Integrable Matrix Theory

(Theory of integrable Hamiltonians with finite number of levels)

What is quantum integrability and who cares?

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KITP Program: Many-Body Localization

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Classical Mechanics

Definition: A classical Hamiltonian $H_0(p, q)$ with n degrees of freedom (n coordinates) is integrable if it has the maximum possible number (n) of functionally independent Poisson-commuting integrals $\{H_i(p, q), H_j(p, q)\}=0;\ i,j=0,1\ldots n$

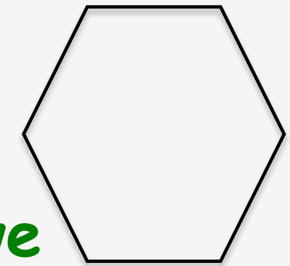
⇓

- ✓ *Unambiguous separation of integrable from nonintegrable (generic)*
- ✓ *Various properties that don't have to be verified on a case by case basis*

Q: What is quantum integrability? How is it defined?

Think finite, N x N, matrix even with very large N

Example: Hubbard model on a ring



$$H = \begin{pmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{pmatrix}$$

Given matrix H how do we tell if it's integrable?

How do we generate (an ensemble of) integrable matrices?

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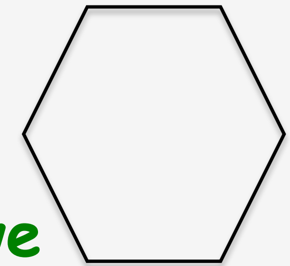


$$H = \begin{pmatrix} \times & 0 & 0 & 0 & 0 \\ 0 & \times & 0 & 0 & 0 \\ 0 & 0 & \times & 0 & 0 \\ 0 & 0 & 0 & \times & 0 \\ 0 & 0 & 0 & 0 & \times \end{pmatrix}$$

Given matrix H how do we tell if it's integrable?

How do we generate (an ensemble of) integrable matrices?

No way! Not even a definition! (See e.g. B. Sutherland, *Beautiful Models* (2004), Caux & Mossel (2011), E.Y. & Shastry (2013) for review)

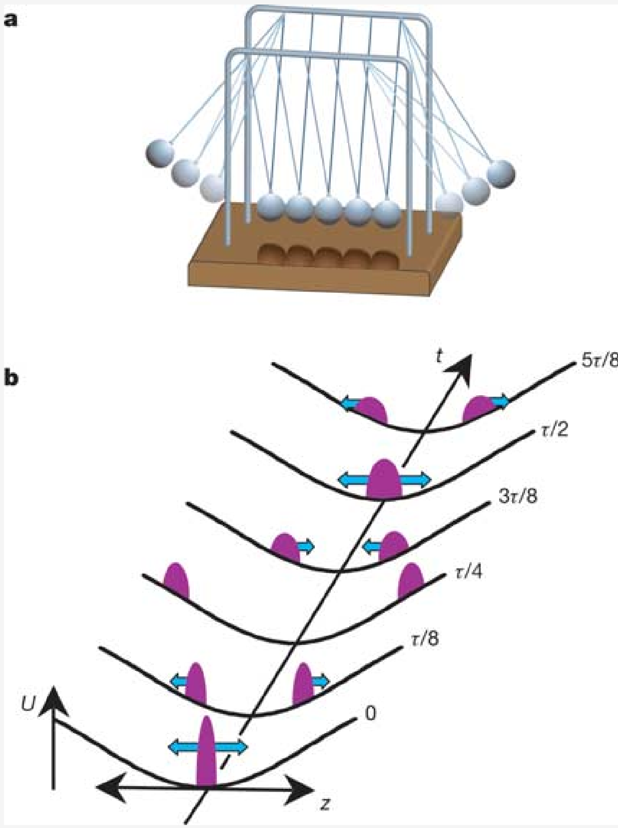
no natural notion of an integral of motion: for any H can find a full set of H_k such that $[H, H_k]=0$

$$H = \sum_{1}^{N} E_n |n\rangle\langle n|, \qquad H_k = |k\rangle\langle k|$$

Alternatively, can consider powers of H_0

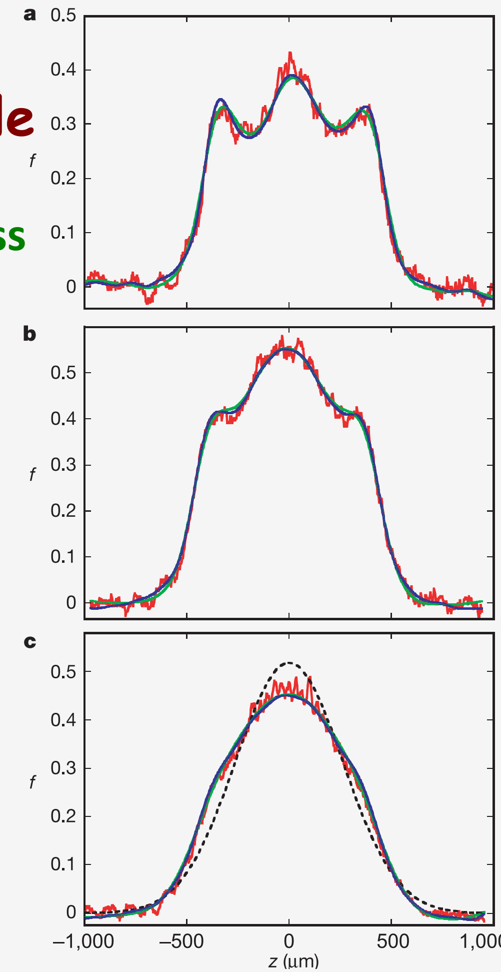
$$H_k = \sum_{n=1}^{N} a_n H_0^n$$

Who cares? – rise of integrability



A quantum Newton's cradle

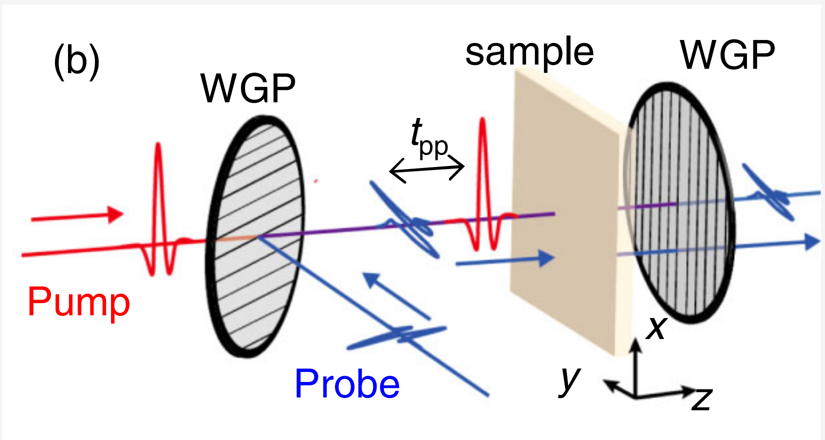
T. Kinoshita, T. Wenger, D. Weiss
Nature (2006)

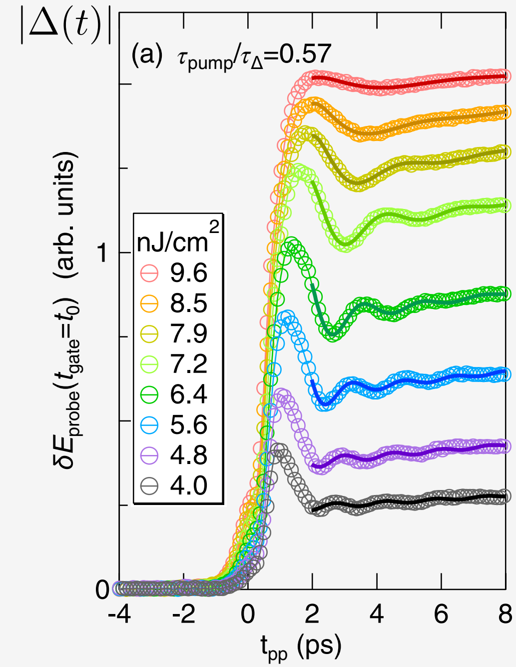


"^{87}Rb atoms … do not noticeably equilibrate even after thousands of collisions. Our results are probably explainable by the well-known fact that a homogeneous 1D Bose gas with point-like collisional interactions is *integrable*."

Higgs Amplitude Mode in the BCS Superconductors $Nb_{1-x}Ti_xN$ Induced by Terahertz Pulse Excitation

Ryusuke Matsunaga,[1] Yuki I. Hamada,[1] Kazumasa Makise,[2] Yoshinori Uzawa,[3]
Hirotaka Terai,[2] Zhen Wang,[2] and Ryo Shimano[1]





$\tau_\Delta = \hbar/\Delta_0 \approx 3\text{ps}$ – **timescale on which $|\Delta(t)|$ evolves**

$$|\psi(0)\rangle = |\textbf{noneq. state produced by the pulse}\rangle$$

$$\hat{H}_{\text{BCS}} = \sum_{i,\sigma} \epsilon_i \hat{c}^\dagger_{i\sigma}\hat{c}_{i\sigma} - u \sum_{i,j} \hat{c}^\dagger_{i\uparrow}\hat{c}^\dagger_{i\downarrow}\hat{c}_{j\downarrow}\hat{c}_{j\uparrow}$$

$$i\frac{d|\psi\rangle}{dt} = \hat{H}_{\text{BCS}}|\psi\rangle$$

$$|\Delta(t)| = \Delta_\infty + a\frac{\cos(2\Delta_\infty t + \alpha)}{\sqrt{\Delta_\infty t}}$$

Yuzbashyan, Tsyplyatyev, Altshuler, PRL (2006)

Integrable systems follow Generalized Gibbs Ensemble?

$$\rho = Z^{-1} e^{-\sum_i \beta_i H_i}$$

$$\langle O(t) \rangle_{t\to\infty} = \mathrm{Tr}\,\rho O$$

$$\langle \mathrm{in} | H_i | \mathrm{in} \rangle = \mathrm{Tr}\,\rho H_i$$

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Does it work?

