1985 (Review)
Aleiner, Altshuler, and Gershenson 1999 (Review)
Zala, Narozhny, and Aleiner 2001

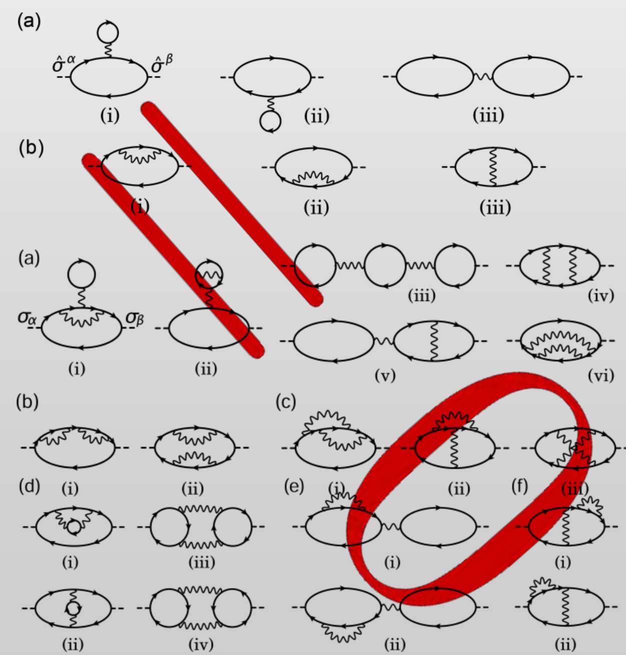
Majorana fluid transport

Surface Majorana fluid can carry spin or heat

In 2D, wave interference dominates transport; quantum conductance corrections due to

1. Multiple scattering off of impurities (weak localization)
2. Scattering off of impurity-induced density Friedel oscillations (Altshuler-Aronov corrections, short-ranged interactions)





1. Anomalous time-reversal symmetry:

$$-\hat{\sigma}^3 \hat{G}^{(A)}(\varepsilon; \mathbf{r_1}, \mathbf{r_2}) \hat{\sigma}^3 = \hat{G}^{(R)}(-\varepsilon; \mathbf{r_2}, \mathbf{r_1})$$

2. Spin U(1) Ward identity:

$$\int_{\mathbf{r}} \hat{G}^{(R)}(\varepsilon; \mathbf{x_1}, \mathbf{r}) \hat{\sigma}^\alpha \hat{G}^{(R)}(\varepsilon; \mathbf{r}, \mathbf{x_2}) = -i\,(\mathbf{x_1} - \mathbf{x_2})^\alpha\, \hat{G}^{(R)}(\varepsilon; \mathbf{x_1}, \mathbf{x_2})$$

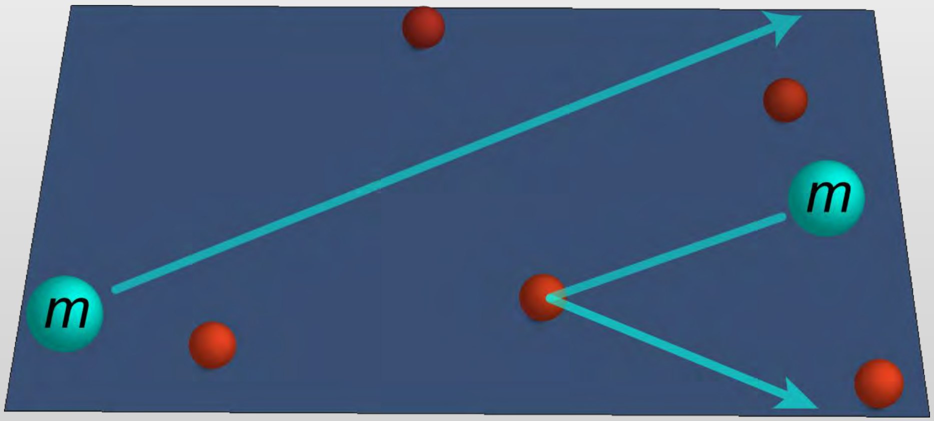
Xie, Chou, and Foster (2015)

Majorana fluid transport

Surface Majorana fluid can carry spin or heat

In 2D, wave interference dominates transport; quantum conductance corrections due to

1. Multiple scattering off of impurities (weak localization)
2. Scattering off of impurity-induced density Friedel oscillations (Altshuler-Aronov corrections)



Anomalous time-reversal symmetry:

No Majorana "density" (mass, spin, color) can ripple (or become non-zero)!

Universal spin, thermal* conductivities!

$$\sigma_s = \frac{|\nu|}{\pi h}\left(\frac{\hbar}{2}\right)^2$$

$$\kappa = \frac{|\nu|}{\pi h}\frac{\pi^2 k_B^2 T}{3}$$

Xie, Chou, and Foster (2015)

Majorana fluid transport redux: Large winding number expansion

- **Clean, non-interacting fermions:**

$$S = \sum_{\omega_n} \int d^2\mathbf{r}\, \bar{\Psi}_{\kappa,a}(\omega_n) \left(-i\omega_n - i\hat{\sigma}^1 \partial_x - i\hat{\sigma}^2 \partial_y\right) \Psi_{\kappa,a}(\omega_n)$$

  - $\kappa \in 1, \ldots, 2k$ (CI) k (AIII, DIII) number of colors (valleys); $k \propto$ winding number
  - $a \in 1, \ldots n$ "replica" index, used for disorder-averaging, $n \to 0$ in the end
  - $\beta \omega_n / 2\pi \in \mathbb{Z} + 1/2$ Matsubara frequency

- **Non-abelian bosonization in 2+1-D: (frequency is an index)**

$$\Psi_{\kappa,a}(\omega_n, \mathbf{r})\, \bar{\Psi}_{\kappa',b}(\omega_n', \mathbf{r}) \sim Q_{\kappa,a\ \kappa',b}(\omega_n, \omega_n'; \mathbf{r})$$

$$S = \frac{1}{8\pi} \int_{\mathbf{r}} \mathrm{Tr}\left[\partial_\mu \hat{Q}^\dagger \partial_\mu \hat{Q}\right] + S_{\mathrm{WZNW}} - \eta \int_{\mathbf{r}} \mathrm{Tr}\left[\hat{\omega}_N \left(\hat{Q} + \hat{Q}^\dagger\right)\right]$$

E.g., chapter 15 of the "big yellow" CFT book Di Francesco, Mathieu, Senechal

Majorana fluid transport redux: Large winding number expansion

- **Clean, non-interacting fermions:**

$$S = \sum_{\omega_n} \int d^2\mathbf{r}\, \bar{\Psi}_{\kappa,a}(\omega_n) \left(-i\omega_n - i\hat{\sigma}^1\partial_x - i\hat{\sigma}^2\partial_y\right) \Psi_{\kappa,a}(\omega_n)$$

$$\Psi_{\kappa,a}(\omega_n)(\mathbf{r})\ \bar{\Psi}_{\kappa',b}(\omega_n')(\mathbf{r}) \sim Q_{\kappa,a\ \kappa',b}(\omega_n, \omega_n'; \mathbf{r})$$

- **Non-abelian bosonization in 2+1-D: (frequency is an index)**

$$S = \frac{1}{8\pi} \int_{\mathbf{r}} \mathrm{Tr}\left[\partial_\mu \hat{Q}^\dagger \partial_\mu \hat{Q}\right] + S_{\mathrm{WZNW}} - \eta \int_{\mathbf{r}} \mathrm{Tr}\left[\hat{\omega}_N \left(\hat{Q} + \hat{Q}^\dagger\right)\right]$$

- **Disorder: relevant perturbation, color sector is massive ("localizes")**

$$S = \frac{k}{8\pi} \int_{\mathbf{r}} \mathrm{Tr}\left[\partial_\mu \hat{Q}^\dagger \partial_\mu \hat{Q}\right] + k S_{\mathrm{WZNW}} - \eta \int_{\mathbf{r}} \mathrm{Tr}\left[\hat{\omega}_N \left(\hat{Q} + \hat{Q}^\dagger\right)\right]$$

- **Technically justified by conformal embedding $SO(4nk)_1$ ➡ $Sp(2n)_k$**

Majorana fluid transport redux: Large winding number expansion

Effective field theory: 2+1-D WZNW Finkelstein Non-linear sigma model

- **Quenched disorder:** Exact treatment via non-abelian bosonization, conformal embedding $SO(4nk)_1$ ➡ $Sp(2n)_k$
- **Interactions:** Controlled in large winding number limit $k = |\nu|/2 \gg 1$

$$S = \frac{1}{8\pi\lambda}\int_{\mathbf{r}} \mathrm{Tr}\left[\partial_\mu \hat{Q}^\dagger \partial_\mu \hat{Q}\right] + k S_{\text{WZNW}} - \eta \int_{\mathbf{r}} \mathrm{Tr}\left[\hat{\omega}_N \left(\hat{Q} + \hat{Q}^\dagger\right)\right]$$

$$- \Gamma_t \sum_a \int_{\tau,\mathbf{r}} \left(\mathrm{Tr}_s\left\{\hat{\vec{S}}\left[\hat{Q}_{aa}(\tau,\tau) + \hat{Q}^\dagger_{aa}(\tau,\tau)\right]\right\}\right)^2$$

$$- \Gamma_c \sum_a \int_{\tau,\mathbf{r}} \left\{\mathrm{Tr}_s\left[\hat{Q}_{aa}(\tau,\tau) - \hat{Q}^\dagger_{aa}(\tau,\tau)\right]\right\}^2$$

c.f. Finkelstein 1983

- **Spin (CI, AIII) or heat resistance (DIII) encoded in** $\lambda = 1/k$

Majorana fluid transport redux: Large winding number expansion

Effective field theory: 2+1-D WZNW Finkelstein Non-linear sigma model

- **Quenched disorder:** Exact treatment via non-abelian bosonization, conformal embedding $SO(4nk)_1$ ➡ $Sp(2n)_k$
- **Interactions:** Controlled in large winding number limit $k = |\nu|/2 \gg 1$
- Spin (CI, AIII) or heat resistance (DIII) encoded in λ

- One-loop RG equations for λ: $(\gamma_{\mathrm{s,c}} \equiv 4\Gamma_{s,c}/\pi\eta)$

