Wir schaffen Wissen – heute für morgen

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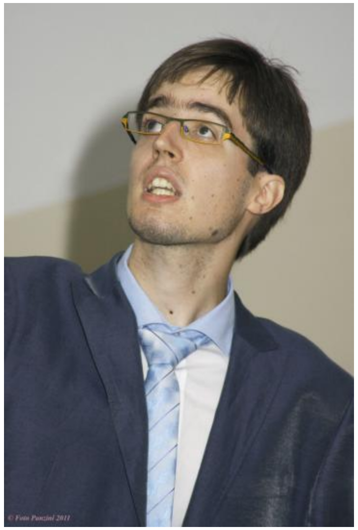
# Absence of mobility edges and the role of rare events in MBL

KITP Santa Barbara, MBL Workshop October 22, 2015

# Collaborators



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# When does MBL exist?

Known: Genuine MBL at all energies exists (almost rigorous proof by J. Imbrie) in a 1d spin chain. This example combines certain ingredients:

- lattice model ↔ continuous space
- quenched disorder ↔ disorder-free localization
- finite local Hilbert space ? ↔ unbounded local Hilbert space
- d = 1 ? ↔ d > 1
- discrete or no symmetry ↔ Continuous symmetry

probably necessary

**Are these ingredients crucial?**

Can there be mobility edges separating loc/delocalized states?

More generally: can one have localization transitions controlled by thermodynamic parameters: T, μ, B?

# Generalization of Mott's argument to MBL?

Mott's argument (single particles)

Generically, loc/deloc states cannot coexist at the same energy!
(except in fine-tuned Hamiltonians) *Mott, 1975 (?)*

Reason: infinitesimal perturbation hybridizes localized and delocalized states (if they are space-filling and non-fractal):

- Matrix elements: $M \sim 1/\sqrt{\mathrm{Vol}}$
- Level spacing of delocalized states: $d \sim 1/\mathrm{Vol}$

$\longrightarrow$ $M >> d$

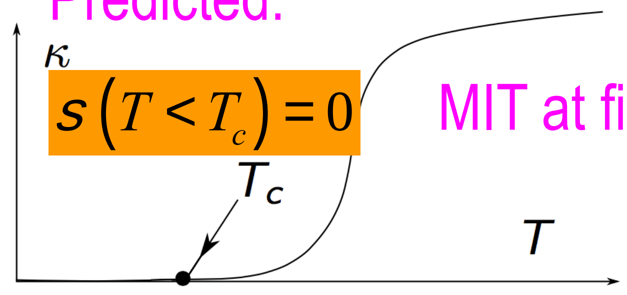
$\longrightarrow$ Use Fermi Golden rule to get decay of localized state!

Basko, Aleiner Altshuler, 2006:

All orders in perturbation theory:

Approximation: retaining the most abundant diagrams at given order

Predicted:




MIT at finite T!

↔ **Mobility edge as function of T (or total E)**

Historic evolution of the notion of MBL:

MBL = absence of any transport AND equilibration

Those are implied by the existence of a complete set of quasi-local integrals of motion! [LIOM]

*Huse, Nandkishore, Oganesyan '13; Serbyn, Papic, Abanin '13;*

Is existence of LIOMs the best definition of MBL?

*(Huse+Nandkishore 2014)*

## How to construct IOM's?

*J. Imbrie (2014)*

Approach (1): Abstract construction of quasilocal unitary conjugation, which block diagonalizes *H*

$$H = -\sum_i h_i s_i^z - \lambda \sum_i s_i^x s_{i+1}^x$$

$$U^{-1} H U = \tilde{H}_{\mathrm{loc}} = \sum_i J_i \tau_i^z + \sum_{i,j} J_{ij} \tau_i^z \tau_j^z + \ldots$$

$$\tilde{s}_i^z \equiv U \tau_i^z U^{-1} \quad \text{integrals of motion!}$$

$$U = \exp[-A], \quad A \text{ quasilocal anti-Hermitian}$$

$$J_{ij} \sim \exp\left[-|i-j|/x\right], \quad \text{quasilocal}$$

## How to construct IOM‘s?

*V. Ros, MM, A. Scardicchio (Nucl. Phys. B 2015)*

Approach (2): Direct local construction

- MBL as localization of a hopping problem in operator space
- Estimate domain of existence of MBL in the BAA model
- Connection to Keldysh Green’s function approach of Basko et al. *(Ann. Phys. 2006)*?
- Both constructions raise questions about mobility edges

# Integrals of Motion

BAA model (disordered electrons)

$$H = \sum_{\alpha} \varepsilon_\alpha n_\alpha + \lambda \sum_{\alpha,\beta,\gamma,\delta} u_{\alpha\beta,\gamma\delta} b^{+}_\alpha b^{+}_\beta b_\gamma b_\delta$$

$$= H_0 + \lambda H_1$$

*V. Ros, MM, A. Scardicchio (Nucl. Phys. B 2015)*

Perturbative ansatz for integrals

$$\longrightarrow \quad I_\alpha = n_\alpha + \sum_{n \geq 1} \lambda^n I_\alpha^{(n)} \qquad [H, I_\alpha] \overset{!}{=} 0$$

$$\left[H_0, I_\alpha^{(n+1)}\right] \equiv ad_{H_0}\left(I_\alpha^{(n+1)}\right) = -\left[H_1, I_\alpha^{(n)}\right]$$

Explicit solution:

$$I_\alpha^{(n+1)} = i \lim_{\eta \to 0} \int_0^\infty dt e^{-\eta t} e^{iH_0 t} [H_1, I_\alpha^{(n)}] e^{-iH_0 t}$$

For large coordination number K: retain only diagrams that create one more particle-hole pair at each successive vertex (likeBAA):

Parametrically justified “forward approximation”

# Integrals of Motion

In "forward approximation":

Can show:

- Convergence for small interaction strength λ
- Estimate of radius of convergence

BUT: Loose control about potential contributions ~ $1/K^x$ to conductivity!