Sometimes yes, sometimes no – depends on the system, observable and the the set of integrals

- ✓ **Works for simple models, e.g. 1D hard-core bosons & Luttinger liquids** Rigol et. al. PRL (2007); Cazalilla PRL (2006)
- ✓ **Fails for models with bound states, e.g. XXZ or attractive Lieb-Liniger** Pozsgay et. al. PRL (2014); Goldstein & Andrei, arXiv:1405.4224
- ✓ **Fails for global observables except for uncorrelated free fermions** Gurarie, J. Stat. Mech. (2013)
- ✓ **Does work for XXZ if new integrals are added** Ilievski et. al. PRL (2015)

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How do we determine if we have the "right" set of integrals and the criteria for the validity of GGE?

Need to know what quantum integrability is! Otherwise, GGE is a mysterious, essentially unfalsifiable conjecture.

Do Classical Mechanics first before going Quantum?!

Properties (??) of quantum integrable models

- ✓ Generalized Gibbs Ensemble: *does it work?*
- ✓ Exact solution via Bethe's Ansatz: *but any matrix can be "exactly solved"* $\det(H - \lambda I) = 0$
- ✓ Commuting integrals: *any matrix has them*
- ✓ Energy level crossings in violation of Wigner-v. Neumann non-crossing rule: *often, but not always. Can have crossings without integrability.*
- ✓ Poisson level statistics: *not always – e.g. BCS model. Non-integrable models can be Poisson?*

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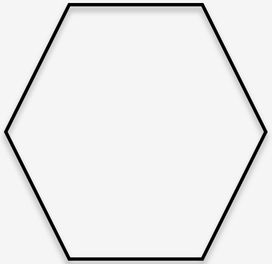


In the absence of a clear notion, have to verify every property separately on a case by case basis

Properties of quantum integrable models: Exact Solution

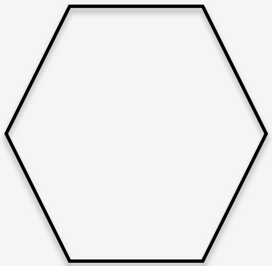
Example: Hubbard model

$$\hat{H} = T \sum_{j,s=\uparrow\downarrow} (c^\dagger_{j\,s} c_{j+1\,s} + c^\dagger_{j+1\,s} c_{j\,s}) + U \sum_j \hat{n}_{j\uparrow} \hat{n}_{j\downarrow}$$

H depends linearly on one parameter ***u=U/T***

tight-binding + onsite interactions, electrons on a ring

N=6 cites, 3 spin-up, ***M=3*** spin-down



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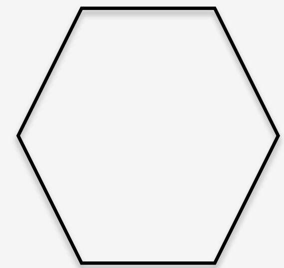

Exact Solution (Bethe's Ansatz):

E.H. Lieb and F.Y.Wu (1969)

$$e^{6ik_j} = \prod_{\alpha=1}^{3} \frac{\Lambda_\alpha - \sin k_j - iu/4}{\Lambda_\alpha - \sin k_j + iu/4}, \qquad \prod_{\alpha=1}^{3} \frac{\Lambda_\alpha - \Lambda_\beta + iu/2}{\Lambda_\alpha - \Lambda_\beta + iu/2} = - \prod_{j=1}^{6} \frac{\Lambda_\beta - \sin k_j - iu/4}{\Lambda_\beta - \sin k_j - iu/4}$$

9 coupled nonlinear equations

$$E = -\sum_{j=1}^{6} 2\cos k_j, \qquad P = \sum_{j=1}^{6} k_j, \qquad |P, S, S_z, \ldots\rangle = \ldots$$

But cf. $\det(H - \lambda I) = 0$

Commuting integrals (conservation laws)
Example: Hubbard model

$$\hat{H} \equiv \hat{H}_0(u) = \sum_{j=1}^{N} \sum_{s=\uparrow\downarrow} (c^\dagger_{js} c_{j+1\,s} + c^\dagger_{j+1\,s} c_{js}) + u \sum_{j=1}^{N} \hat{n}_{j\uparrow}\hat{n}_{j\downarrow}$$

$$\hat{n}_{j\sigma} = c^\dagger_{j\,s} c_{j\,s}$$

$$\hat{H}_1(u) = -i \sum_{j=1}^{N} \sum_{s=\uparrow\downarrow} (c^\dagger_{j+2s} c_{js} - c^\dagger_{js} c_{j+2s}) - iu \sum_{j=1}^{N} \sum_{s=\uparrow\downarrow} (c^\dagger_{j+1s} c_{js} - c^\dagger_{js} c_{j+1s})(\hat{n}_{j+1,-s} + \hat{n}_{j,-s} - 1)$$

$$[\hat{H}_0(u), \hat{H}_1(u)] = 0 \quad \text{for all } u$$

B. S. Shastry, PRL (1986)

$H_2(u)$, $H_3(u)$, $H_4(u)$,… **- in principle, infinitely many integrals of motion can be found from Shastry's transfer matrix (but not all of them are nontrivial for finite *N*)**

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But any Hamiltonian has commuting integrals. So what's special about Hubbard?

The Hamiltonian and the first integral are linear in a real parameter *u*.
Higher integrals are polynomial in *u*.

Properties of quantum integrable models: Level crossings

Example: Hubbard model

$$\hat{H} = T \sum_{j,s=\uparrow\downarrow} (c^{\dagger}_{j\,s} c_{j+1\,s} + c^{\dagger}_{j+1\,s} c_{j\,s}) + U \sum_{j} \hat{n}_{j\uparrow} \hat{n}_{j\downarrow}$$

H depends linearly on one parameter ***u=U/T***

Q: How do eigenvalues look like as functions of u?

For a typical *H(u)* energy levels with same quantum numbers (spin, momentum etc.) never cross – noncrossing rule

Hund (1927), Neumann & Wigner (1929)

Properties of quantum integrable models: Level crossings

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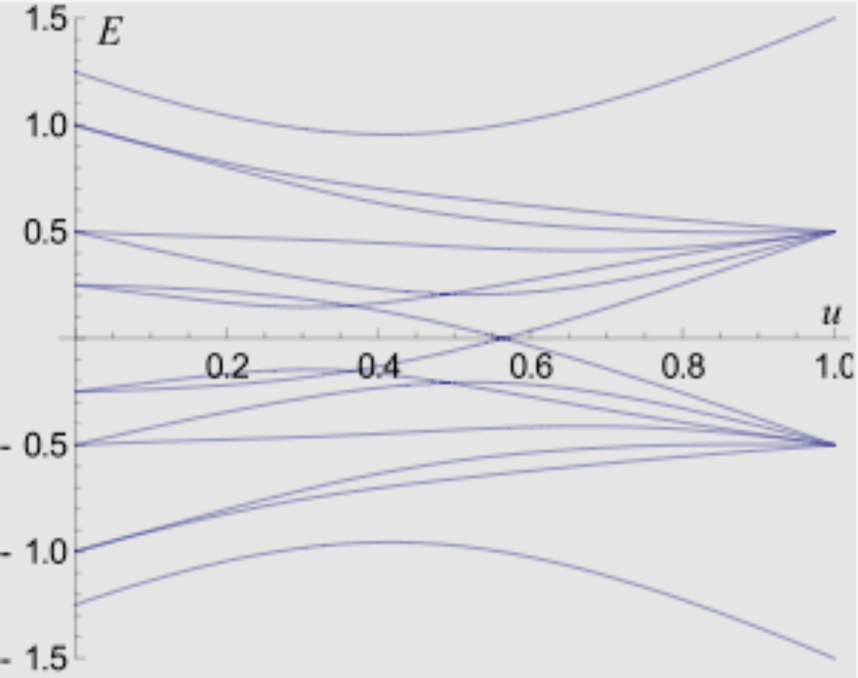


Energies for a 14 x 14 block of 1d Hubbard on six sites characterized by a complete set of quantum numbers

H(u)=A+uB is a **14 x 14** Hermitian matrix linear in real parameter ***u***

"The noncrossing rule is apparently violated in the case of the 1d Hubbard Hamiltonian for benzene molecule *[six sites]*…"

Heilmann and Lieb (1971)

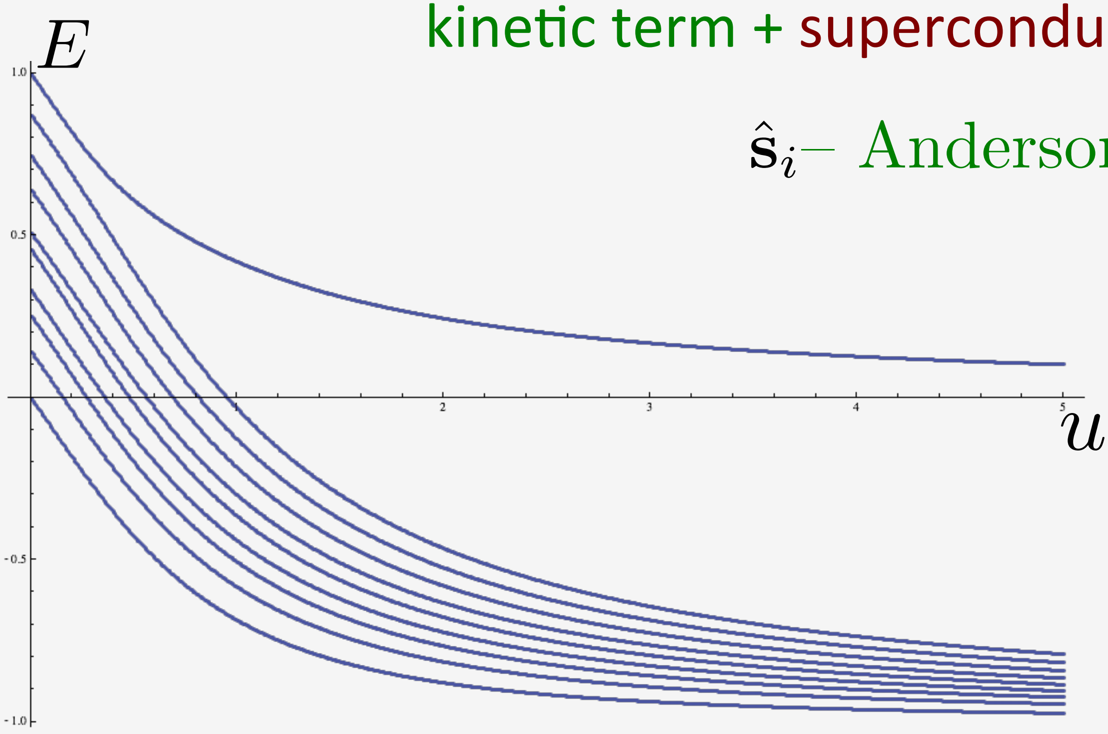
Properties of quantum integrable models: Level crossings