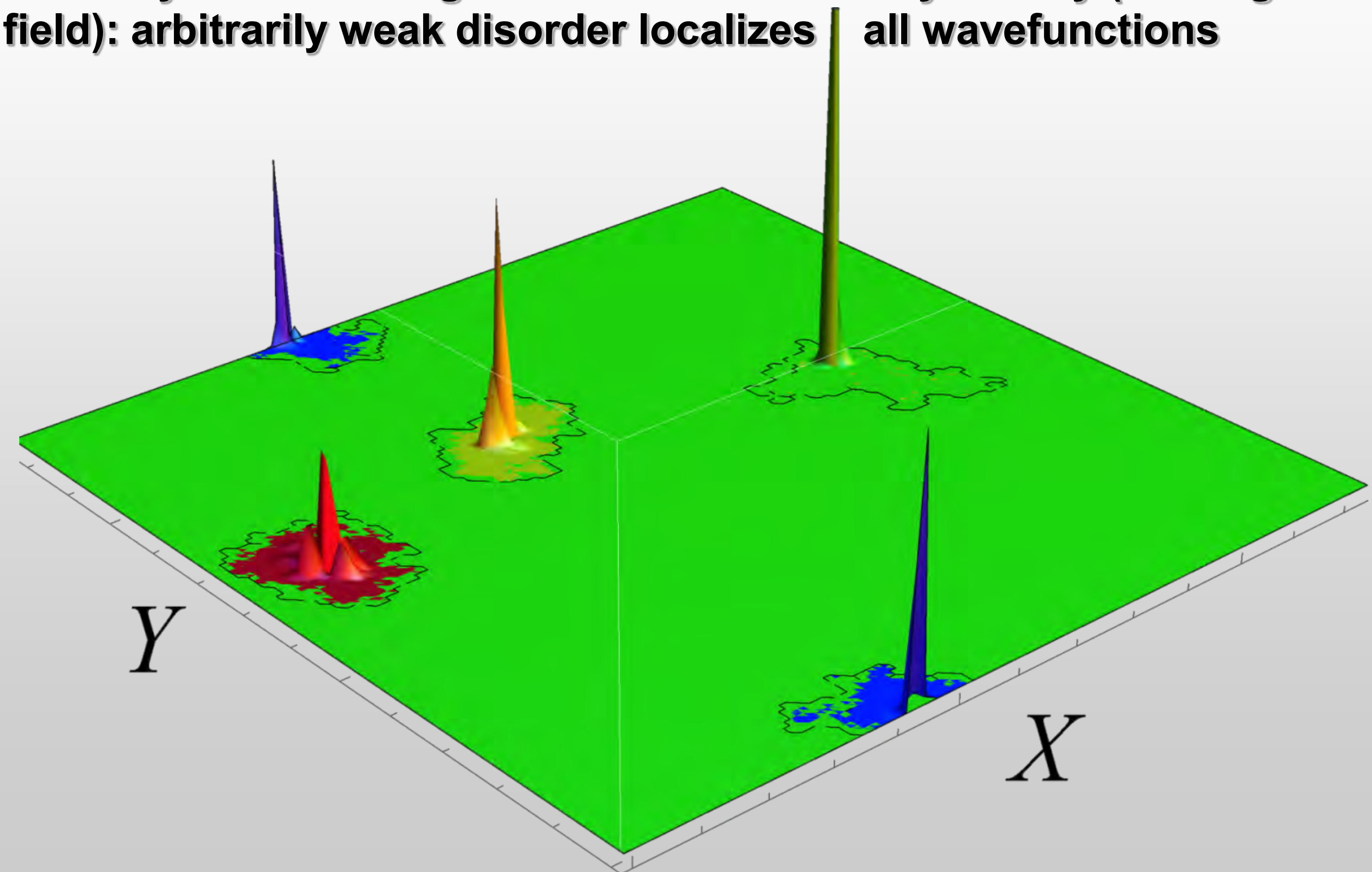
$$
\begin{aligned}
\text{CI:} \quad & d\lambda/dl = \lambda^2\left[1-(k\lambda)^2\right]\left[1+\mathcal{J}(\gamma_{\mathrm{s}},\gamma_{\mathrm{c}})\right], \\
\text{AIII:} \quad & d\lambda/dl = \lambda^2\left[1-(k\lambda)^2\right]\mathcal{I}(\gamma_{\mathrm{s}},\gamma_{\mathrm{c}}), \\
\text{DIII:} \quad & d\lambda/dl = -\lambda^2\left[1-(k\lambda)^2\right]\left[2+\mathcal{K}(\gamma_{\mathrm{c}})\right].
\end{aligned}
$$

➡ All corrections (incl Altshuler-Aronov) vanish for $\lambda = 1/k$!

Majorana fluid wavefunctions

Universal transport: Disorder has no effect?

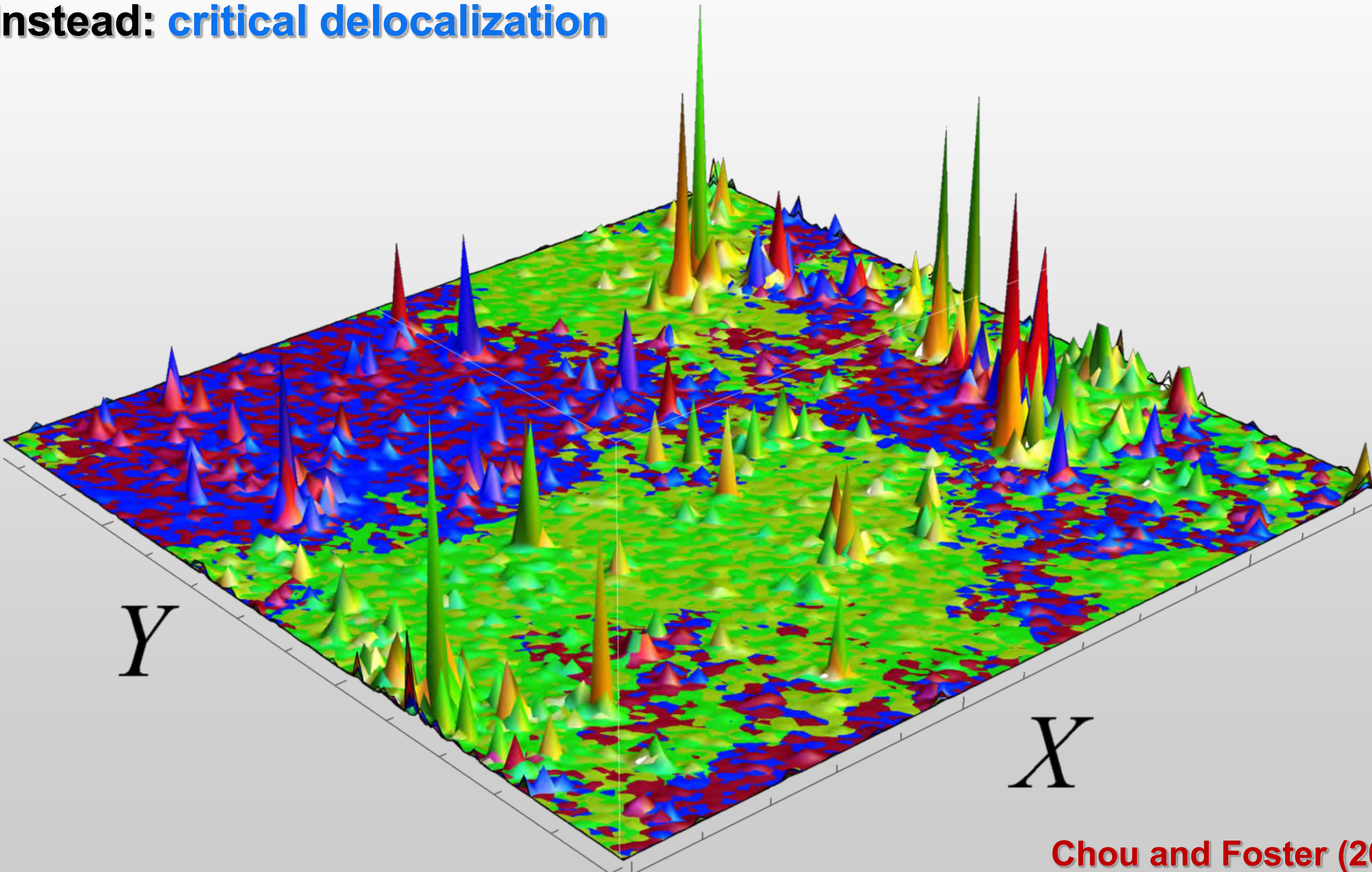
Ordinary 2D electron gas with time reversal symmetry (no magnetic field): arbitrarily weak disorder localizes all wavefunctions



Majorana fluid wavefunctions

Universal transport: Disorder has no effect?

Surface Majorana states cannot be localized (topological protection).
Instead: critical delocalization

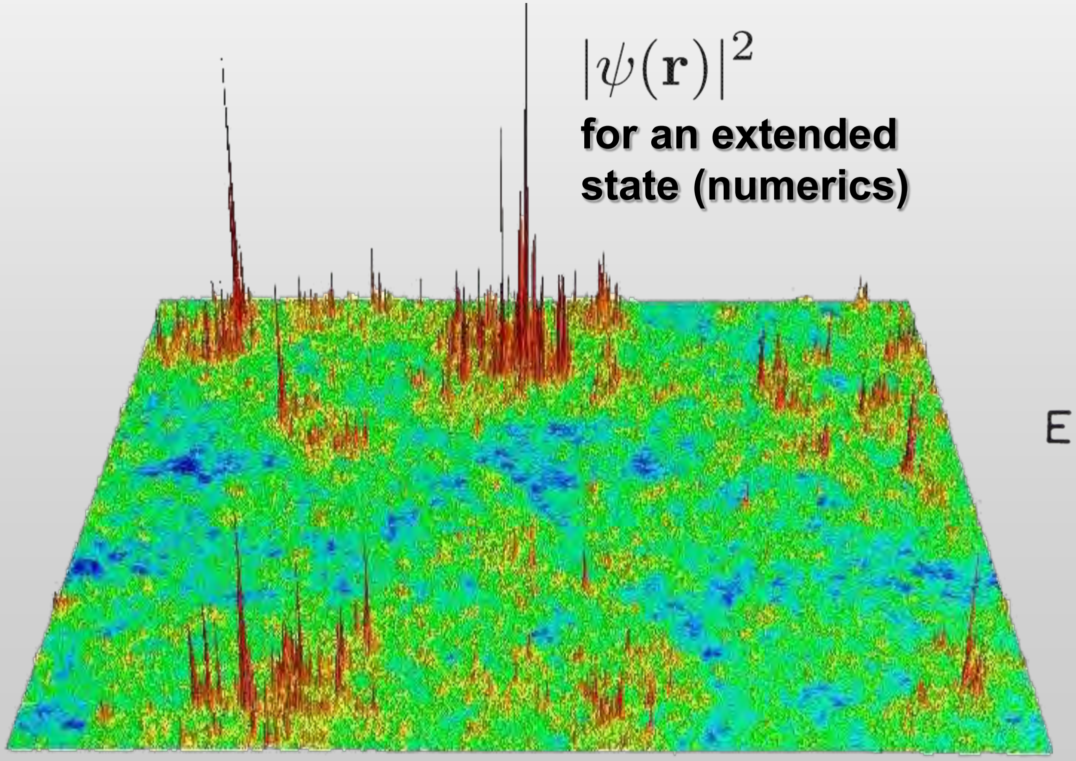


Chou and Foster (2014)