Note: Convergent construction of LIOM implies MBL at ALL temperatures.

What happens to LIOM if there were a finite T transition?

a) LIOM are operator series that weakly converge only on low T states
(this happens in forward approximation)

? b) MBL exists only if it exists at all T ←

Analogous question for disorder-free systems, as function of ρ.

Since ρ<<1 is always deloc → (b) rules out genuine MBL without disorder

# Delocalization through rare energy configurations

Main idea:

Sponataneous local fluctuations ("ergodic bubbles") are mobile and delocalize the whole system

Word of caution:

Absence of a genuine mobility edge:

→ putative transition → sharp crossover

BUT: the extremely badly conducting phase is nevertheless very interesting and behaves localized on parametrically long time scales!

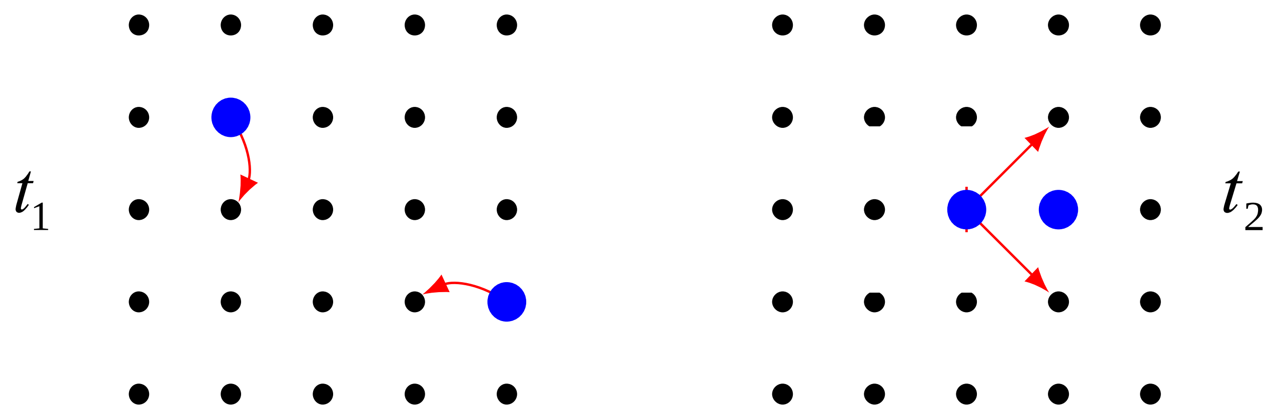
# Instructive toy problem

## Strongly assisted hopping on cubic lattice (d>2)

$$H = -t_1 \sum_{\langle x,y\rangle} (c_x^\dagger c_y + \text{h.c.}) + \sum_x \epsilon_x n_x \quad (1)$$

$$-t_2 \sum_x \sum_{s,s'=\pm} \sum_{1\le\alpha<\beta\le d} n_x (c^\dagger_{x+s\vec{e}_\alpha} c_{x+s'\vec{e}_\beta} + \text{h.c.}).$$

$$t_1 << W = \sqrt{\langle e^2 \rangle} << t_2$$



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$$t_1 << W = \sqrt{\langle e^2 \rangle} << t_2$$

1) analyze 2-particle problem: delocalization through rare events
2) finite but low density

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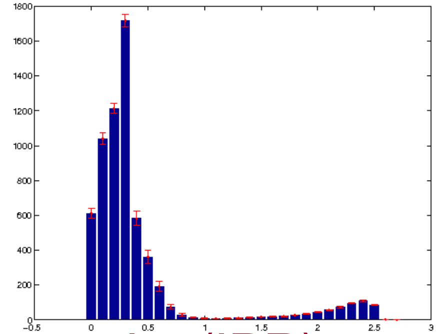

-log(IPR)

### First: 2-particle problem:

Two types of eigenstates:

(i) States with 2 particles at distance [> log(L)] are trivially localized.

(ii) States with a close pair are delocalized: Interaction-induced delocalization! Particles diffuse *together:* Separating they localize, get back together and move on.

# Instructive toy problem

## Strongly assisted hopping on cubic lattice (d>2)

$$H = -t_1 \sum_{\langle x,y \rangle} (c_x^\dagger c_y + \text{h.c.}) + \sum_x \epsilon_x n_x \qquad (1)$$

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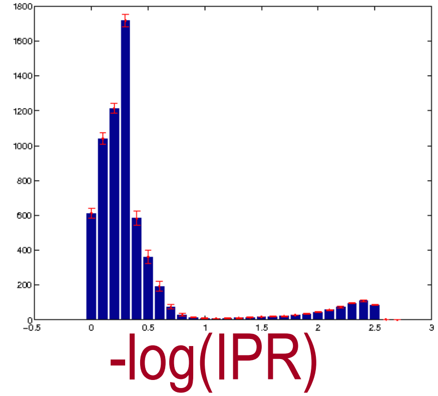


Low but finite density problem:

Finite density of delocalized pairs form a bath and scatter single particles

→ All states at finite density should be fully delocalized and ergodic

# Instructive toy problem

## Strongly assisted hopping on cubic lattice (d>2)

$$
\begin{aligned}
H \;=\; & -t_1 \sum_{\langle x,y\rangle} (c_x^\dagger c_y + \text{h.c.}) + \sum_x \epsilon_x n_x \qquad (1) \\
& -t_2 \sum_x \sum_{s,s'=\pm} \sum_{1\le\alpha<\beta\le d} n_x (c^\dagger_{x+s\vec{e}_\alpha} c_{x+s'\vec{e}_\beta} + \text{h.c.}).
\end{aligned}
$$