Counterexample: BCS (Richardson) model

$$\hat{H}_{\mathrm{BCS}} = \sum_i 2\varepsilon_i \hat{s}_i^z - u \sum_{i,j} \hat{s}_i^- \hat{s}_j^+ = \sum_i 2\varepsilon_i \hat{H}_i$$

kinetic term + superconducting interactions

$\hat{\mathbf{s}}_i$– Anderson pseudospins



Gaudin magnet integrable family

$$\hat{H}_i(u) = \hat{s}_i^z - u \sum_{j \neq i} \frac{\hat{\mathbf{s}}_i \cdot \hat{\mathbf{s}}_j}{\epsilon_i - \epsilon_j}$$

$$[\hat{H}_i(u), \hat{H}_j(u)] = 0$$

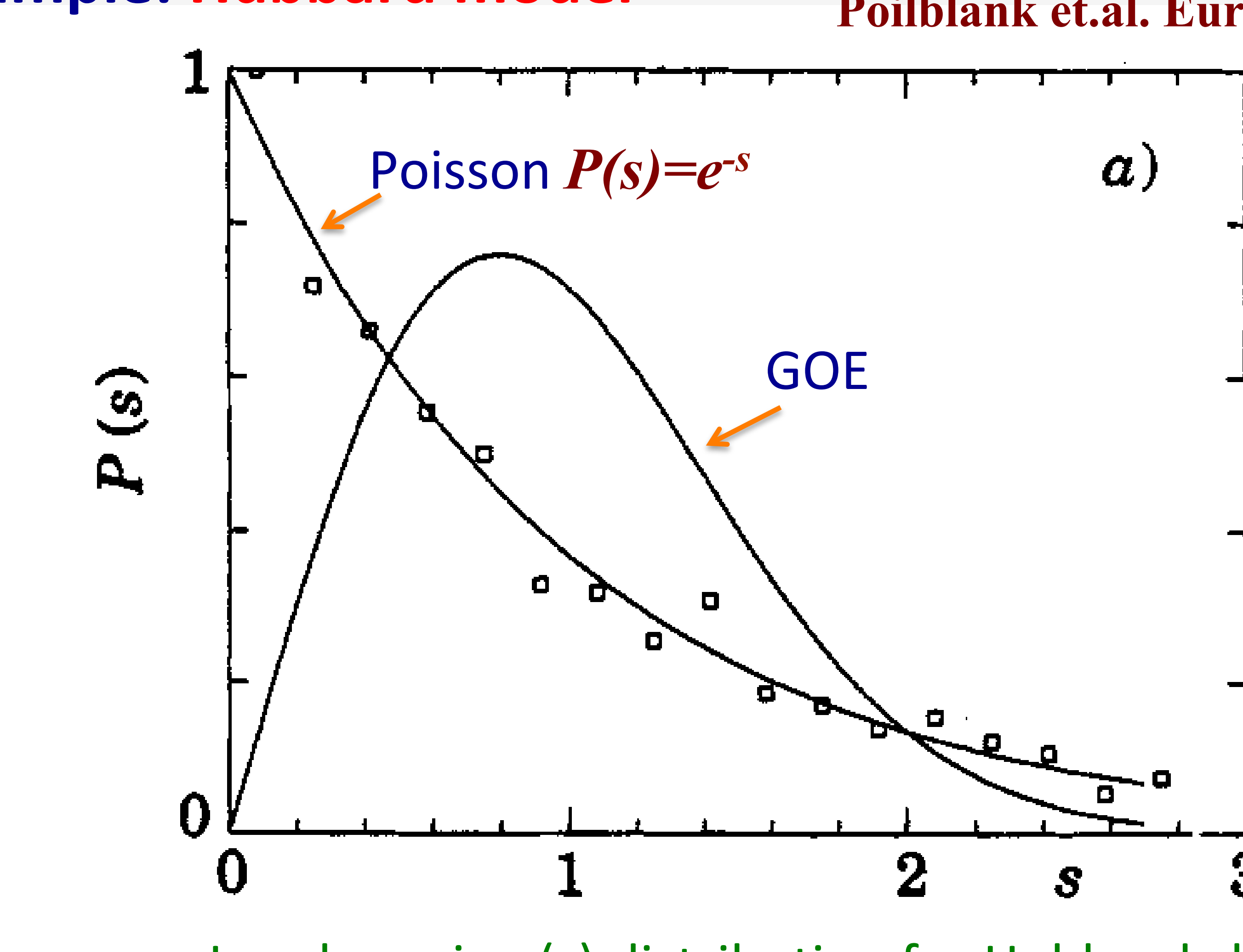
$$[\hat{H}_{\mathrm{BCS}}(u), \hat{H}_i(u)] = 0$$

Energies for a 10 x 10 block of the BCS model for 10 levels characterized by a complete set of quantum numbers

Properties of quantum integrable models: Poisson statistics

Example: Hubbard model

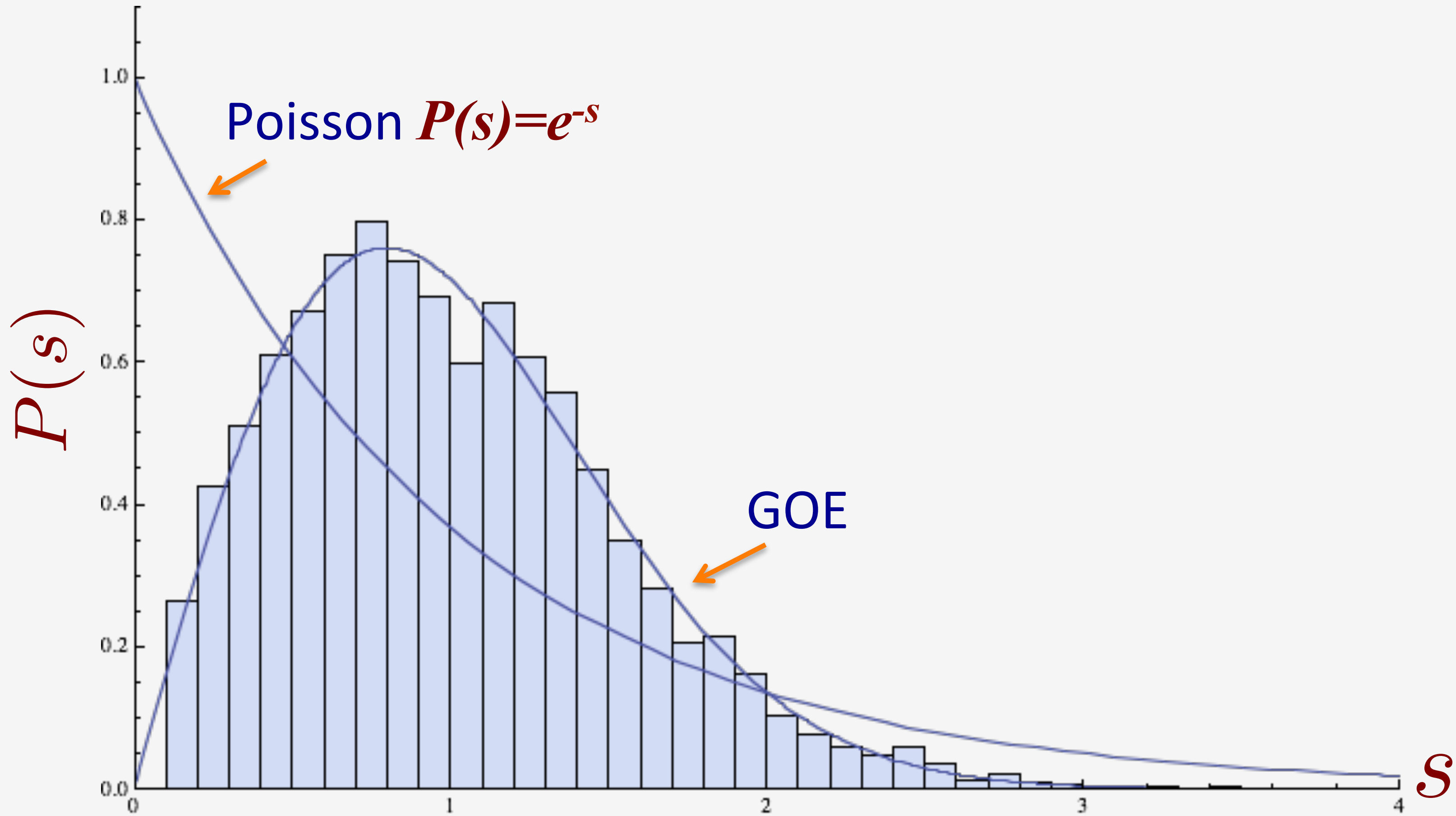
Poilblank et.al. Europhys. Lett. (1993)



Level spacing (*s*) distribution for Hubbard chain with 12 sites at ¼ filling, total momentum $P=\pi/6$, spin $S=0$

Properties of quantum integrable models: Poisson statistics

Counterexample: BCS (Richardson) model



Level spacing (s) distribution for the BCS model for $N=5000$ levels and 1 Copper pair

See also **Relano, Dukelsky et. al. PRE (2004)**

Notion of Quantum Integrability: What are we looking for?

Definition: Quantum Hamiltonian H_0 is integrable if…

Consequences:

1. Exact Solution
2. Generate (ensembles of) integrable models
3. Commuting integrals $[H_i, H_j]=0;\ i, j=0,1\ldots$
4. Energy level crossings?
5. Poisson level statistics *and exceptions*
6. Generalized Gibbs Ensemble for dynamics?