2D critical delocalization: Universal wavefunction multifractality

Bulk wavefunctions at the integer quantum Hall plateau transition

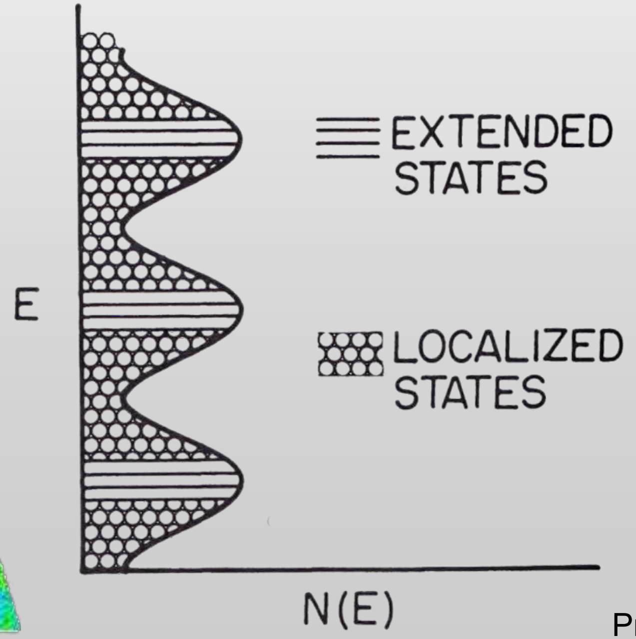
- States away from LL center are Anderson localized ("Topological Anderson Insulator")
- *Delocalized states* at the transition are extended (non-zero regular conductance), but highly rarified and inhomogeneous



$|\psi(\mathbf{r})|^2$

for an extended state (numerics)

F. Evers

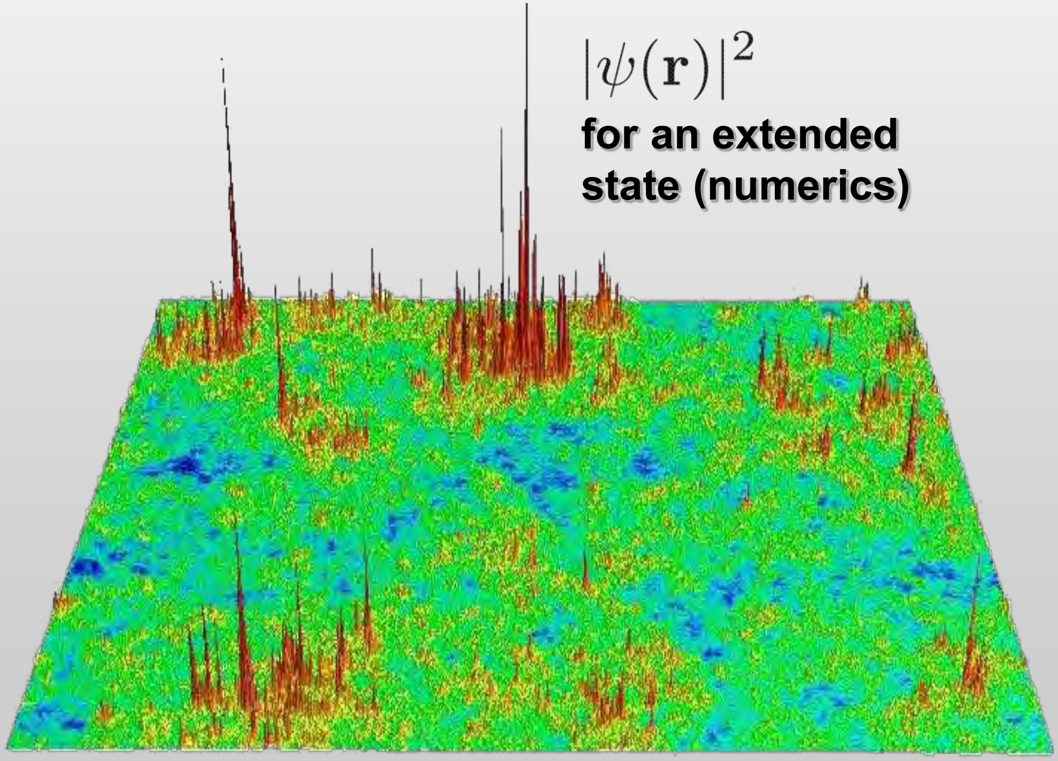


Prange 1987

2D critical delocalization: Universal wavefunction multifractality

Wavefunction multifractality: integer quantum Hall plateau transition

Statistics of rare peaks and valleys can be encoded the "singularity spectrum" $f(\alpha)$



$|\psi(\mathbf{r})|^2$

for an extended state (numerics)

F. Evers

***Interpretation*: over a *fractal level set* of measure $L^{f(\alpha)}$,**

$$(0 \leq f(\alpha) \leq 2)$$

the wavefunction probability scales as

$$|\psi|^2 = c\, L^{-\alpha}$$

2D critical delocalization: Universal wavefunction multifractality

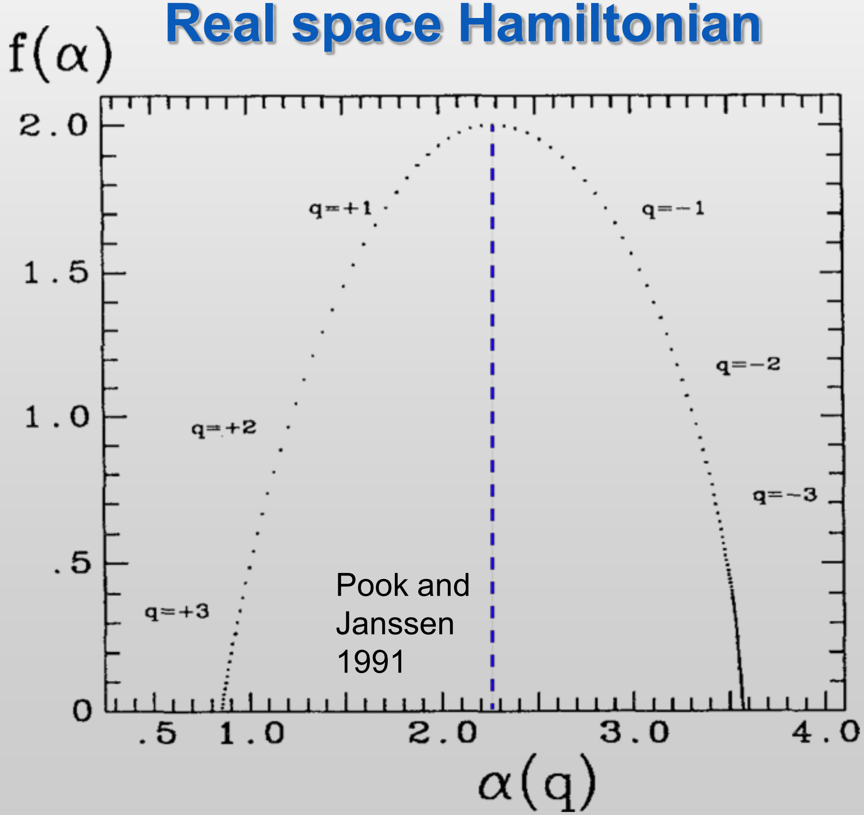
Wavefunction multifractality: integer quantum Hall plateau transition

Statistics of rare peaks and valleys can be encoded the "singularity spectrum" $f(\alpha)$

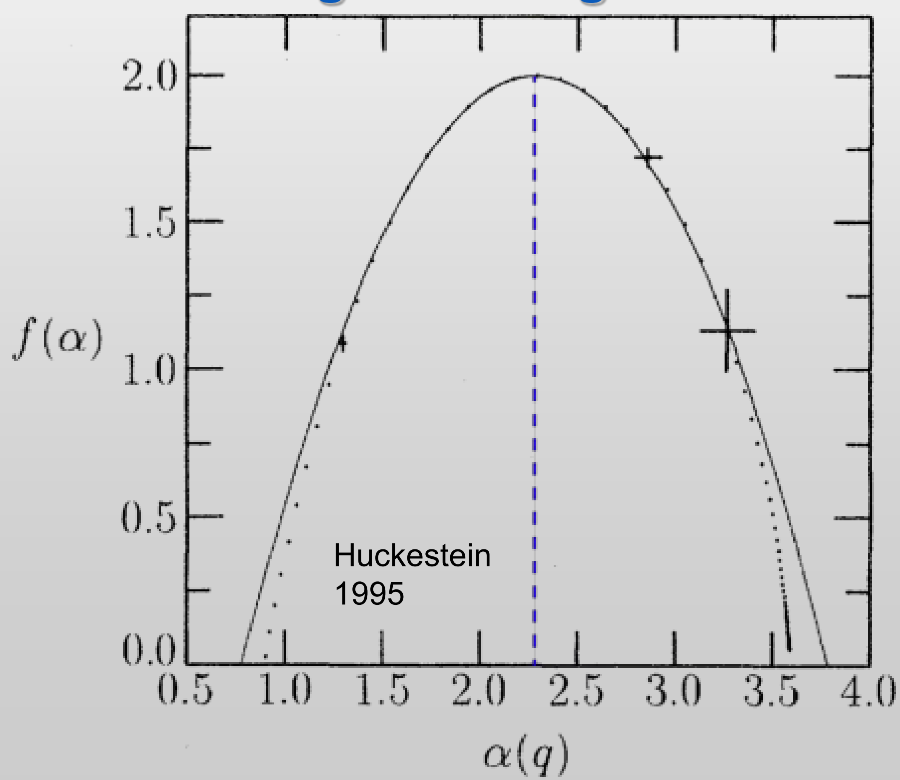
Chamon, Mudry, Wen 1996;
Mirlin and Evers 2000;
Obuse, Subramaniam, Furusaki, Gruzberg, Ludwig 2008;
Evers, Mildenberger, Mirlin 2008