Logical steps of the argument:

(i) Construct a resonant, delocalized subgraph in configuration space (close pairs) $\longrightarrow$ expect delocalized eigenstates

(ii) Argue that coupling to configurations outside the subgraph do *not* reinstate localization

(iii) Argue that (rare, but finite density) pairs delocalize everything

# Instructive toy problem

## Strongly assisted hopping on cubic lattice (d>2)

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\begin{aligned}
H &= -t_1 \sum_{\langle x,y \rangle} (c_x^\dagger c_y + \mathrm{h.c.}) + \sum_x \epsilon_x n_x \qquad (1) \\
&\quad -t_2 \sum_x \sum_{s,s'=\pm} \sum_{1 \le \alpha < \beta \le d} n_x (c^\dagger_{x+s\vec{e}_\alpha} c_{x+s'\vec{e}_\beta} + \mathrm{h.c.}).
\end{aligned}
$$

**Now repeat these logical steps for MBL!**

(i) Construct a resonant, delocalized subgraph in configuration space (close pairs) $\longrightarrow$ expect delocalized eigenstates

(ii) Argue that coupling to configurations outside the subgraph do *not* reinstate localization

(iii) Argue that (rare, but finite density) pairs delocalize everything

# Rare events in MBL problems

Assumption: At high T there is delocalization/ergodicity

↔ Large systems at high T are ergodic and satisfy ETH

Aim: Show that this implies delocalization at any T > 0!

Analogy:

| | | |
|---|---|---|
| Assisted hopping at low ρ | ↔ | MBL at low T |
| Rare pairs | ↔ | Rare high energy bubbles |
| Pairs dissociate | ↔ | Bubbles spread their energy |
| Pairs get back together | ↔ | Bubbles reshape |

# Resonant delocalization of ergodic spots?

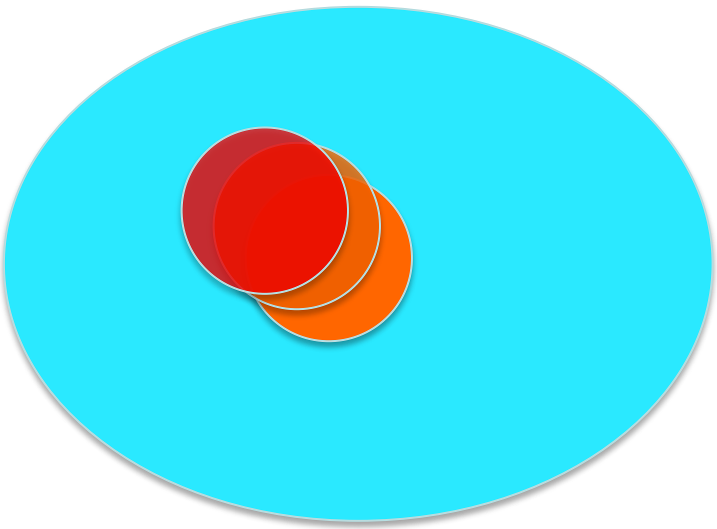
## (i) Constructing a resonant graph of “ergodic spots”:

Disordered case: regions of high energy density

Disorder free case: low density spots

*W. de Roeck and F. Huveneers (2014)*

*W. de Roeck, F. Huveneers, MM, M. Schiulaz (2015)*



Resonant, delocalized subgraph in configuration space

**Displacement of ergodic spot**

Matrix element for bubble displacement

$$M \sim \sqrt{1/\dim\left[H_{\text{bubble}}\right]}$$

Internal level spacing

$$d \sim 1/\dim\left[H_{\text{bubble}}\right] << M$$

→ Resonant transition!

# Resonant delocalization of ergodic spots?

(ii) Bubbles do not get localized by diluting themselves:

Assume that there is no energy transport

→ the extra energy cannot diffuse away

→ the bubble can and will reshape and moves on resonantly

This takes long, but the time is independent of system size

→ very slow, but finite diffusion via bubble configurations!

Note about dynamics: Bubbles cannot disappear completely. At any given time there must be a finite density of bubbles in a typical state at energy E, due to time-invariance of the thermal ensemble.

# Resonant delocalization of ergodic spots?

(iii) Mobile bubbles delocalize the whole system

Mobile bubbles form a good bath with continuous spectrum

Level spacing of translation modes: $d \sim 1/\mathrm{Vol}$

Coupling to any other degrees of freedom: $M \sim 1/\sqrt{\mathrm{Vol}}$

Conclusion:

Existence of an ergodic phase at high T implies delocalization at any finite T.

Remark aside: The same type of reasoning seems to rule out delocalized non-ergodic phases (except for spontaneous sym. breaking)

# Potential divergences in operator expansion for IOM

How did bubbles hide in the BAA analysis,

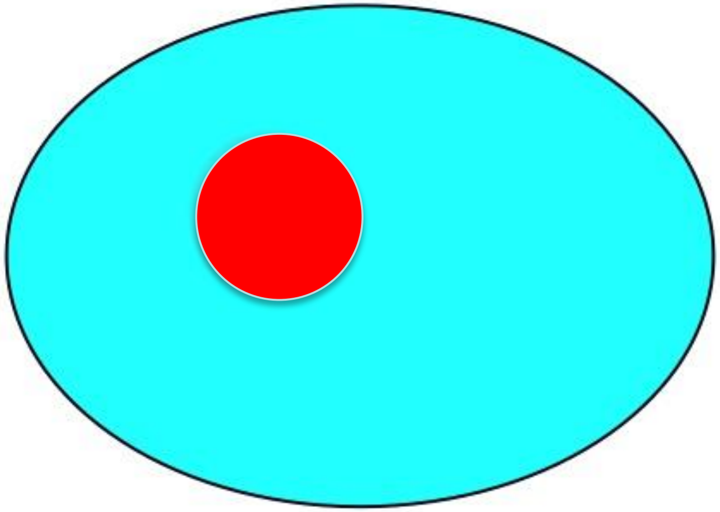
and in the construction of LIOMs ?