Classical integrability has it all

Definition: A classical Hamiltonian $H_0(p\,,\,q\,)$ with n degrees of freedom (n coordinates) is integrable if it has the maximum possible number (n) of functionally independent Poisson-commuting integrals $\{H_i\,,\,H_j\}=0;\ i,j=0,1\ldots n$

⇩

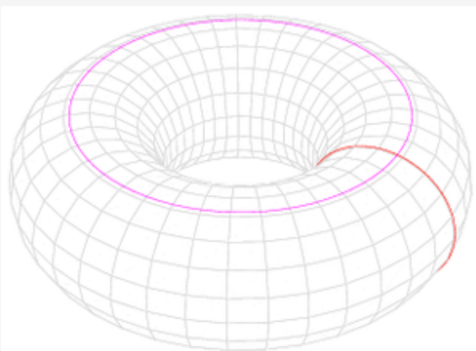
Consequences:

1. **Exact solution:** the dynamics of $H_i(p\,,\,q\,)$ is exactly solvable by quadratures **(Liouville-Arnold theorem)**
2. Poisson level statistics semi-classically *[Berry & Tabor (1976)]* except when $E(n_1,\,n_2,\,\ldots)$ is flat in $n_1,\,n_2,\,\ldots$, i.e. decoupled harmonic oscillators
3. **Generalized Microcanonical Ensemble typically holds for dynamics** *[Arnold, Math. Methods of CM, E.Y. ArXiv:1509.06351]*

Generalized Gibbs Ensemble DeMystified in Classical Mechanics

Dynamics is on "invariant torus" – n-dim portion of $2n$-dim phase-space cut out by integrals of motion $H_1(p,q,)$=const, $H_2(p,q,)$ =const, …, $H_n(p,q,)$=const

There are n typically incommensurate frequencies $\omega_1, \omega_2, \ldots, \omega_n$ (non-resonant torus)



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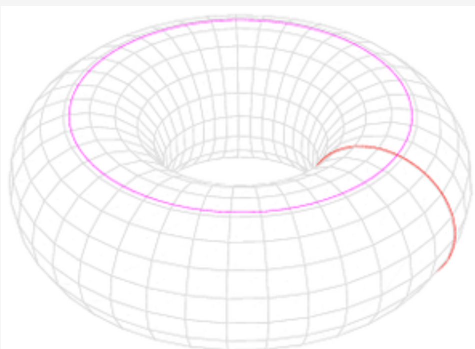
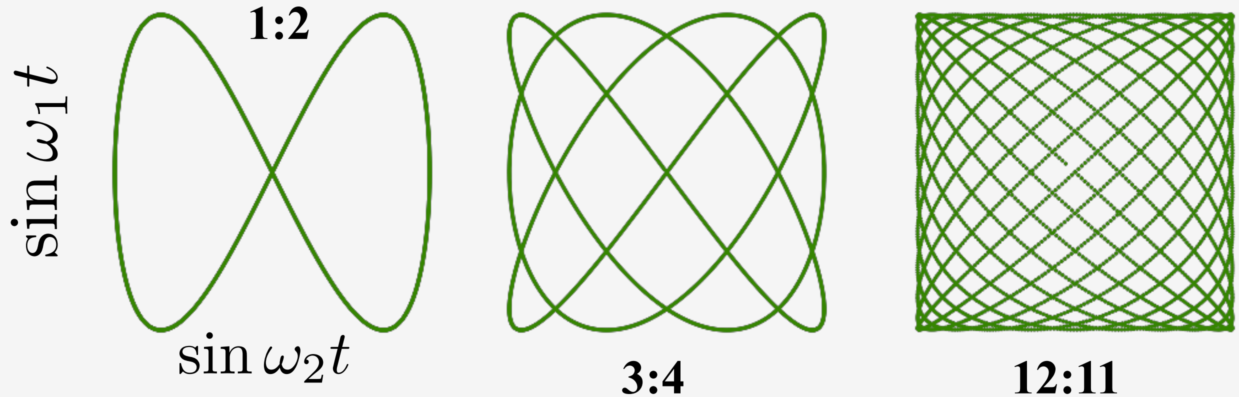


Lissajous figures



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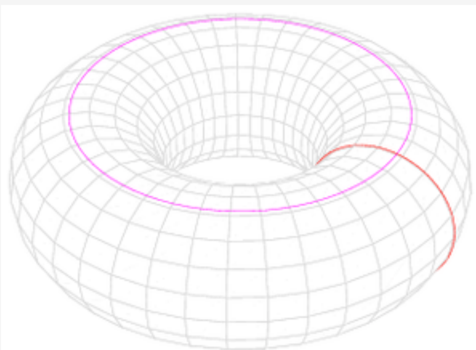

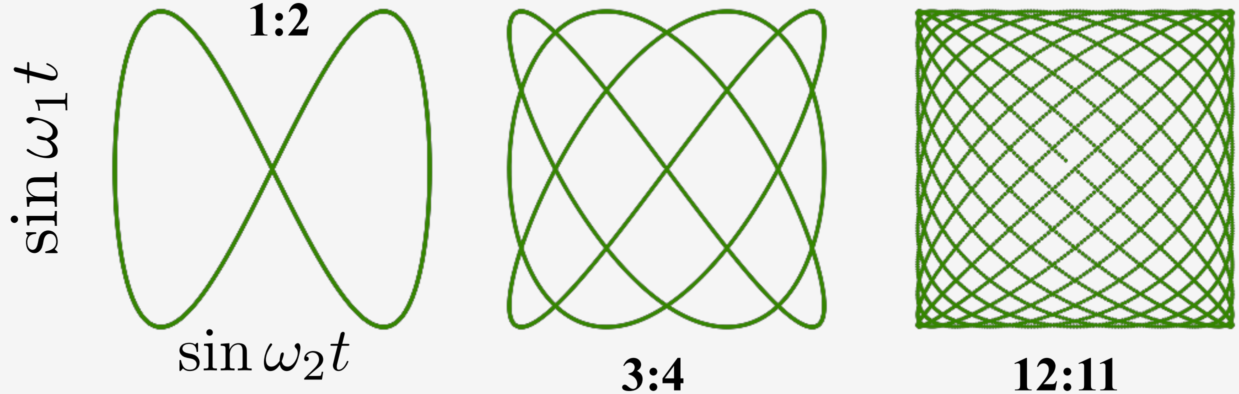

Theorem about averages (Arnold, *Math. Methods of CM*):

For a non-resonant torus and any "reasonable" observable $O(p,q,)$

time average = phase-space average over the torus

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T O\left(t\right)dt=\int O(\varphi)\frac{d\varphi}{(2\pi)^n}$$

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Going back to the original variables *p & q* and using the fact that this is a canonical transform can prove Generalized Microcanonical distribution

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T O\left(t\right)dt = \int O(p,q)\rho(p,q)dpdq$$

E.Y. ArXiv:1509.06351

$$\rho(p,q) = V^{-1}\prod_{k=1}^{n}\delta\left(H_k(p,q)-\alpha_k\right)$$

Works for any system size (any *n*)!

Exceptions: resonant tori

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Additive integrals, thermodynamic limit

$$H_k \propto n$$

$$n\to\infty$$

$\Longrightarrow$

Generalized (canonical) Gibbs

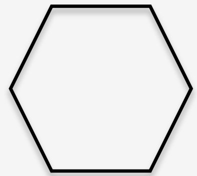
$$\rho(p,q) = Z^{-1}\exp\left(-\sum_k \lambda_k H_k(p,q)\right)$$

See e.g. Ruelle, Stat. Mech.: Rigorous Results (1999)

$\boxed{\textbf{not always the case???}}$

Can we develop a similar sound notion of integrability in Quantum Mechanics – for N x N Hermitian matrices (Hamiltonians)?

Hints from Hubbard study, $u=U/T$:
Yuzbashyan, Altshuler, Shastry (2002)



$$H(u) = T + uV$$

u – real parameter,
T, V – N x N Hermitian matrices

Nontrivial integrals depend on a real parameter (interaction or external field) in a certain fixed way. *Always at least one linear integral. Same is the case for other known parameter-dependent models*

- 1d Hubbard, XXZ spin chain (u = anisotropy): integrals are polynomial in u
- Gaudin magnets (all integrable pairing models): u=hyperfine interaction, Hamiltonian and all integrals are linear in u

$$\hat{H}_i(u) = \hat{s}_i^z - u \sum_{j \neq i} \frac{\hat{\mathbf{s}}_i \cdot \hat{\mathbf{s}}_j}{\epsilon_i - \epsilon_j}$$

$$[\hat{H}_i(u), \hat{H}_j(u)] = 0$$

Proposed solution: fix parameter dependence

Let $H(u) = T + uV$ u – real parameter, T, V – N x N Hermitian matrices

Suppose we require a commuting partner also linear in u**:**

$$H_1(u) = T_1 + uV_1$$

$$[H(u), H_1(u)] = 0$$

$$\Updownarrow$$

$$[V, V_1] = 0, \quad [T, V_1] = [T_1, V], \quad [T, T_1] = 0$$

These commutation relations severely constraint matrix elements of T. **For a generic/typical** $H(u)$ **– no commuting partners except itself and identity.** ***Now can separate generic (no partners) from special (integrable).***

Proposed solution: fix parameter dependence

Let $H(u) = T + uV$ ***u*** **– real parameter, *T,V* – *N* x *N* Hermitian matrices**

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$$\Updownarrow$$

$$[V, V_1] = 0, \quad [T, V_1] = [T_1, V], \quad [T, T_1] = 0$$

In the simplest 3 x 3 case – single algebraic constraint on matrix elements T_{ij}

Xing condition: $\exists u_0 : \text{Discriminant}_\lambda \, |H(u_0) - \lambda I| = 0$ **also single constraint**

Moreover, xing condition = commutation condition, i.e.

$$[H_0(u), H_1(u)] = 0 \Longleftrightarrow \text{xings in } 3 \times 3 \text{ case!}$$

N x N Hamiltonians linear in a parameter separate into *two distinct classes = good notion of integrability*

$H(u) = T + uV$

No commuting partners linear in u other than itself and identity (typical) – nonintegrable, need $N^2/2$ real parameters to specify $H(u)$

Nontrivial commuting partners $H_k(u)=T_k+uV_k$ exist – integrable, turns out need less than $4N$ parameters – measure zero in the space of linear Hamiltonians

Classification by the number n of commuting partners

$n = N\text{-}1$ (maximum possible) – type 1 integrable system
$n = N\text{-}2$ – type 2
$n = N\text{-}3$ – type 3
…
$n = N\text{-}M$ – type M
…

Definition: A Hamiltonian operator $H \equiv H_0(u) = T_0 + uV_0$ is integrable if it has $n \geq 1$ nontrivial linearly independent commuting partners $H_i(u) = T_i + uV_i$

$[H_i(u), H_j(u)] = 0$ for all u and $i, j = 0, \ldots, n-1$

General member of the commuting family: $h(u) = \sum_{i=1}^{n} d_i H_i(u)$

Known parameter-dependent integrable models fall under this definition:

- **1d Hubbard model: *u=U/T*, Hamiltonian and first integral are linear in *u***
- **integrable XXZ spin chain: *u* = anisotropy, *$H_0(u)$* and *$H_1(u)$* are linear in *u***
- **Gaudin magnets (all integrable pairing models): *u*=spin exchange,**

  Hamiltonian and all integrals are linear in *u*

$$\hat{H}_i(u) = \hat{s}_i^z - u \sum_{j \neq i} \frac{\hat{\mathbf{s}}_i \cdot \hat{\mathbf{s}}_j}{\epsilon_i - \epsilon_j} \qquad [\hat{H}_i(u), \hat{H}_j(u)] = 0$$

$\mathbf{s}_i$ – quantum spins ϵ_i – real parameters

What can we achieve with this notion of quantum integrability? – quite a lot!!