Self-averaging and universal

Real space Hamiltonian



Tight-binding model



Majorana fluid wavefunctions

2D Conformal field theory solution via *conformal embeddings*

Reviewed in (e.g.)
J. Fuchs, *Affine Lie Algebras and Quantum Groups*

Class	Embedding	$\lvert\nu\rvert$	
CI	$SO(4nk)_1 \supset Sp(2n)_k \oplus Sp(2k)_n$	$2k$	$(k \geq 1)$
AIII	$U(nk)_1 \supset U(n)_k \oplus SU(k)_n$	k	$(k \geq 2)$
DIII	$SO(nk)_1 \supset SO(n)_k \oplus SO(k)_n$	k	$(k \geq 3)$

Winding number |ν| = # colors

- "Fractionalization": Color sector "localizes"

Nersesyan, Tsvelik, Wenger 94

Wave functions are multifractal; exact spectra computed via CFT:

Exact Results

$$f(\alpha) = 2 - \frac{(\alpha - 2 - \theta_k)^2}{4\theta_k}$$

Class	$\lvert\nu\rvert$	θ_k	
CI	$2k$	$\frac{1}{2(k+1)}$	**Foster, Yuzbashyan 12**
AIII	$k \geq 1$	$\frac{k-1}{k^2} + \lambda_A$	Mudry, Chamon, Wen 96 Caux, Kogan, Tsvelik 96
DIII	$k \geq 3$	$\frac{1}{k-2}$	**Foster, Xie, Chou 14**

Numerical tests: Critical DOS, multifractal scaling

Minimal case: 2 valley Dirac (Classes CI and AIII)

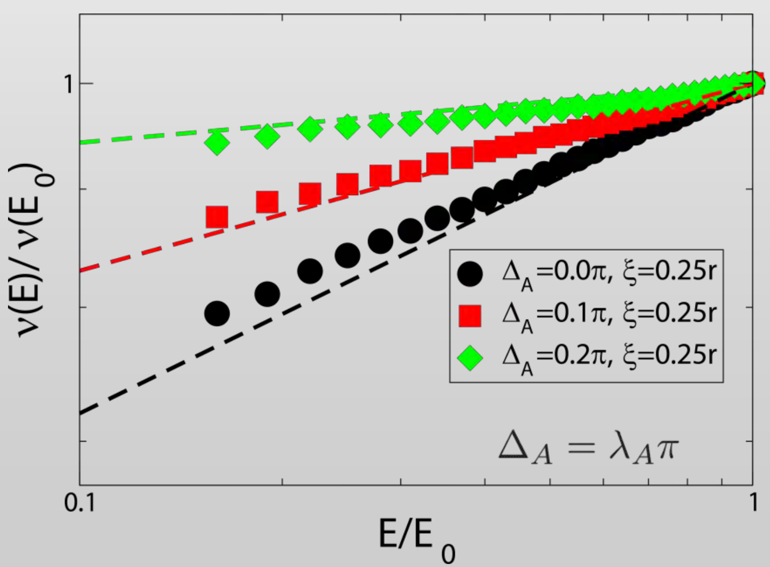
CFT predictions:

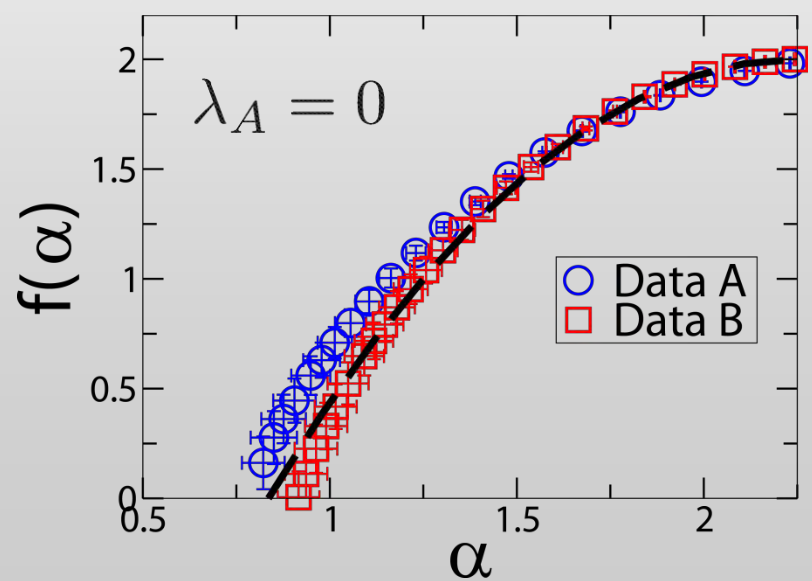
- **Global density of states** $\nu(\varepsilon) \sim |\varepsilon|^{\eta}, \quad \eta = \frac{1-4\lambda_A}{7+4\lambda_A}$
- **Multifractal spectrum** $f(\alpha) = 8\frac{(\alpha_+ - \alpha)(\alpha - \alpha_-)}{(\alpha_+ - \alpha_-)^2}, \quad \alpha_\pm = (\sqrt{2} \pm \sqrt{\theta_2})^2$

Numerical scheme:

Momentum-space disordered Dirac fermion (avoids fermion doubling)

Bardarson, Tworzydlo, Brouwer, Beenakker 07
Nomura, Koshino, Ryu 07





Chou and Foster 2014

Physical picture: Chalker scaling, multifractality, and interactions

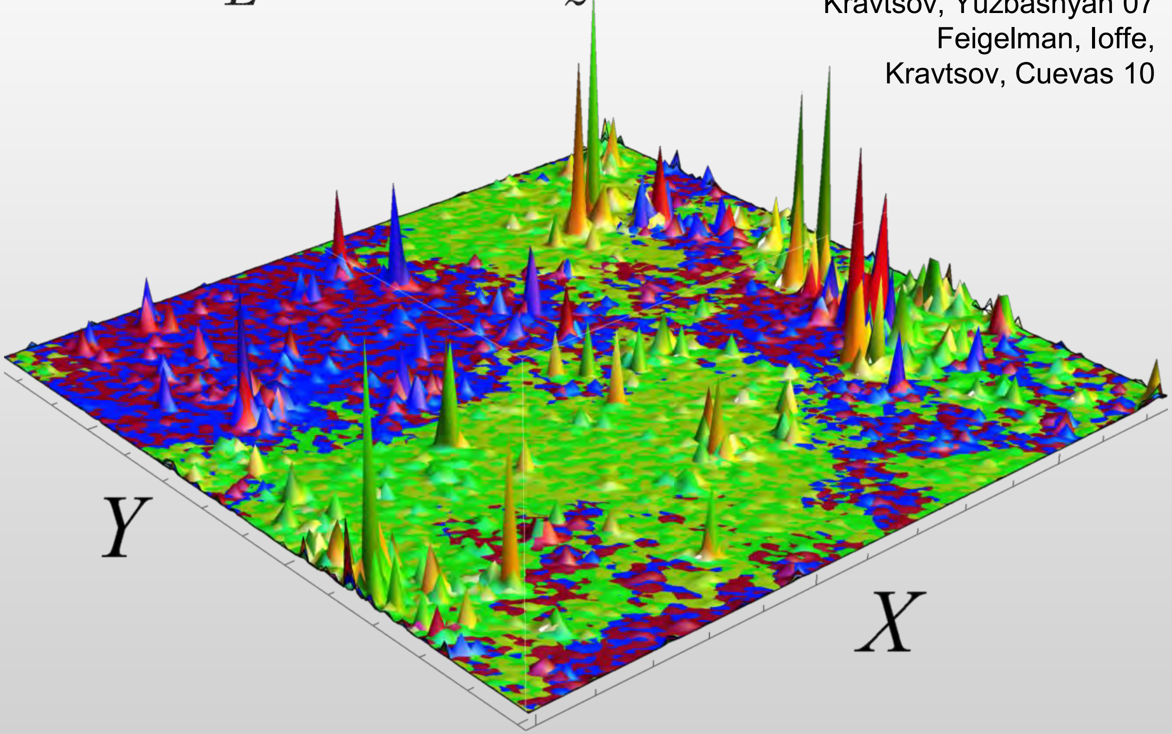
- **Chalker scaling:** **Overlapping peaks and valleys in multifractal eigenstates with nearby energies**

Chalker, Daniell 88
Chalker 90
Cuevas, Kravtsov 07

Feigelman, Ioffe, Kravtsov, Yuzbashyan 07
Feigelman, Ioffe, Kravtsov, Cuevas 10

$$\lim_{L\to\infty} \int d^2\mathbf{r}\, |\psi_0(\mathbf{r})|^2 |\psi_\varepsilon(\mathbf{r})|^2 \sim \frac{\varepsilon^{-\mu}}{L^2}, \quad \mu = \frac{2-\tau(2)}{z}$$

Probability peaks in *different* wavefunctions tend to cluster



Chou and Foster (2014)