# Potential divergences in operator expansion for IOM

?

a) LIOM are operator series that weakly converge only on low T states (which happens in forward approximation)

b) MBL exists only if it exists at all T



Divergence of Type I:
Forward approximation

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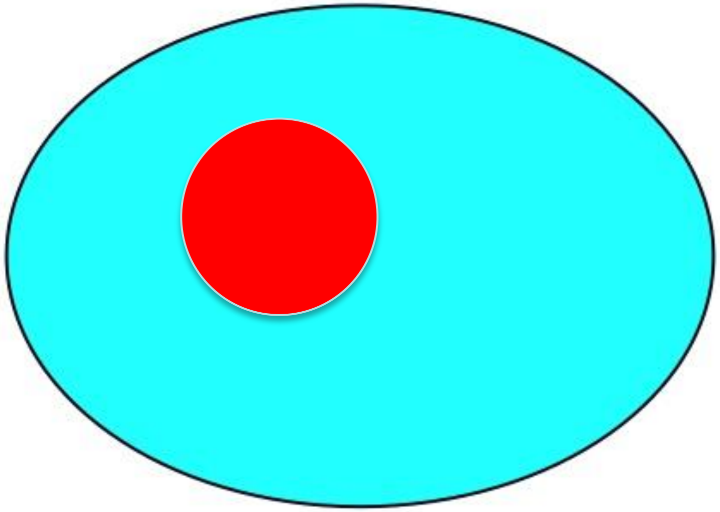

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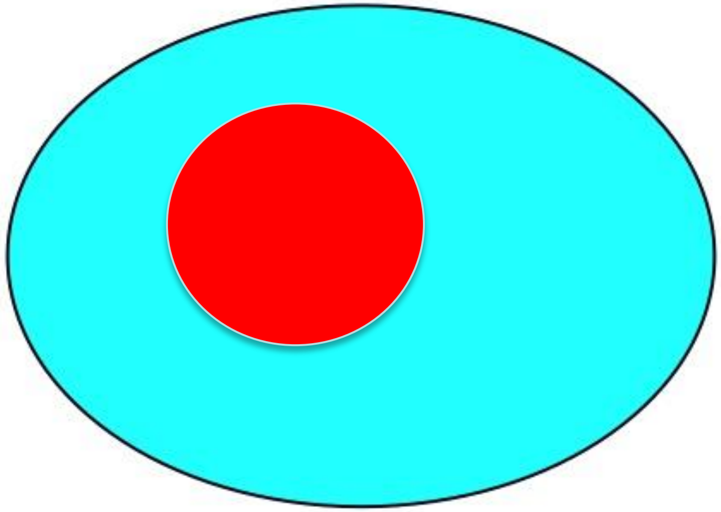

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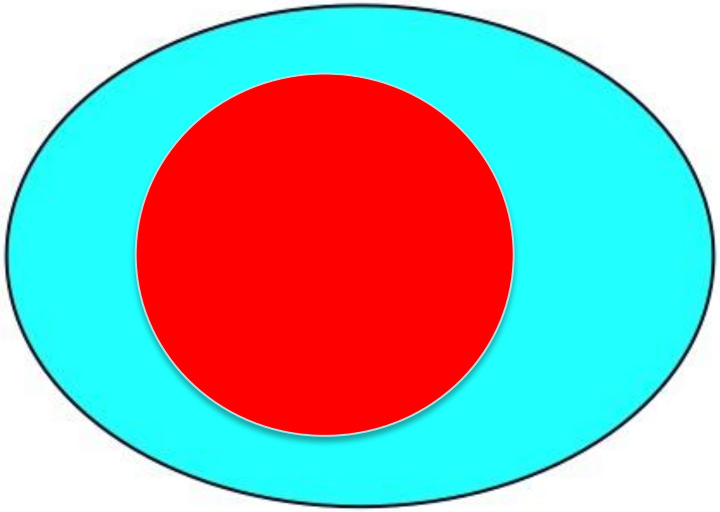

Divergence of Type I:
Forward approximation

Can potentially be killed by application to low T states

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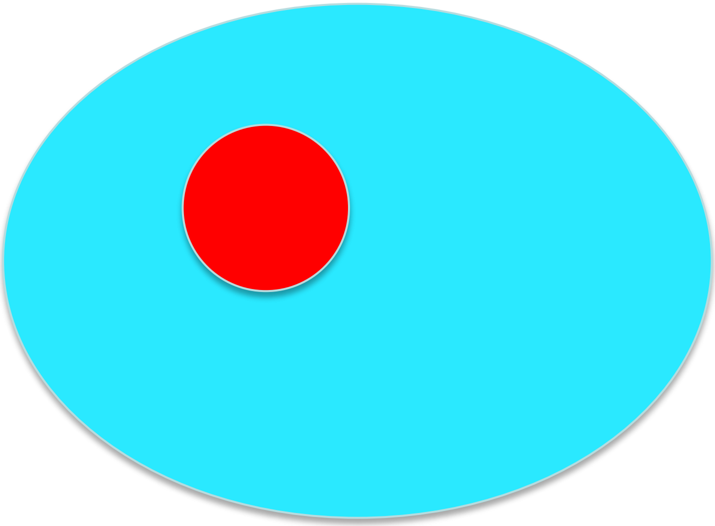