Definition: Quantum Hamiltonian H_0 is integrable if…

Consequences:

1. Exact Solution
2. Generate (ensembles of) integrable models
3. Commuting integrals $[H_i, H_j]=0; i, j=0,1\ldots$
4. Energy level crossings?
5. Poisson level statistics *and exceptions*
6. Generalized Gibbs distribution for dynamics?

What can we achieve with this notion of quantum integrability? – quite a lot!!

✓ Construct (ensembles of) integrable models with any given number n of integrals!

$$[V, V_1] = 0, \quad [T, V_1] = [T_1, V], \quad [T, T_1] = 0$$

Simplest case: $n=N-1$ (type 1 – max # of integrals – analog of classical integrability)

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Every type-1 family contains a "reduced" Hamiltonian

$$\Lambda(u) = E + u|\gamma\rangle\langle\gamma|$$

Hermitian matrix E Arbitrary vector $|\gamma\rangle$

$\Downarrow$

N commuting $N \times N$ Hermitian matrices $H_i(u)$

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⇓

N commuting $N \times N$ Hermitian matrices $H_i(u)$

General member of the commuting family: $H(u) = \sum_{i=1}^{N} d_i H_i(u) = T + uV$

$$[H(u)]_{km} = u\gamma_k\gamma_m \left(\frac{d_k - d_m}{\varepsilon_k - \varepsilon_m}\right), \quad [H(u)]_{mm} = d_m - u\sum_{j\neq m}\gamma_j^2\left(\frac{d_j - d_m}{\varepsilon_j - \varepsilon_m}\right)$$

ϵ_k - eigenvalues of E, γ_k - components of $|\gamma\rangle$ ($2N$ arbitrary real parameters)

d_k- eigenvalues of T - another N arbitrary real numbers to fix a linear combination *within* the family. By construction $[T, E] = 0$.

Constructed *all* $n = N-1, N-2, N-3$ (types 1, 2, 3) and *some* for arbitrary other n

What can we achieve with this notion of quantum integrability? – quite a lot!!

✓ **Exact solution through a single algebraic equation for all types (cf. Bethe Ansatz)**

(type 1) $$\sum_j \frac{\gamma_j^2}{\lambda - \epsilon_j} = \frac{1}{u}, \quad E_k = \frac{u\gamma_k^2}{\lambda - \epsilon_k}, \quad |\lambda\rangle = \sum_j \frac{\gamma_j |j\rangle}{\lambda - \epsilon_j}$$

γ_j, ϵ_j - given; solve for λ

✓ **Number of level crossings as a function of the # (n) of commuting partners in an integrable family**

$$\text{\# of xings} = (N^2 - 5N + 2)/2 + n - 2k, \quad k = 1, 2, \ldots$$

Typically $\sim N^2/2$ xings

But it's also possible to have no xings

✓ **Yang-Baxter formulation**

scattering matrix $$S_{ij} = \frac{(\epsilon_j - \epsilon_i)I + 2g\Pi_{ij}}{(\epsilon_j - \epsilon_i) + g\,(\gamma_i^2 + \gamma_j^2)}$$

$$S_{ik}S_{jk}S_{ij} = S_{ij}S_{jk}S_{ik}$$

Applications: 1d Hubbard model (6 sites, 3 up/3 down spins

- Each block is characterized by a complete set of quantum #s (P, S^2, S_z…)
- We determine the type of each block

# of nontrivial integrals = Size – Type

Momenta $P = \pi/6, 5\pi/6$	
Size of the block	Its Type
8×8	Type 3
3×3	Type 1
16×16	Type 12
14×14	Type 3
3×3	Type 1

Momenta $P = \pi/3, 2\pi/3$	
Size of the block	Its Type
12×12	Type 7
14×14	Type 11
4×4	Type 1
2×2	—
16×16	Type 6

Results for Hubbard:

- In most blocks – exact solution in terms of a single equation – vast simplification over Bethe Ansatz (9 equations)!
- New symmetries in 1d Hubbard! # of nontrivial integrals linear in u=U/T is 14-3-1=10. Only one such integral was identified before

Applications: BCS (Richardson) and Gaudin models

$$\hat{H}_{\text{BCS}} = \sum_i 2\varepsilon_i \hat{s}_i^z - u \sum_{i,j} \hat{s}_i^- \hat{s}_j^+ = \sum_i 2\varepsilon_i \hat{H}_i$$

Gaudin magnet integrable family

$$\hat{H}_i(u) = \hat{s}_i^z - u \sum_{j \neq i} \frac{\hat{\mathbf{s}}_i \cdot \hat{\mathbf{s}}_j}{\epsilon_i - \epsilon_j}$$

One spin-flip sector $J_z = \{\max -1, \min +1\}$ is type-1 with $\gamma_i^2 = 2s_i$.
Other sectors – other types.

General member of the commuting family: $H(u) = \sum_{i=1}^{N} d_i H_i(u) = T + uV$

$$[H(u)]_{km} = u\gamma_k\gamma_m \left(\frac{d_k - d_m}{\varepsilon_k - \varepsilon_m}\right), \quad [H(u)]_{mm} = d_m - u \sum_{j \neq m} \gamma_j^2 \left(\frac{d_j - d_m}{\varepsilon_j - \varepsilon_m}\right)$$

Set $d_i = \varepsilon_i$ and $\gamma_i = 1$ to get BCS, $\hat{H}_{\text{BCS}} = \Lambda(u) = E + |\gamma\rangle\langle\gamma|$

Every type-1 family contains a "reduced" Hamiltonian

Integrable Matrix Theory (IMT) – ensemble theory of quantum integrability

$\text{Two matrices } [T, E] = 0 \text{ \& vector } |\gamma\rangle \iff \text{type 1 } H(u) = T + uV$

Other types similarly given in terms of two commuting matrices and a vector $|\gamma>$

To generate an integrable matrix with any prescribed number of integrals – generate T, E **and** $|\gamma>$

Integrable Matrix Theory (IMT) – ensemble theory of quantum integrability

Two matrices $[T, E] = 0$ & vector $|\gamma\rangle \Longleftrightarrow$ type 1 $H(u) = T + uV$

Other types similarly given in terms of two commuting matrices and a vector $|\gamma>$

To generate an *ensemble* of integrable matrices with any prescribed number of integrals – generate an *ensemble* of *T, E* and $|\gamma>$

Type 1 in the shared eigenbasis of *T & E:*

$$[H(u)]_{km} = u\gamma_k\gamma_m\left(\frac{d_k - d_m}{\varepsilon_k - \varepsilon_m}\right), \quad [H(u)]_{mm} = d_m - u\sum_{j\neq m}\gamma_j^2\left(\frac{d_j - d_m}{\varepsilon_j - \varepsilon_m}\right)$$

d_k, ε_k – eigenvalues of T, E. γ_k – components of $|\gamma\rangle$

Q: What is the natural probability density function for this ensemble? How do we generate most typical/random integrable models?

$$P(T, E, \gamma) = ?$$

Two matrices $[T,E]=0$ & vector $|\gamma\rangle \Longleftrightarrow$ type 1 $H(u)=T+uV$

Q: What is the natural probability density function for this ensemble? How do we generate most typical/random integrable models?

$$P(T,E,\gamma)=?$$

Similar to Random Matrix Theory, two ways to derive ***P(T, E, γ)***

1. Maximize the entropy of the distribution (least information, most unbiased choice. Generalized Gibbs Ensemble follows from the same principle)

$$S[P]=-\langle\ln(P)\rangle=-\int P(T,E,\gamma)\ln(P(T,E,\gamma))d\gamma\, dT\, dE$$

$$\langle\mathrm{Tr}\,T\rangle,\langle\mathrm{Tr}\,T^2\rangle,\langle\mathrm{Tr}\,E\rangle,\langle\mathrm{Tr}\,E^2\rangle=\mathrm{const}$$

Integration over constrained space: $[T,E]=0, \quad |\gamma|=1$

2. Statistical independence + rotational invariance of ***P(T, E, γ)***. ***T, E, γ*** are given by RMT results projected onto the constrained space ***[T, E] = 0***

Integrable Matrix Theory (IMT)