Physical picture: Chalker scaling, multifractality, and interactions

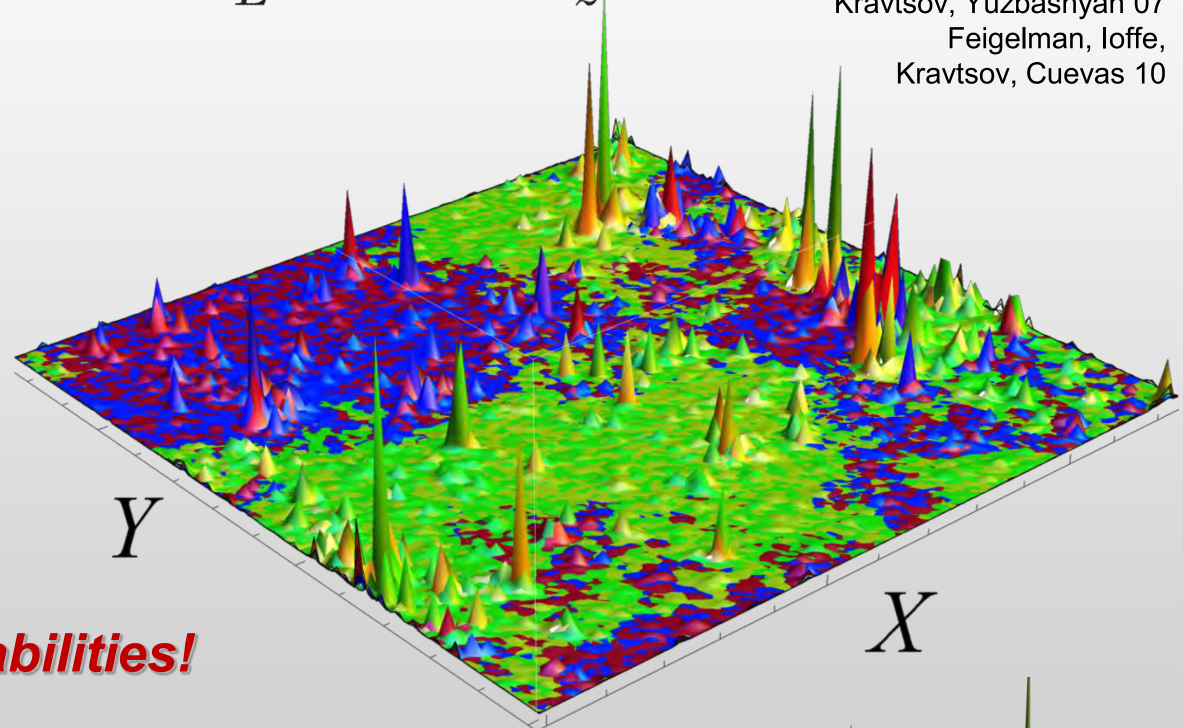
- **Chalker scaling:** **Overlapping peaks and valleys in multifractal eigenstates with nearby energies**

Chalker, Daniell 88
Chalker 90
Cuevas, Kravtsov 07

Feigelman, Ioffe, Kravtsov, Yuzbashyan 07
Feigelman, Ioffe, Kravtsov, Cuevas 10

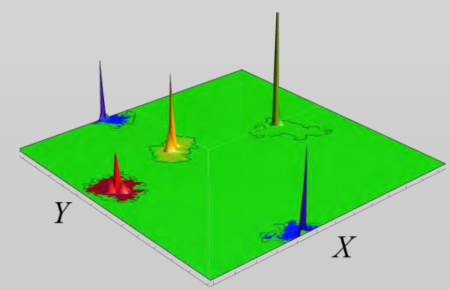
$$\lim_{L\to\infty} \int d^2\mathbf{r}\, |\psi_0(\mathbf{r})|^2 |\psi_\varepsilon(\mathbf{r})|^2 \sim \frac{\varepsilon^{-\mu}}{L^2}, \quad \mu = \frac{2-\tau(2)}{z}$$

Probability peaks in *different* wavefunctions tend to cluster



- **Why scaling theory of localization works**
- **Enhances interaction matrix elements *–instabilities!***
- **Anderson insulator: No overlap for nearby energies**

$$|\psi_\varepsilon(\mathbf{r})|^2 |\psi_{\varepsilon'}(\mathbf{r})|^2 \sim 0, \quad 0 < |\varepsilon - \varepsilon'| \ll \delta_l$$



Topological protection? Disorder and interactions

Extended, multifractal surface states:
No Anderson localization = topological protection! **…BUT**

Add generic, *weak* **interparticle interactions**, consistent with bulk symmetries [time-reversal, spin SU(2) for CI]

$$H_I = U \int d^2\mathbf{r}\, \Psi^\dagger_\alpha \Psi_\beta \Psi^\dagger_\gamma \Psi_\delta$$

Method 1: Scaling interactions with disorder

Clean limit: DoS determines relevance of short-ranged interactions

$$\frac{dU}{dl} = (\Delta_1 - \Delta_2^{(U)})U = -\Delta_1 U + O\left(U^2\right), \quad \Delta_2^{(U)} = 2\Delta_1$$

Clean Dirac: $\Delta_1 = 2 - z = 1$ ➡ **interactions irrelevant!**

Method 1: Scaling interactions with disorder

Clean limit: DoS determines relevance of short-ranged interactions

$$\frac{dU}{dl} = (\Delta_1 - \Delta_2^{(U)})U = -\Delta_1 U + O\left(U^2\right), \quad \Delta_2^{(U)} = 2\Delta_1$$

Clean Dirac: $\Delta_1 = 2 - z = 1$ ➡ **interactions irrelevant!**

Dirty case:

- $\Delta_1 \equiv$ scaling dimension of disorder-averaged LDoS
- $\Delta_2^{(U)} \equiv$ scaling dimension of disorder-averaged interaction

Constraint: $\Delta_2^{(U)} \geq \Delta_2$

- $\Delta_2 \equiv$ multifractal scaling dimension of second LDoS moment

Compute $\{\Delta_1, \Delta_2, \Delta_2^{(U)}\}$ **exactly via CFT**

IQHP:
Lee and Wang 96

Method 1: Scaling interactions with disorder

Clean limit: DoS determines relevance of short-ranged interactions

$$\frac{dU}{dl} = (\Delta_1 - \Delta_2^{(U)})U = -\Delta_1 U + O\left(U^2\right), \quad \Delta_2^{(U)} = 2\Delta_1$$

Clean Dirac: $\Delta_1 = 2 - z = 1$ ➡ **interactions irrelevant!**

Dirty case:

- $\Delta_1 \equiv$ scaling dimension of disorder-averaged LDoS
- $\Delta_2^{(U)} \equiv$ scaling dimension of disorder-averaged interaction

Maximally relevant interaction: $\Delta_2^{(U)} = \Delta_2$

Convexity property for a multifractal extended surface state:

$\Delta_2 < 2\Delta_1$ **(independent dimensions!)** Duplantier and Ludwig 1991

∴ Wavefunction multifractality can amplify short-ranged interactions!

Method 2: Many valleys, WZNW-FNLsM

$$S = \frac{k}{8\pi} \int_{\mathbf{r}} \mathrm{Tr}\left[\partial_\mu \hat{Q}^\dagger \partial_\mu \hat{Q}\right] + k S_{\mathrm{WZNW}} - \eta \int_{\mathbf{r}} \mathrm{Tr}\left[\hat{\omega}_N \left(\hat{Q} + \hat{Q}^\dagger\right)\right]$$

$$- \Gamma_t \sum_a \int_{\tau,\mathbf{r}} \left(\mathrm{Tr}_s \left\{\hat{\vec{S}} \left[\hat{Q}_{aa}(\tau,\tau) + \hat{Q}^\dagger_{aa}(\tau,\tau)\right]\right\}\right)^2$$

$$- \Gamma_c \sum_a \int_{\tau,\mathbf{r}} \left\{\mathrm{Tr}_s \left[\hat{Q}_{aa}(\tau,\tau) - \hat{Q}^\dagger_{aa}(\tau,\tau)\right]\right\}^2$$

Class CI (Spin SU(2) symmetry): Disorder and interactions

Hamiltonian

$$H_{\text{CI}}^{(I)} = \int d^2\mathbf{r} \left[\begin{array}{l} U\left(m_a m_a - 4\vec{S}_a \cdot \vec{S}_a\right) + V J_{Sa}^{\gamma} \bar{J}_{Sa}^{\gamma} \\ + W\left(3 m_a m_a + 4\vec{S}_a \cdot \vec{S}_a - \frac{1}{k} J_{Sa}^{\gamma} \bar{J}_{Sa}^{\gamma}\right) \end{array} \right]$$

Interaction channels:

- $m\,m$ — Cooper pairing of surface quasiparticles (time-reversal invariance)
- $\vec{S} \cdot \vec{S}$ — Spin exchange (spin is conserved = hydrodynamic mode)
- $J_S^{\gamma} \bar{J}_S^{\gamma}$ — Spin current-current

Order parameters break time-reversal:

- **Spin polarization** $\vec{S} = c^{\dagger} \frac{\vec{\sigma}}{2} c$
- ***Imaginary* s-wave pairing mass** $m \sim -i c_{\uparrow}^{\dagger} c_{\downarrow}^{\dagger} + i c_{\downarrow} c_{\uparrow}; \quad \tilde{\sigma}_s^{xy} = k \operatorname{sgn}(m)$

Class C
Spin QHE

Class CI (Spin SU(2) symmetry): Disorder and interactions

Hamiltonian

$$H_{\mathrm{CI}}^{(I)} = \int d^2\mathbf{r} \left[\begin{array}{l} U\left(m_a m_a - 4\vec{S}_a \cdot \vec{S}_a\right) + V J_{Sa}^{\gamma} \bar{J}_{Sa}^{\gamma} \\ + W\left(3 m_a m_a + 4\vec{S}_a \cdot \vec{S}_a - \frac{1}{k} J_{Sa}^{\gamma} \bar{J}_{Sa}^{\gamma}\right) \end{array} \right]$$

CFT:

$$\frac{dU}{dl} = \frac{1}{2(k+1)} U + \ldots$$ **Relevant! (Multifractal enhancement)**

$$\frac{dV}{dl} = -\frac{(4k+3)}{2(k+1)} V + \ldots$$ **Irrelevant**

$$\frac{dW}{dl} = -\frac{3}{2(k+1)} W + \ldots$$ **Irrelevant**

Order parameters break time-reversal:

- **Spin polarization** $\vec{S} = c^{\dagger} \frac{\vec{\sigma}}{2} c$
- ***Imaginary* s-wave pairing mass** $m \sim -i c_{\uparrow}^{\dagger} c_{\downarrow}^{\dagger} + i c_{\downarrow} c_{\uparrow};\quad \tilde{\sigma}_s^{xy} = k\,\mathrm{sgn}(m)$