Divergence of Type II:
Beyond forward approximation

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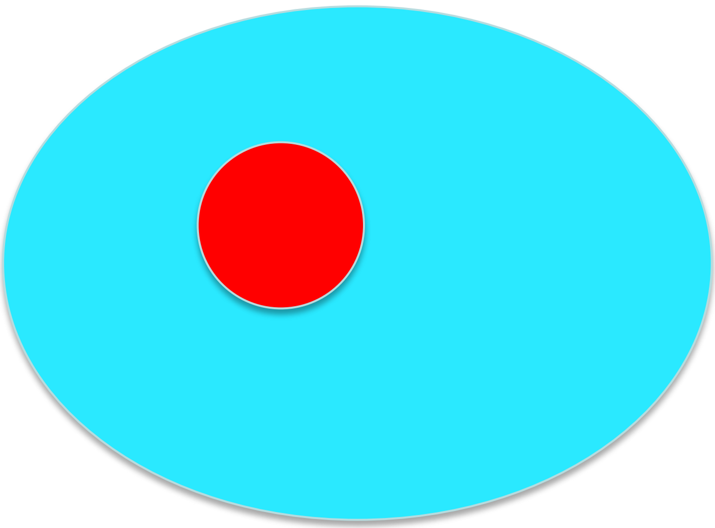

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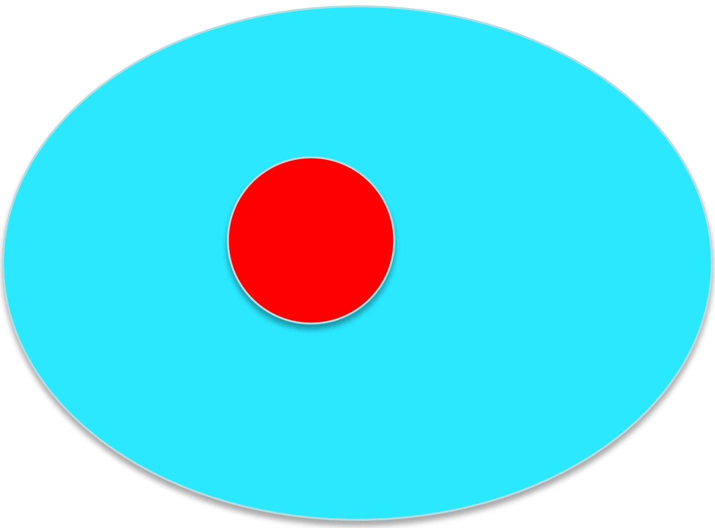

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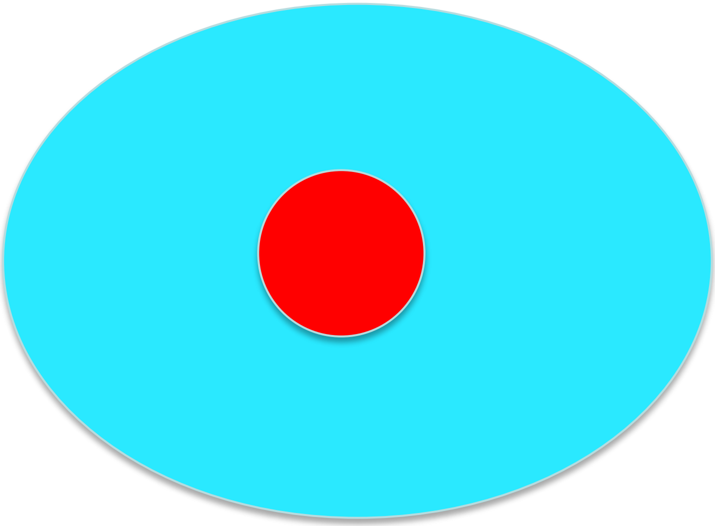

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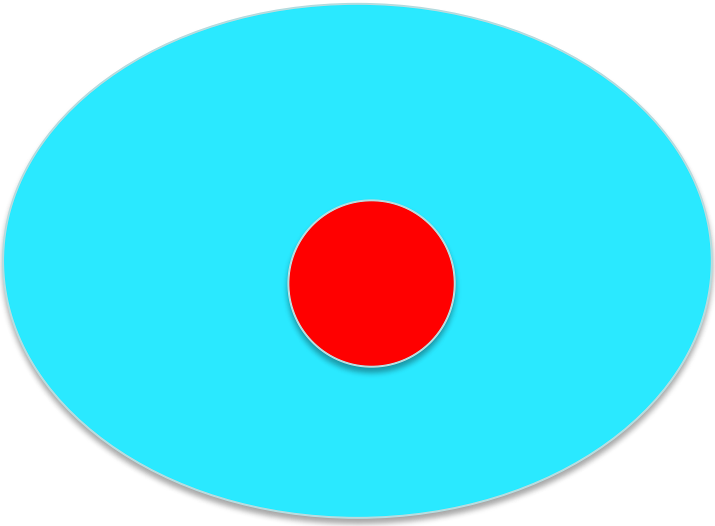