Both approaches yield the same answer, *β =1,2* for Hermitian, real-symmetric

$$P(d,\varepsilon,\gamma) \propto \delta\left(1-|\gamma|^2\right)\prod_{i<j}|\varepsilon_i-\varepsilon_j|^\beta |d_i-d_j|^\beta e^{-\sum_k \varepsilon_k^2} e^{-\sum_k d_k^2}$$

d_k, ε_k – eigenvalues of T, E. γ_k – components of $|\gamma\rangle$

T, E - random matrices with uncorrelated eigenvalues

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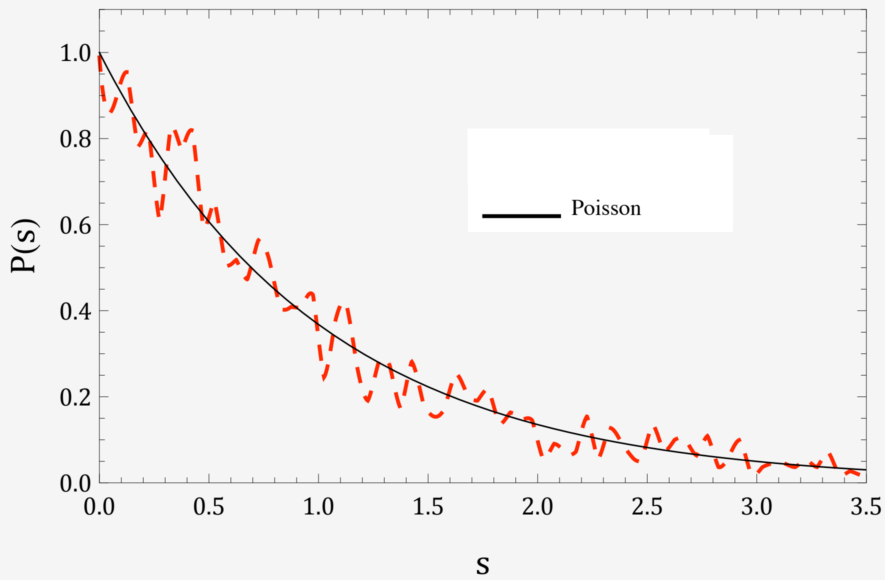
$$[H(u)]_{km} = u\gamma_k\gamma_m\left(\frac{d_k-d_m}{\varepsilon_k-\varepsilon_m}\right), \quad [H(u)]_{mm} = d_m - u\sum_{j\neq m}\gamma_j^2\left(\frac{d_j-d_m}{\varepsilon_j-\varepsilon_m}\right)$$

Similar but more involved construction for other types, see arXiv:1511.02446

Now can study *ensembles of integrable matrices* and obtain integrable counterparts of RMT results as opposed to only a spectral statistics of specific integrable models

Integrable Matrix Theory, Level Statistics (numerics)

I. **Statistics are typically Poisson as long as the # of integrals (=size-type) isn't too small**



Level spacing distribution for a 4000 x 4000 real symmetric integrable matrix *H(u)=T+uV* at *u=1*

Integrable Matrix Theory, Level Statistics

I. **Statistics are typically Poisson as long as the # of integrals (=size-type) isn't too small**

II. **There are two exceptions to Poisson statistics**

   A. **At $u=0$ the statistics is Wigner-Dyson. Can engineer any statistics in $H(u)=T+uV$ at isolated value of the coupling $u=u_0$**

T, E - random matrices with uncorrelated eigenvalues d_i, ε_i

Can arbitrarily chose either T or V, but not both, i.e. can have a desired statistics e.g. at $u=0$, but not at all u

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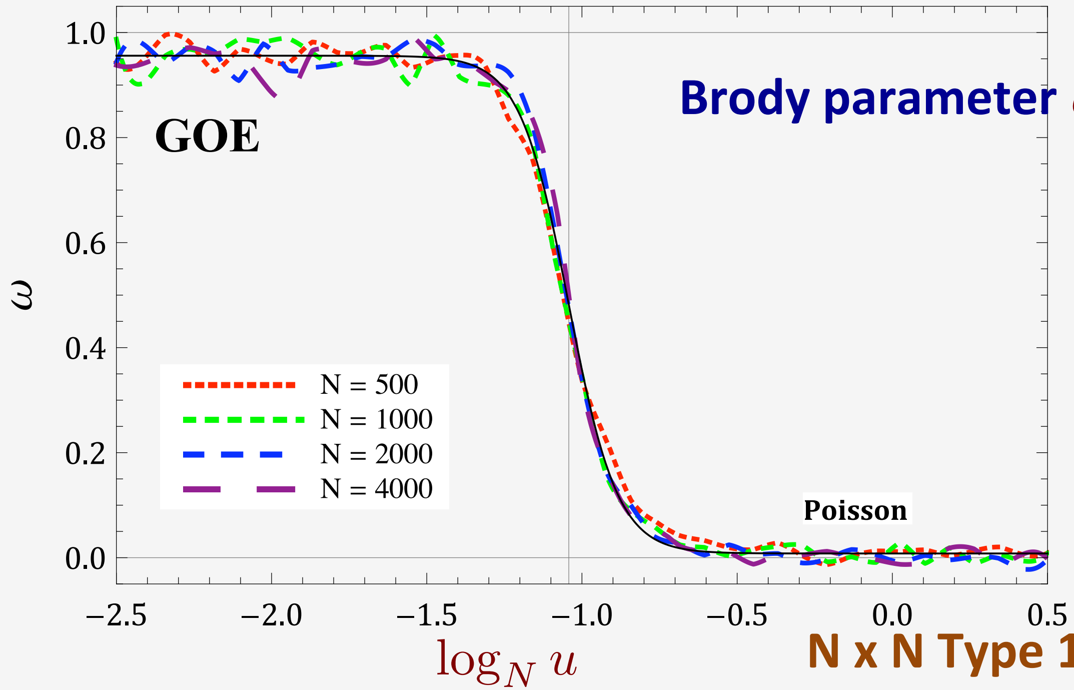
But it becomes Poisson already at $(u-u_0) \propto 1/N$



Brody parameter ω as a function of $\log_N (u)$

Brody distribution:

$$P(s,\omega) = as^\omega e^{-bs^{\omega+1}}$$

$$P(s,1) = \frac{\pi}{2} s e^{-\frac{\pi}{4}s^2} \text{ - Wigner}$$
$$P(s,0) = e^{-s} \text{ - Poisson}$$

N x N Type 1, # of integrals = N - 1

Exceptions to Poisson Statistics in IMT

A. **At $u=0$ the statistics is Wigner-Dyson. Can engineer any statistics in $H(u)=T+uV$ at isolated value of the coupling $u=u_0$**

T, E - random matrices with uncorrelated eigenvalues d_i, ε_i

B. **Statistics is non-Poisson when normally uncorrelated parameters become correlated (atypical integrable models)**

$T=f(E)$, $d_i=f(\varepsilon_i)$ - non-Poisson with strong level repulsion, e.g. BCS model has $d_i=\varepsilon_i$

General member of the commuting family: $H(u)=\sum_{i=1}^{N} d_i H_i(u) = T+uV$

Type 1 in the shared eigenbasis of *T & E*:

$$[H(u)]_{km} = u\gamma_k\gamma_m\left(\frac{d_k-d_m}{\varepsilon_k-\varepsilon_m}\right), \quad [H(u)]_{mm} = d_m - u\sum_{j\neq m}\gamma_j^2\left(\frac{d_j-d_m}{\varepsilon_j-\varepsilon_m}\right)$$

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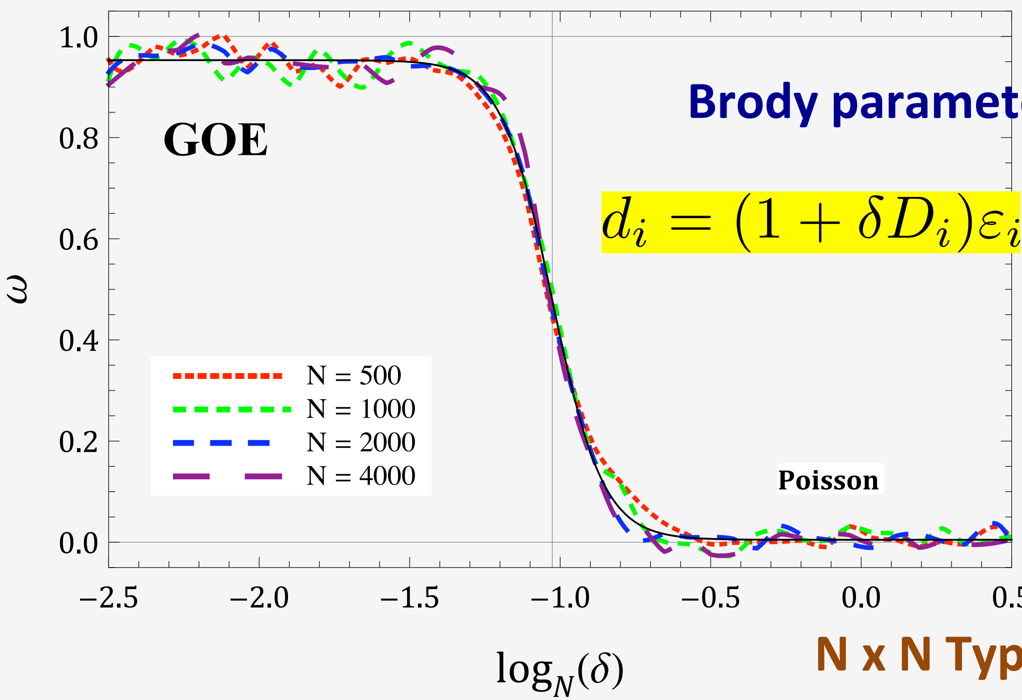
B. Statistics is non-Poisson when normally uncorrelated parameters become correlated (atypical integrable models)

Reverts to Poisson at deviations $\delta \propto 1/N$ from such points



Brody parameter ω as a function of $\log_N(\delta)$

$d_i = (1 + \delta D_i)\varepsilon_i$ $\quad D_i - \mathcal{O}(1)$ random number

N x N Type 1, # of integrals = N – 1, $u=1$

Exceptions to Poisson Statistics in IMT

A. At *u=0* the statistics is Wigner-Dyson. Can engineer any statistics in *H(u)=T+uV* at isolated value of the coupling $u=u_0$

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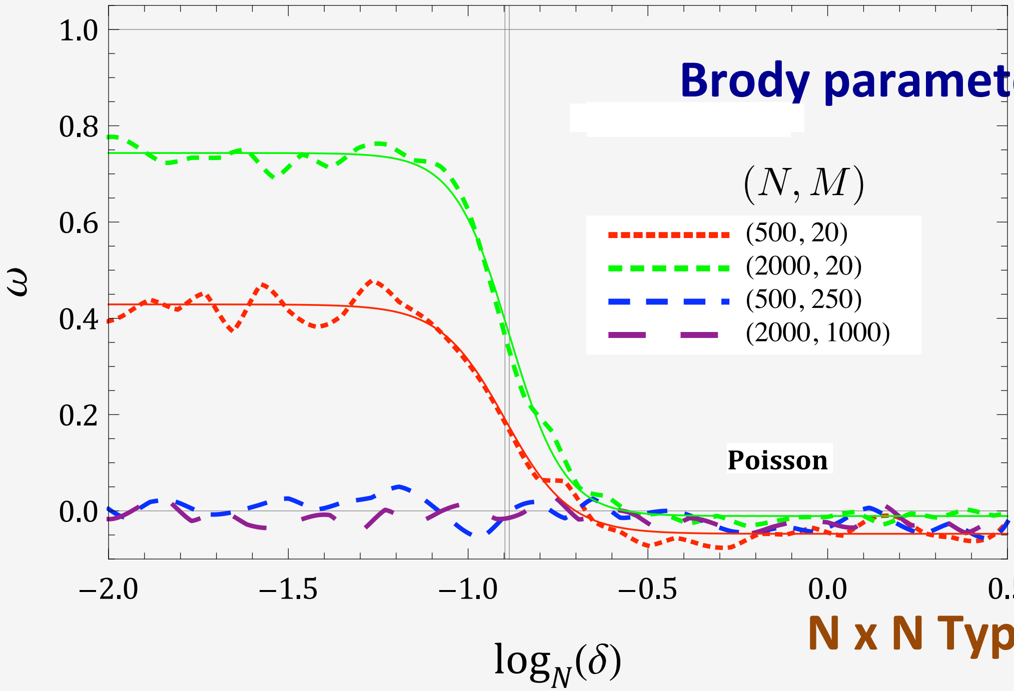