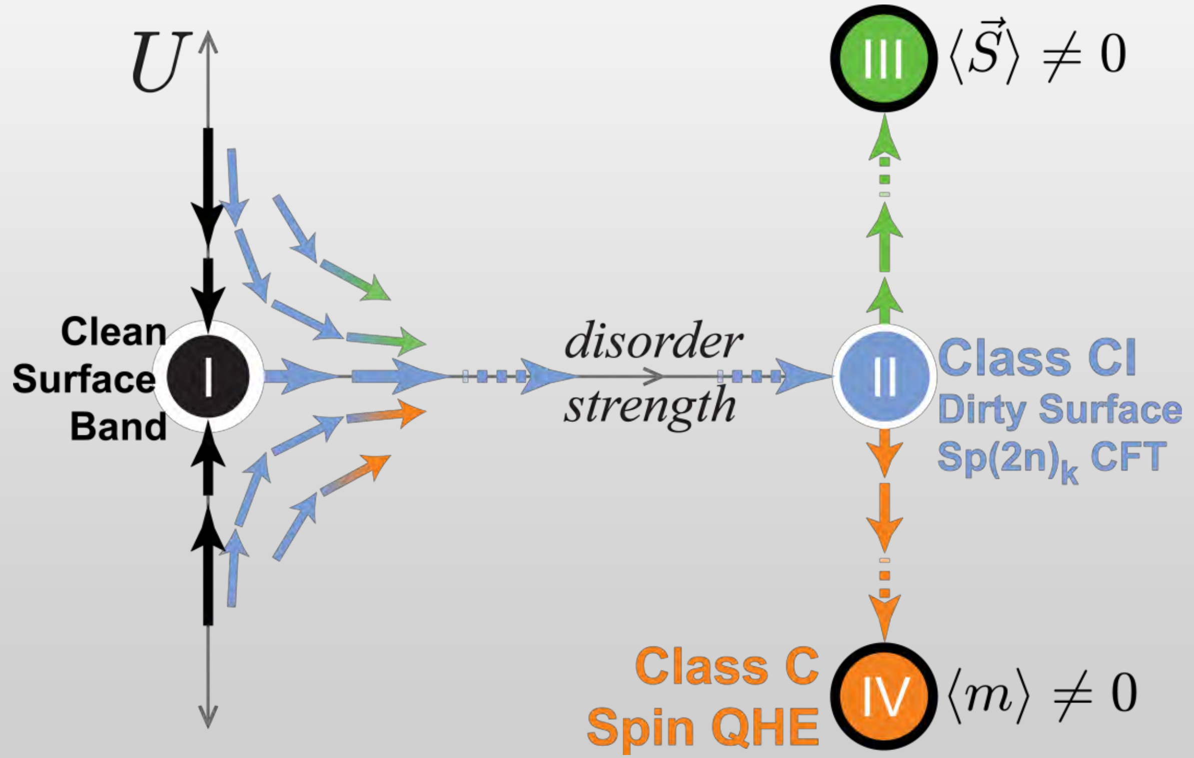
Class C
Spin QHE

Weak disorder and interactions can sabotage topological protection!

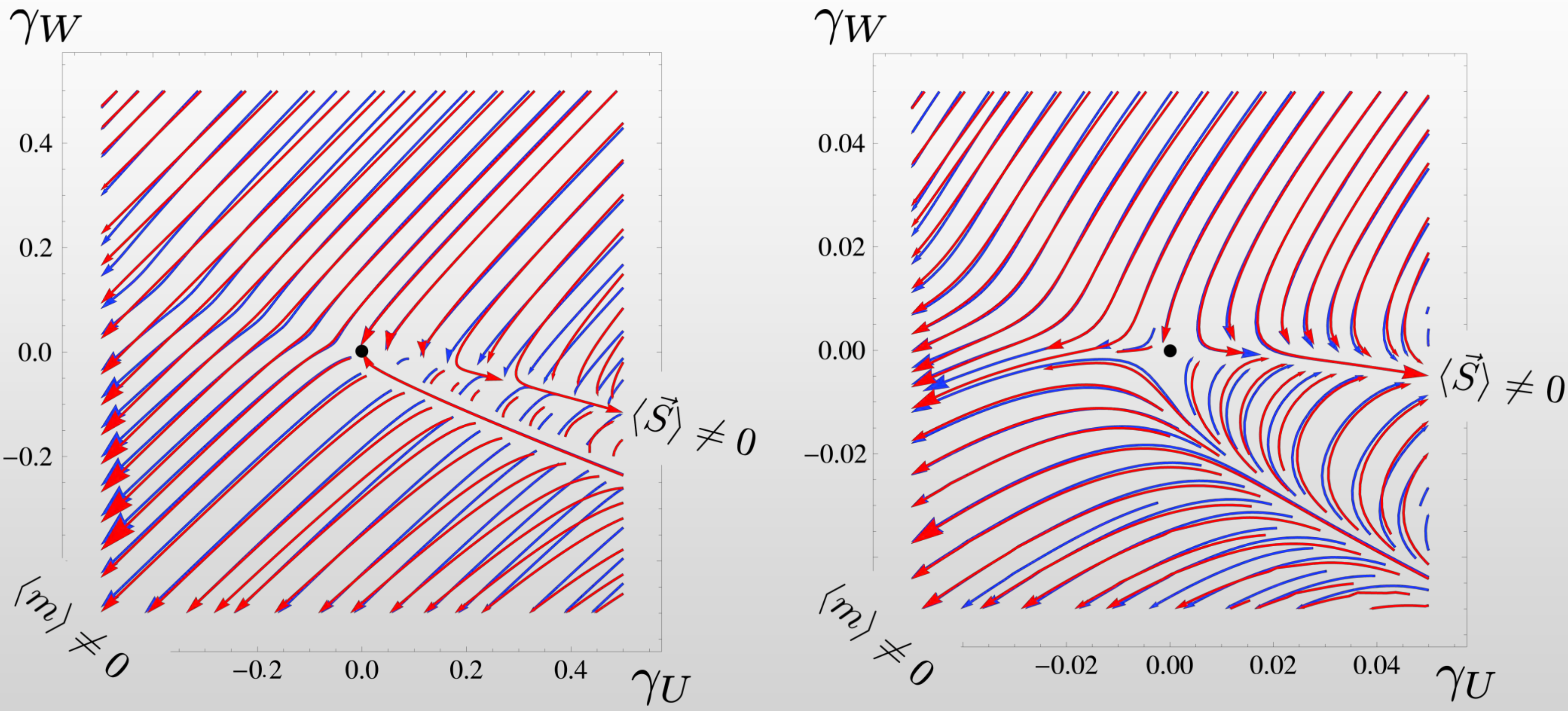
- **Result: Not always protected. Even weak disorder and weak interactions can destroy some surface states**

Class CI Topological superconductors: Majorana surface fluid always unstable for any disorder, interactions, winding number

Foster, Yuzbashyan (2012)



Class CI: WZNW-FNLsM (many valleys)



➢ **Class CI: Interaction-mediated instability**

Class AIII (Spin U(1) symmetry): Disorder and interactions

Hamiltonian

$$H_{\text{AIII}}^{(I)} = \int d^2\mathbf{r} \left[\frac{U}{2}\left(m_a m_a - 4 S_a^z S_a^z - \frac{4}{k} J_a \bar{J}_a \right) + V J_a \bar{J}_a + \frac{W}{2}\left(m_a m_a + 4 S_a^z S_a^z \right) \right]$$

Interaction channels:

$m\,m$ Cooper pairing of surface quasiparticles (time-reversal invariance)

$S^z S^z$ z-spin exchange (z-spin is conserved = hydrodynamic mode)

$J\bar{J}$ z-spin current-current

Order parameters break time-reversal:

- **Spin polarization** $\vec{S} = c^\dagger \frac{\vec{\sigma}}{2} c$
- ***Imaginary* s-wave pairing mass** $m \sim -i c_\uparrow^\dagger c_\downarrow^\dagger + i c_\downarrow c_\uparrow; \quad \tilde{\sigma}_s^{xy} = k\,\mathrm{sgn}(m)$

Class C
Spin QHE

Class AIII: Disorder and interactions

Hamiltonian

$$H^{(I)}_{\text{AIII}} = \int d^2\mathbf{r} \left[\frac{U}{2}\left(m_a m_a - 4S^z_a S^z_a - \frac{4}{k} J_a \bar{J}_a \right) + V J_a \bar{J}_a + \frac{W}{2}\left(m_a m_a + 4 S^z_a S^z_a \right) \right]$$

CFT:

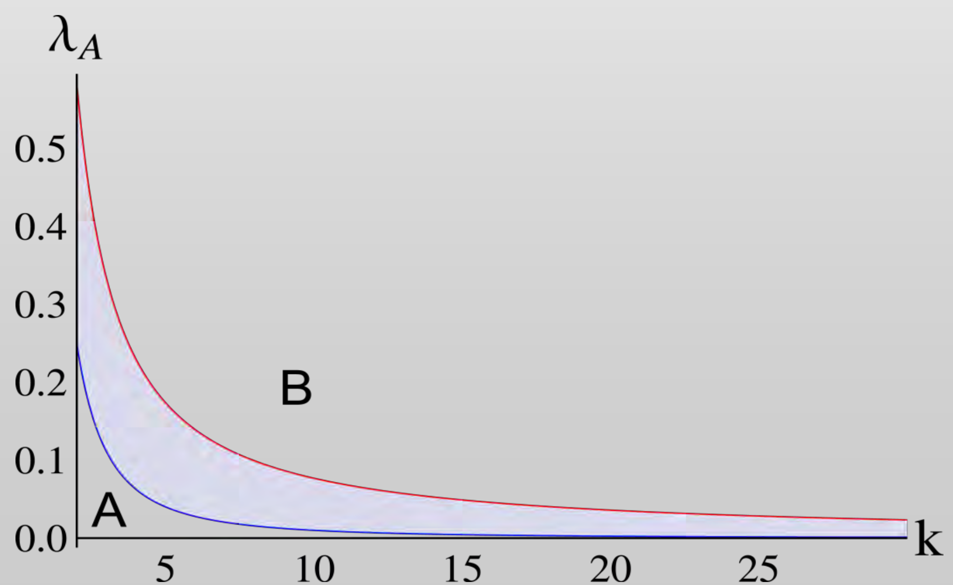
$$\frac{dU}{dl} = \left(\frac{1}{k^2} - \lambda_A \right) U + \ldots$$

$$\frac{dV}{dl} = \left(\frac{1-2k^2}{k^2} - \lambda_A \right) V + \ldots$$

$$\frac{dW}{dl} = \left(-\frac{3+2k}{k^2} + 3\lambda_A \right) W + \ldots$$

Window of stability:

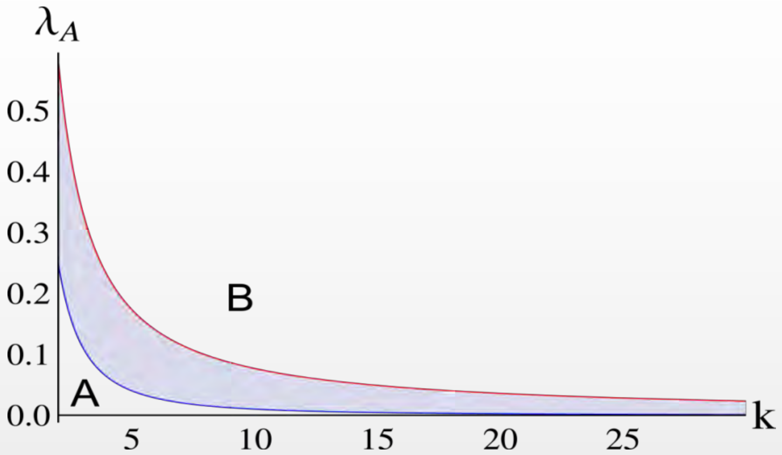
$$\frac{1}{k^2} < \lambda_A < \frac{3+2k}{3k^2}$$



Class AIII: WZNW-FNLsM (many valleys)

CFT: Window of Stability

$$\frac{1}{k^2} < \lambda_A < \frac{3+2k}{3k^2}$$



Simplified interaction plane flow ($\lambda = 1/k$):

- **Retain only BCS non-linearity (Anderson's theorem)**
- **Qualitatively the same as full WZNW-FNLsM results for $\lambda > 1/k^2$**
- **In both cases: New interaction-stabilized fixed point**

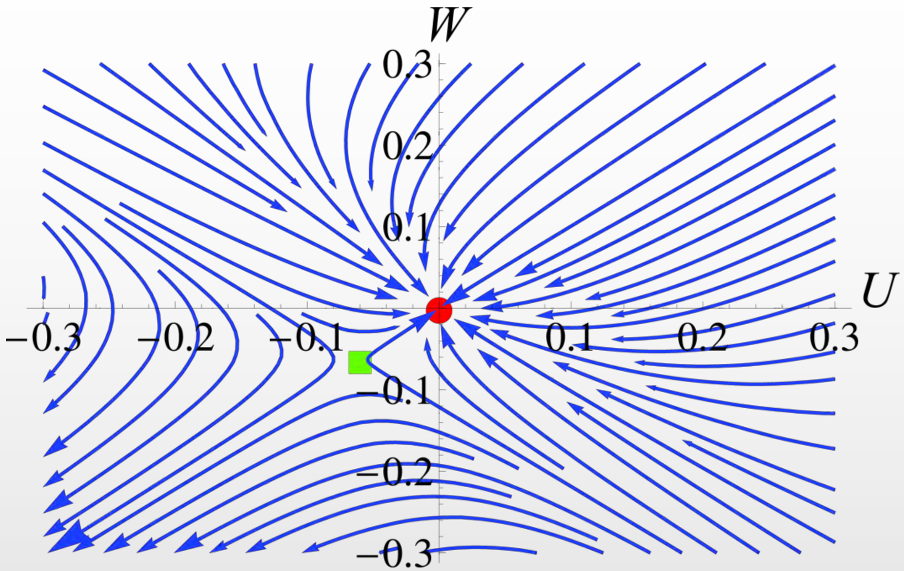
$$\frac{dU}{dl} = \left(\frac{1}{k^2} - \lambda_A\right) U - \frac{1}{2}(U+W)^2$$

$$\frac{dW}{dl} = \left(-\frac{3+2k}{k^2} + 3\lambda_A\right) W - \frac{1}{2}(U+W)^2$$

Foster, Xie, Chou 2014

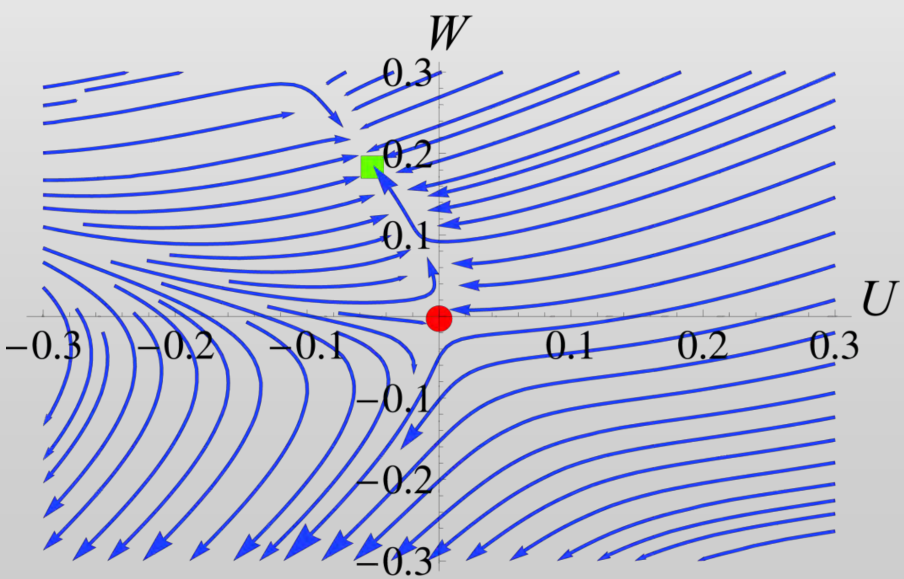
Class AIII: Disorder and interactions

Foster, Xie, Chou
PRB 2014



- **CFT: For weak disorder, interactions irrelevant. Stable surface.**

$$1/k^2 < \lambda_A < (3+2k)/3k^2$$



- **CFT: Stronger disorder, interactions relevant.**

$$\lambda_A > (3+2k)/3k^2$$

- **WZNW-FNLsM: critical interacting fixed point. Stable surface is possible.**

Weakly-coupled (perturbatively accessible) for finite window of disorder

Class DIII (No spin symmetry): Disorder and interactions

Hamiltonian

$$H^{(I)}_{\mathrm{DIII}} = U \int d^2\mathbf{r}\, m_a m_a$$

Interaction channel:

$m\,m$ Cooper pairing of surface quasiparticles (time-reversal invariance)

CFT: $$\frac{dU}{dl} = -\left(\frac{1}{k-2}\right) U + \ldots$$

$$|\nu| = k \geq 3$$

Always Irrelevant!

No multifractal enhancement

Knowing behavior of average density of states is not enough!

Class CI: $\nu(\varepsilon) \sim |\varepsilon|^{\eta}, \quad \eta = \frac{1}{4k+3}$ $\qquad \frac{dU}{dl} = \frac{1}{2(k+1)} U$

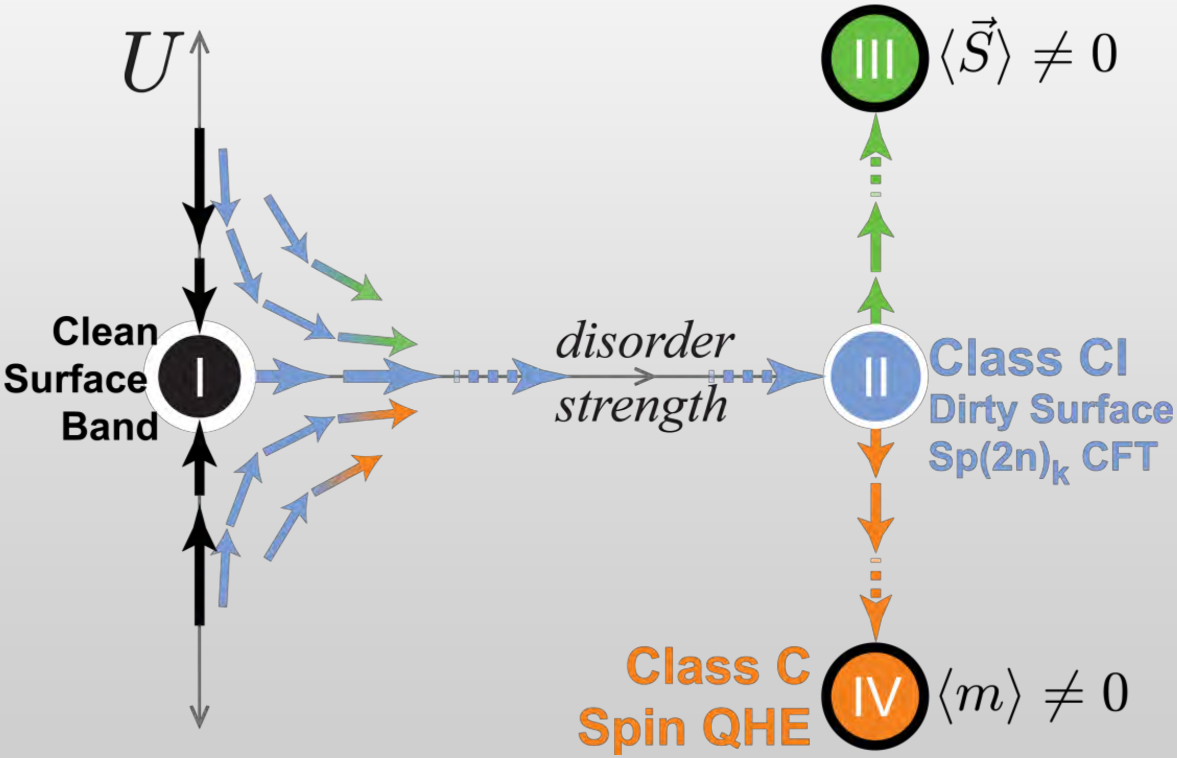
Class DIII: $\nu(\varepsilon) \sim |\varepsilon|^{\eta}, \quad \eta = -\frac{1}{2k-3}$ $\qquad \frac{dU}{dl} = -\left(\frac{1}{k-2}\right) U$

Weak disorder and interactions can sabotage topological protection!