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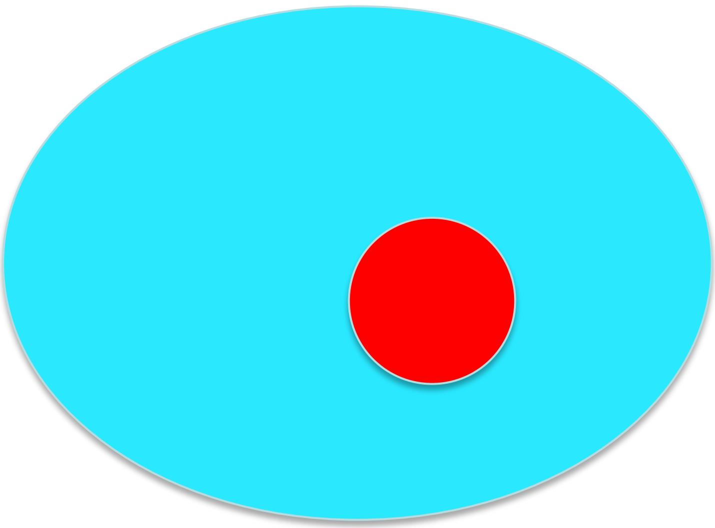


Divergence of Type II:
Beyond forward approximation

Cannot be made to converge by application to low T states

But: Type I divergence suggests presence of Type II subsequence, too!

→ b) MBL exists only if it exists at all T

# Further caveats for expert discussions

- Weak localization of bubbles in d<3?
- Localization in d=1 due to very bad disorder regions?
- Unidentified sources that kill the resonances on the subgraph [like in Lifshitz model, quantum percolation]

  or: that drive the dynamic time scales to infinity?
  (via Anderson orthogonality, as in spin-boson localization)

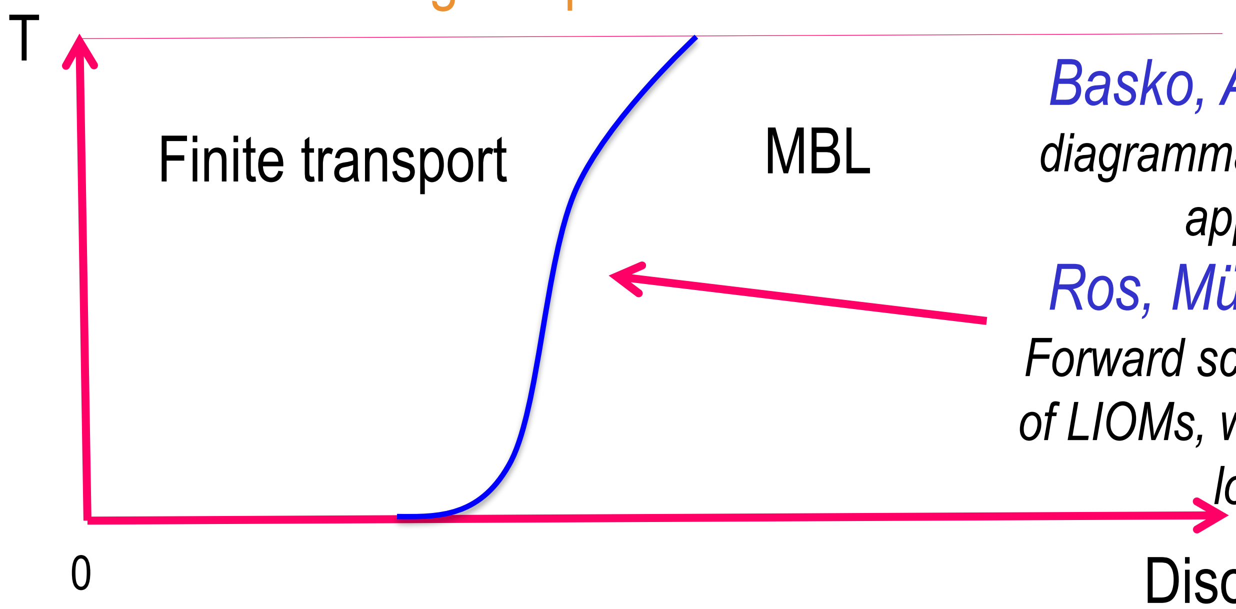
We just can't see this work in MBL.
The situation is not so different from diffusion in 3d Anderson...

# Consequences

## Disordered case:

Genuine MBL either at all T or only at T = 0 -
only strong crossover as a function of T:

Phase diagram predicted without bubbles




*Basko, Aleiner, Altshuler*
*diagrammatics within "forward approximation"*

*Ros, Müller, Scardicchio*
*Forward scattering construction of LIOMs, weakly converging on low T states*