Brody parameter ω as a function of $\log_N(\delta)$

$$d_i = (1 + \delta D_i)\varepsilon_i$$

D_i – $\mathcal{O}(1)$ random number

N x N Type M, # of integrals = N - M, *u=1*

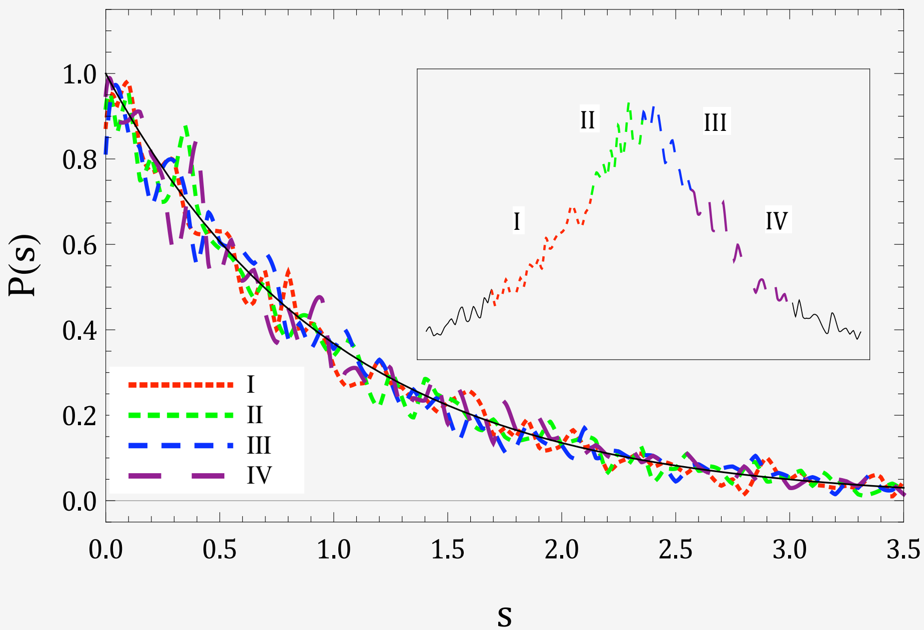
Integrable Matrix Ensembles are ergodic (numerics)

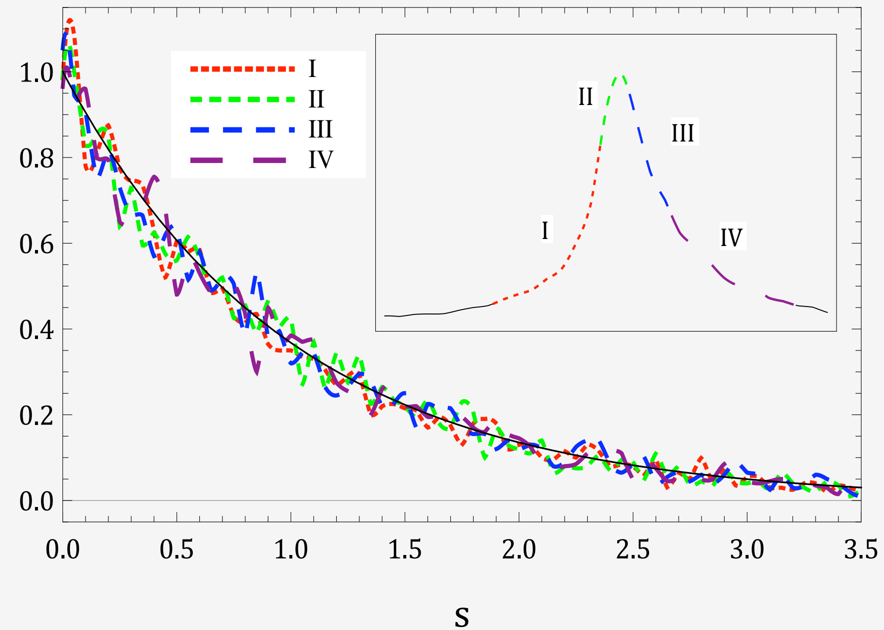
At large N, spectral statistics is independent of the region R of the spectrum and coincides with the ensemble distribution of j^{th} spacing

$$\lim_{N\to\infty} P_{i,N,R}(s) \approx e^{-s} \approx \lim_{N\to\infty} p_{N,j}(s)$$

i^{th} matrix (member) of the ensemble

j^{th} spacing across the entire ensemble





Single N x N Type 1 matrix,
$N = 20000$, $u=1$, # of integrals = 19999

Single N x N Type 10000 matrix,
$N = 20000$, $u=1$, # of integrals = 10000

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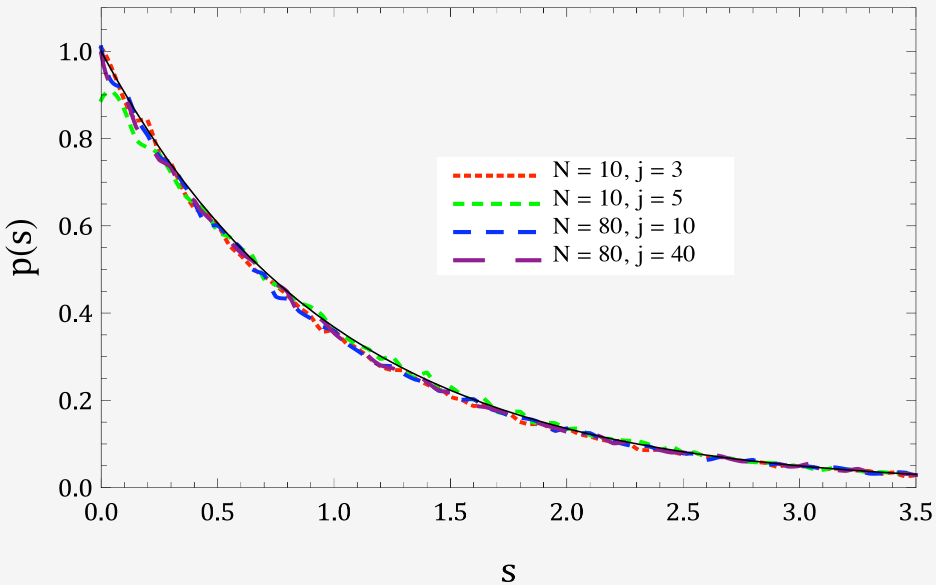
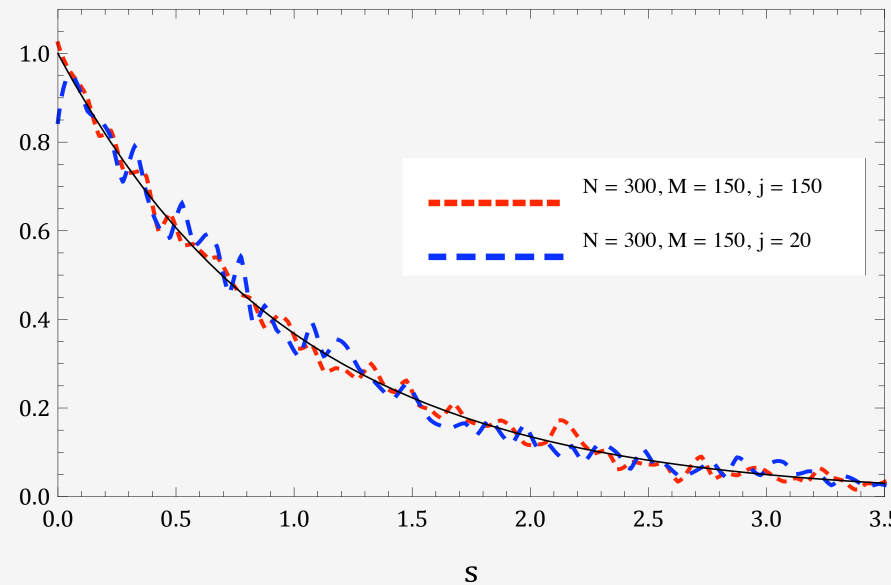


$p_{N,j}(s)$ - distribution of $j^{\rm th}$ spacing in $\sim 10^5$ type 1 $N\times N$ matrices