- **Result: Not always protected. Even weak disorder and weak interactions can destroy some surface states**

Class CI Topological superconductors: Majorana surface fluid always unstable for any disorder, interactions, winding number

Foster, Yuzbashyan (2012)

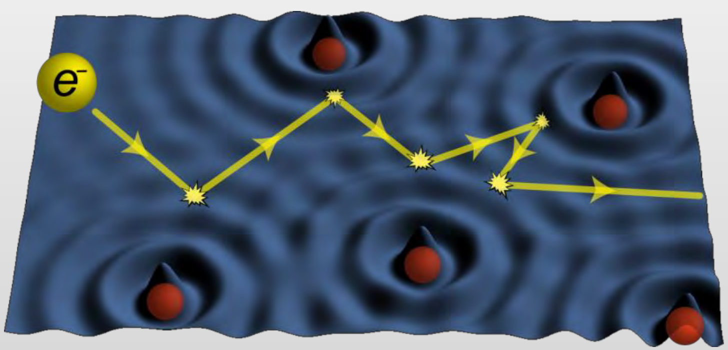


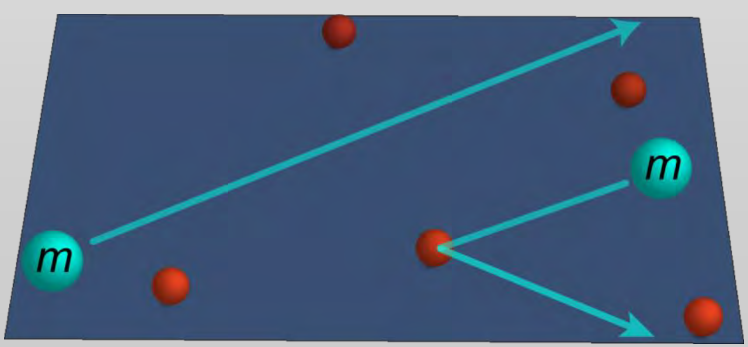
Classes AIII, DIII: Stable surface states

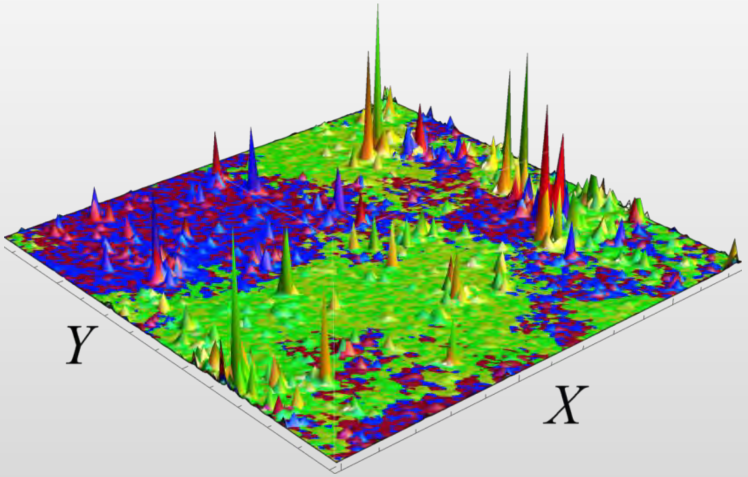
Foster, Xie, Chou (2014)

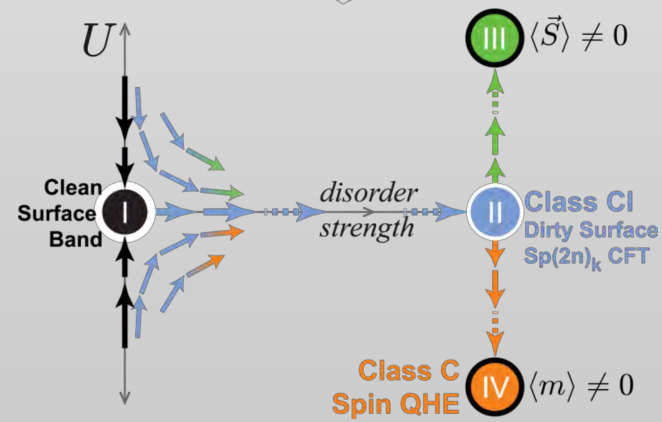
2D Majorana liquid theory

- **Surface states of a bulk topological superconductor**
- **Universal transport coefficients encode bulk winding number**
- **Combined effects of disorder and interactions can lead to instabilities**









Summary

2D Majorana liquid theory

- Surface states of a bulk topological superconductor
- Universal transport coefficients encode bulk winding number
- Combined effects of disorder and interactions can lead to instabilities

3D Topological superconductivity: close analog of the integer quantum Hall effect

- Is there a fractional analog? (Bulk with topological order; gapless surface fluid with fractionalized transport coefficients)

- What about gapless (nodal) "topological" superconductor surface states?

Sato 2006
Beri 2010
Schnyder and Ryu 2011
Matsuura, Chang, Schnyder, Ryu 2013
Zhao and Wang 2013

- **Materials?**

Topological Insulators and Superconductors: The 10-Fold Way

Schnyder, Ryu, Furusaki, Ludwig 08, 10; Kitaev 09:

- **5 symmetry classes of topological materials**
- **These are a *subset* of the 10 random matrix classes (also used in Anderson localization)**

Zirnbauer 96; Altland and Zirnbauer 97

Green:
2D, 3D Z_2 Topological Insulator (Bi_2Se_3, etc)

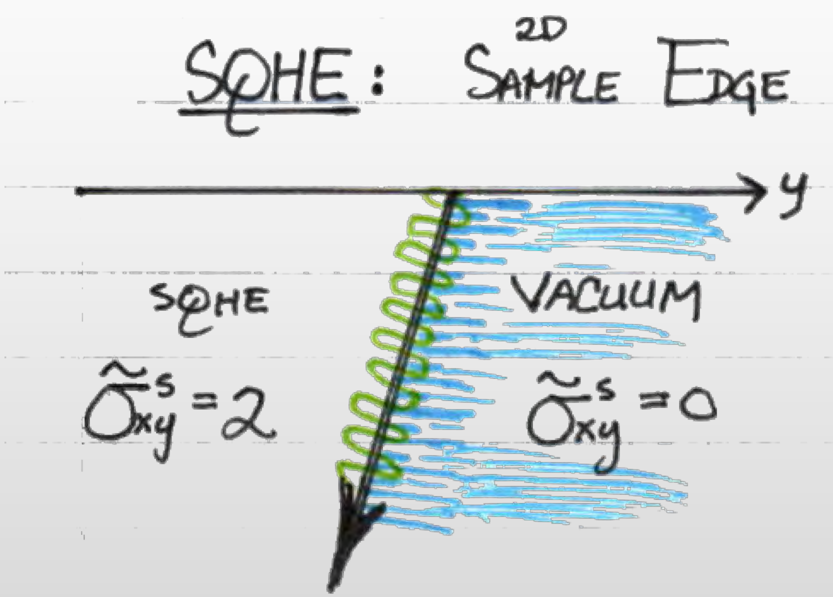
Red:
3D Topological Superconductors

Cartan	d = 0	1	2	3	4	5	6	7	8
Complex case:									
A	Z	0	Z	0	Z	0	Z	0	Z
AIII	0	Z	0	Z	0	Z	0	Z	0
Real case:									
AI	Z	0	0	0	2Z	0	Z_2	Z_2	Z
BDI	Z_2	Z	0	0	0	2Z	0	Z_2	Z_2
D	Z_2	Z_2	Z	0	0	0	2Z	0	Z_2
DIII	0	Z_2	Z_2	Z	0	0	0	2Z	0
AII	2 Z	0	Z_2	Z_2	Z	0	0	0	2Z
CII	0	2 Z	0	Z_2	Z_2	Z	0	0	0
C	0	0	2Z	0	Z_2	Z_2	Z	0	0
CI	0	0	0	2 Z	0	Z_2	Z_2	Z	0

Schnyder, Ryu, Furusaki, Ludwig 2010

Massive aside: Topological "Half-(twice)-integer" Spin QHE

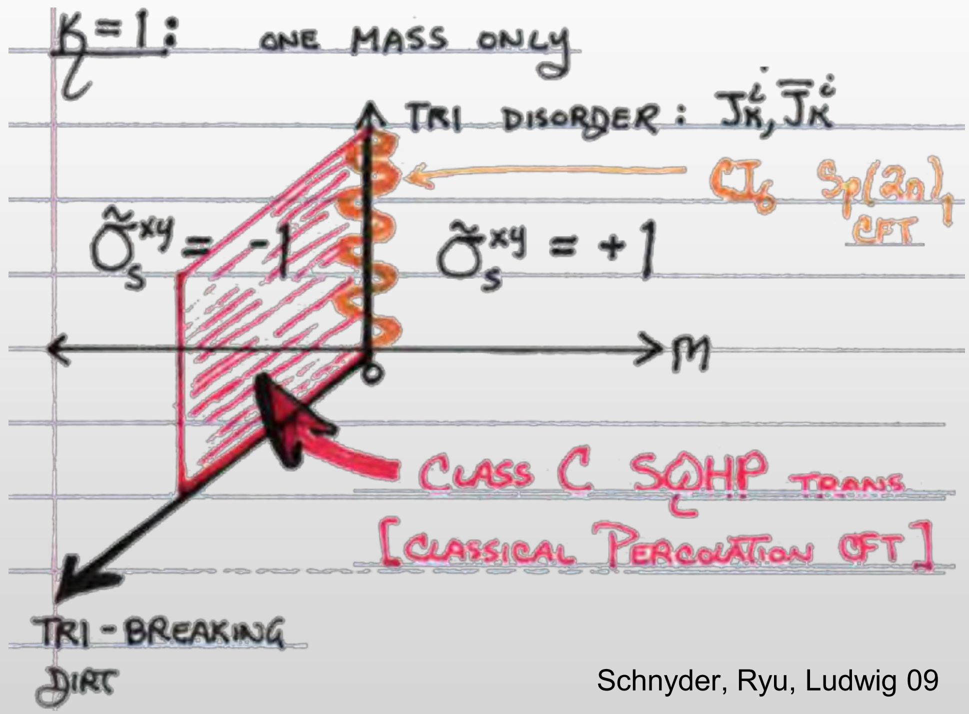
"Regular" 2D Spin QHE



Senthil, Marston, Fisher 99

Gruzberg, Ludwig, Read 99

"Half"-Spin QHE (Top SC surface)



Schnyder, Ryu, Ludwig 09

Ryu, Moore, Ludwig 12

***Imaginary* s-wave pairing mass**

$$m = \Psi^\dagger \hat{\sigma}^3 \Psi \sim -i c_\uparrow^\dagger c_\downarrow^\dagger + i c_\downarrow c_\uparrow$$

Class C
Spin QHE

$$\tilde{\sigma}_s^{xy} = k \,\mathrm{sgn}(m)$$