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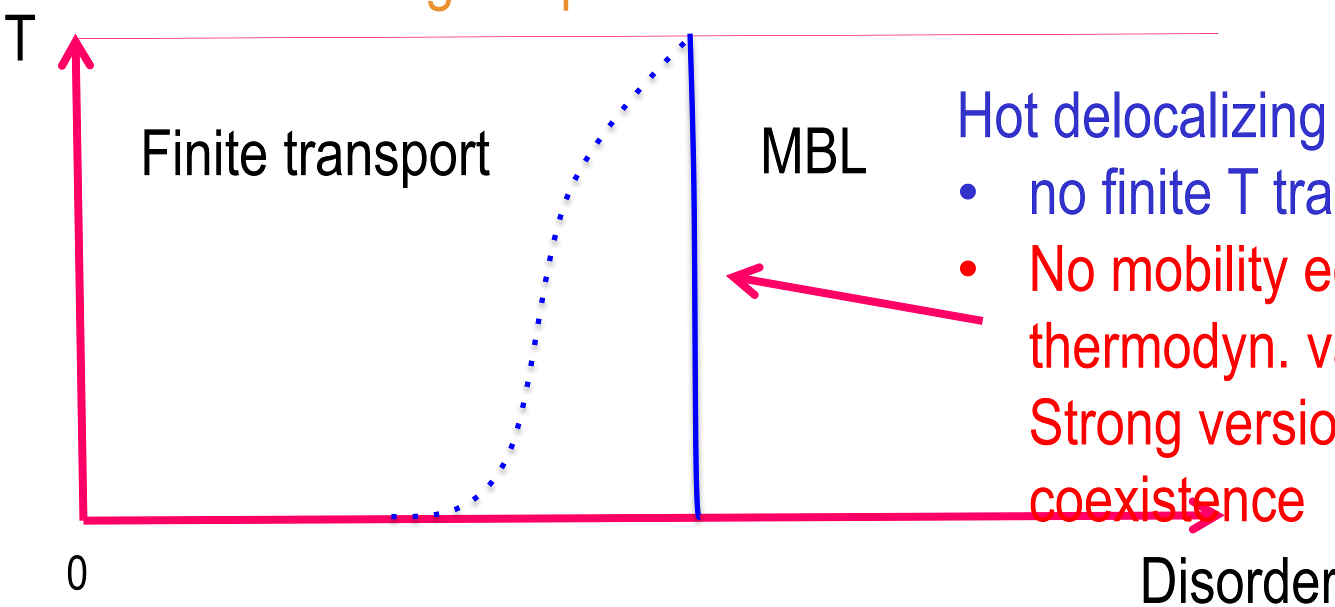



Hot delocalizing bubbles:

- no finite T transition!
- No mobility edges in thermodyn. variables (T,μ): Strong version of Mott non-coexistence

- Genuine MBL in the continuum is impossible:

At high energy: infinitesimal interaction will delocalize, as $\xi \longrightarrow \infty$.

# Consequences

Disorder-free localization (on configurational disorder)?

Only asymptotic localization

Finite transport due to rare low-density bubbles

BUT: highly suppressed: non-perturbative in the hopping t

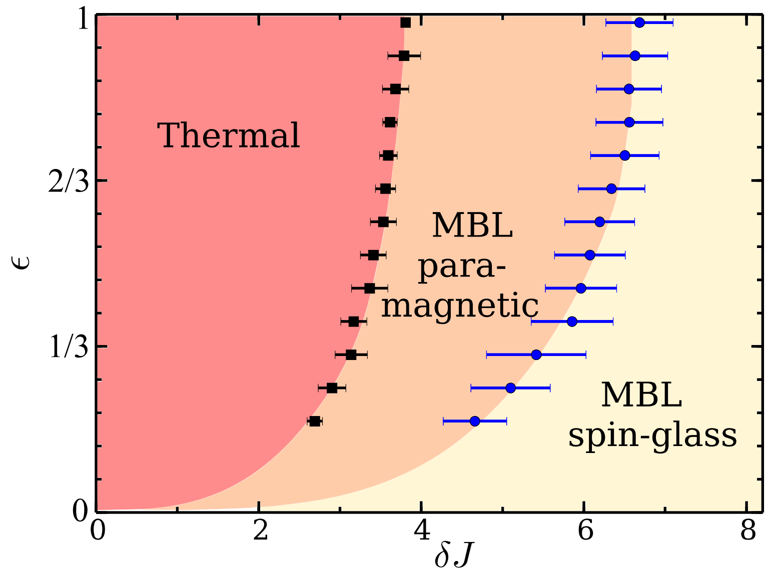
$$s \leq \exp\left[-\frac{const}{t^{a}}\right]$$

(for models with power law interactions)

*Schiulaz, Müller, Silva PRB (2015)*

# Why are mobility edges seen in numerics then?

$$H = -\sum_{i=1}^{L}\left[(J+\delta J_i)\,\sigma_i^z\sigma_{i+1}^z + J_2\sigma_i^z\sigma_{i+2}^z + h_z\sigma_i^z + h_x\sigma_i^x\right]$$




*Kjäll, Bardarson, and Pollmann, PRL 113, 107204 (2014).*

*Also: Luitz, Laflorencie, Alet PRB 2015;*

# Numerics and finite size effects

## Why are mobility edges seen in numerics then?

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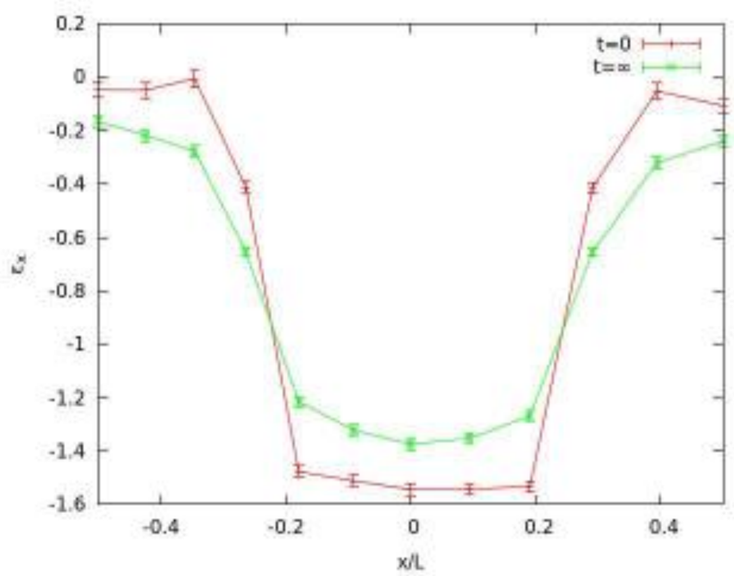
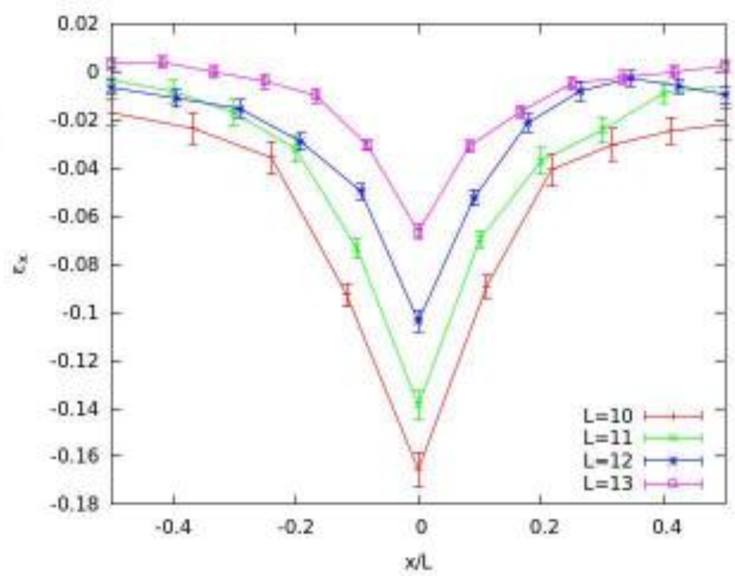
A: Small ergodic systems are not ideal baths! Bubbles must be large to move!




δJ=3J



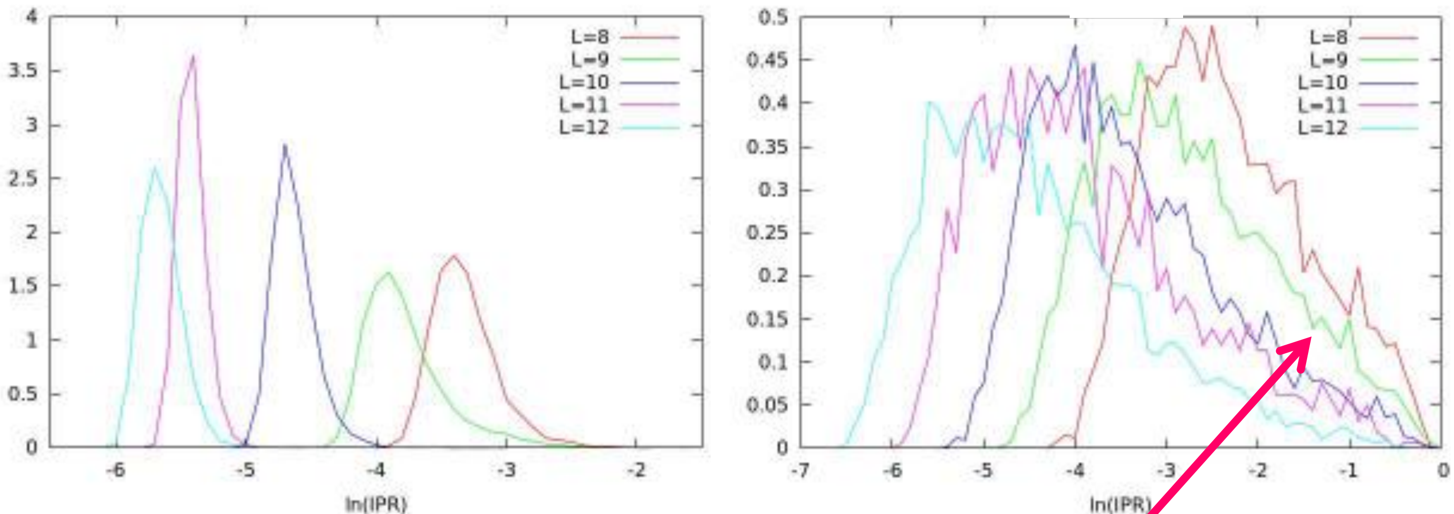

δJ=J

# Numerics and finite size effects

## How good a bath is an 'ergodic' finite size system?

$$\mathrm{IPR}_\alpha \equiv \sum_\beta \left| \langle\beta|\,\sigma_1^z\,|\alpha\rangle \right|^4 \sim \frac{1}{\#\left(\text{peaks in spectral function of } s_1^z\right)}$$

δJ=0.1J    δJ=J




- Typical 1/IPR ~ $2^L$, but with strong fluctuations for moderate disorder
- Even at δJ=J << $\delta J_c$~ 3J, small IPR are not rare:

  → Too small bubbles are often bad baths and get stuck.
- The critical bubble size is substantial!

  → Finite size effects are strong even far from MBL transition (δJ~3J).

# Aren't there known exceptions??

- Fine-tuned Hamiltonians may have mobility edges
  (Y. Huang, 2015) – no robustness to perturbations of H

- Quantum Random Energy Model (mean field system)

  *Warzel/Aizenmann:* coexistence of deloc/loc states
  *Laumann/Pal/Scardicchio:* computation of a mobility edge

  Crucial ingredient: energy is a completely non-local function of spins
  $\longrightarrow$ There is no notion of bubbles!
  $\longrightarrow$ Localization is rather abstract

# Conclusion

Genuine MBL is rather restrictive: It's all or none!

| | | |
|---|---|---|
| • lattice model | ↔ | continuous space |
| • quenched disorder | ↔ | disorder-free localization |
| • finite local Hilbert space ? | ↔ | unbounded local Hilbert space |
| • $d = 1$ ? | ↔ | $d > 1$ |
| • discrete or no symmetry | ↔ | Continuous symmetry |
| probably necessary | | |

BUT:

strong 'asymptotic localization' remains interesting and useful to explore!

- Disorder free case
- T- or ρ- tuned crossovers