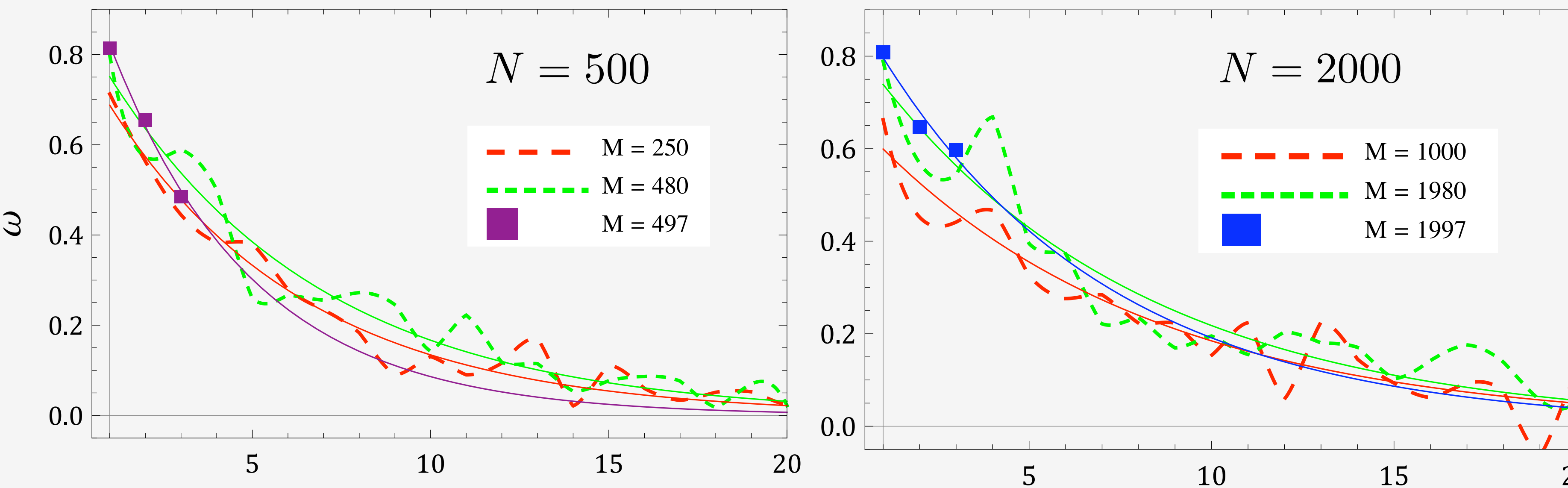


$p_{N,j}(s)$ - distribution of $j^{\rm th}$ spacing in $\sim 10^4$ type M $N\times N$ matrices

Q: How many nontrivial integrals should a system have so that its level statistics is Poisson? (numerics)

# of nontrivial integrals = Size – Type
= *N - M*

$$H(u) = \sum_{i=1}^{k} d_i H_i(u), \quad k \le N - M$$



Brody parameter ω as a function of k for $N \times N$ type M matrices.
Fit: $a \exp(-bk/\ln N)$. $b = (1.13, 1.04; 0.99, 1.03)$ for $M = (250, 480; 1000, 1980)$

$\omega = 1$ – GOE, $\omega = 0$ – Poisson

# of integrals needed $\propto \ln N$ (log of Hilbert space dim)?

Type 1 and short-range impurity problem

Every type-1 family contains a "reduced" Hamiltonian

$$\Lambda(u) = E + u|\gamma\rangle\langle\gamma|$$

$\equiv \hat{H}_{\mathrm{BCS}}$ in 1 Cooper pair sector,
GOE (exception from typical Poisson)

Type 1 *H(u)*: # of integrals =*N-1* (max # – analog of classical integrability)

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$\equiv \hat{H}_{\mathrm{BCS}}$ in 1 Cooper pair sector,
GOE (exception from typical Poisson)

Also, $\equiv \hat{H}_{\mathrm{imp}}$ short-range impurity, $u\delta(r)$, in a quantum dot

Aleiner & Matveev, PRL (1998)
Bogomolny et. al. PRL (2000)

$$\sum_i \frac{\gamma_i^2}{\lambda_m - \epsilon_i} = \frac{1}{u}$$

ε_i - eigenvalues of E
λ_m - eigenvalues of $\Lambda(u)$

$$P(\{\lambda_m, \varepsilon_i\}) = \ldots, \; P(\{\lambda_m\}) = \mathrm{GOE?}\;\; \text{At least } P(s) \propto s^{\beta}$$

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ε_i - eigenvalues of E
λ_m - eigenvalues of $\Lambda(u)$

$P(\{\lambda_m, \varepsilon_i\}) = \dots,\ P(\{\lambda_m\}) = \mathrm{GOE?}$ At least $P(s) \propto s^\beta$

General member of the commuting family: $H(u) = \sum_{i=1}^{N} d_i H_i(u) = T + uV$

Eigenvalues of $H(u)$: $E_m = u \sum_i \frac{d_i \gamma_i^2}{\lambda_m - \varepsilon_i}$, d_i - GOE

***Q:* Can we determine the statistics of eigenvalues of *H(u)* analytically?**

Type 1: Second "Hamiltoniazation" & Localization

Every type-1 family contains a "reduced" Hamiltonian

$$\Lambda(u) = E + u|\gamma\rangle\langle\gamma|$$

All members of a commuting family have the same eigenstates – can consider any one of them

$$\Lambda(u) \to \hat{H}(\Lambda) = \sum_{ij} \Lambda_{ij}(u) c_i^\dagger c_j$$

$$[A, B] = 0 \Longleftrightarrow [\hat{H}(A), \hat{H}(B)] = 0$$

Type 1: Second “Hamiltoniazation” & Localization

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$$\Lambda(u) \to \hat{H}(\Lambda) = \sum_{ij} \Lambda_{ij}(u) c_i^\dagger c_j$$

$$\Lambda(u) \to \hat{H}(u) = \sum_i \varepsilon_i \hat{n}_i + u \sum_{ij} \gamma_i \gamma_j c_i^\dagger c_j$$

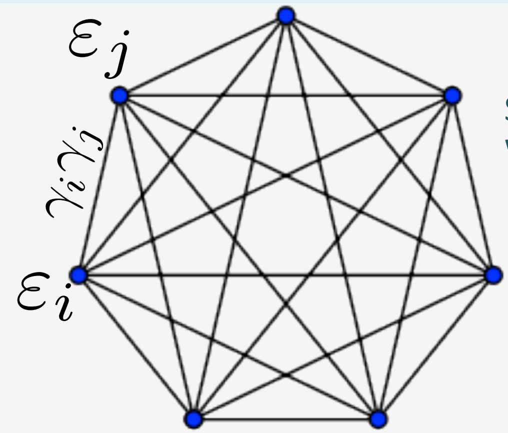
Infinite range hopping in the Hilbert space between the eigenstates of ***u=0*** or generally ***u=u₀*** Hamiltonian

$$H_i(u) \to \hat{H}_i(u) = \hat{n}_i + u \sum_{j \neq i} \frac{\gamma_i \gamma_j (c_i^\dagger c_j + c_j^\dagger c_i) - \gamma_i^2 \hat{n}_j - \gamma_j^2 \hat{n}_i}{\varepsilon_i - \varepsilon_j}$$

$$[\hat{H}_i(u), \hat{H}_j(u)] = 0, \quad \hat{H}(u) = \sum_i \varepsilon_i \hat{H}_i(u) + \text{const}$$

Type 1: Second "Hamiltoniazation" & Localization

$$\hat{H}(u) = \sum_i \varepsilon_i \hat{n}_i + u \sum_{ij} \gamma_i \gamma_j c_i^\dagger c_j \quad u < 0$$



Source: Wikipedia

ε_i, γ_i - random (arbitrary)

Complete graph, (N-1)-simplex

Exact solution: $\displaystyle \sum_{i=1}^{N} \frac{\gamma_i^2}{\lambda_m - \epsilon_i} = \frac{1}{u}, \quad |\lambda_m\rangle = \sum_{i=1}^{N} \frac{\gamma_i c_i^\dagger}{\lambda_m - \epsilon_i} |0\rangle$

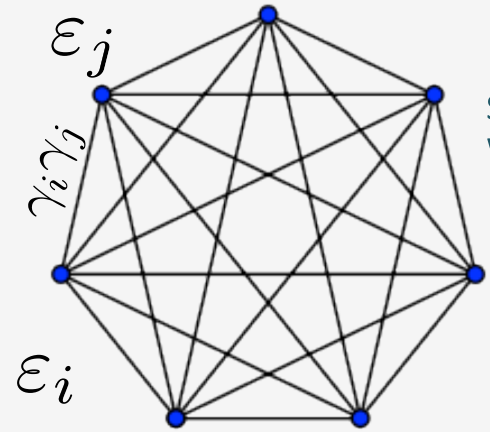
Participation ratio: $\displaystyle \mathrm{PR}_{\lambda_m} = \frac{\left[\sum_i \frac{\gamma_i^2}{(\lambda_m - \varepsilon_i)^2}\right]^2}{\sum_i \frac{\gamma_i^4}{(\lambda_m - \varepsilon_i)^4}}$

All states are localized except the ground state. Ground state delocalizes at $|u_c|/\delta \sim 1/\log(N)$

δ – average level spacing between ε_i

$$\hat{H}(u) = \sum_i \varepsilon_i \hat{n}_i + u \sum_{ij} \gamma_i \gamma_j c_i^\dagger c_j \quad u < 0$$

ε_i, γ_i - random (arbitrary)



Source: Wikipedia

Complete graph, (N-1)-simplex

Excited states localized at any *u* [see also Ossipov (2013)]

Ground state extended for $|u| >> 1/\log(N)$. Delocalization of the ground state at $|u_c|/\delta \sim 1/\log(N)$ corresponds to the superconducting transition in H_{BCS}

Can explicitly determine exact PR in $N \to \infty$ limit when ε_i, γ_i are distributed with a smooth density, i.e. neglecting mesoscopic fluctuations in the DoS

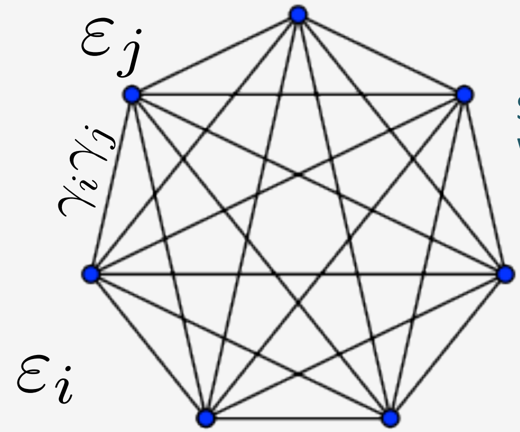
e.g. for $\varepsilon_i \in [-W/2, W/2]$ with $\rho(\varepsilon_i) = \mathrm{const}$ and $\gamma_i = 1$

Excited states: $\mathrm{PR}_{\lambda_m} = \dfrac{3 + 3f^2(\varepsilon_m)}{1 + 3f^2(\varepsilon_m)}, \quad f(x) = -\dfrac{\delta}{\pi u} + \dfrac{1}{\pi} \ln \dfrac{2x + W}{W - 2x}, \quad 1 \le \mathrm{PR}_{\lambda_m} \le 3$

Ground state: $\mathrm{PR}_{g.s.} = \dfrac{3N}{1 + 2\cosh(\delta/u)} \propto N$

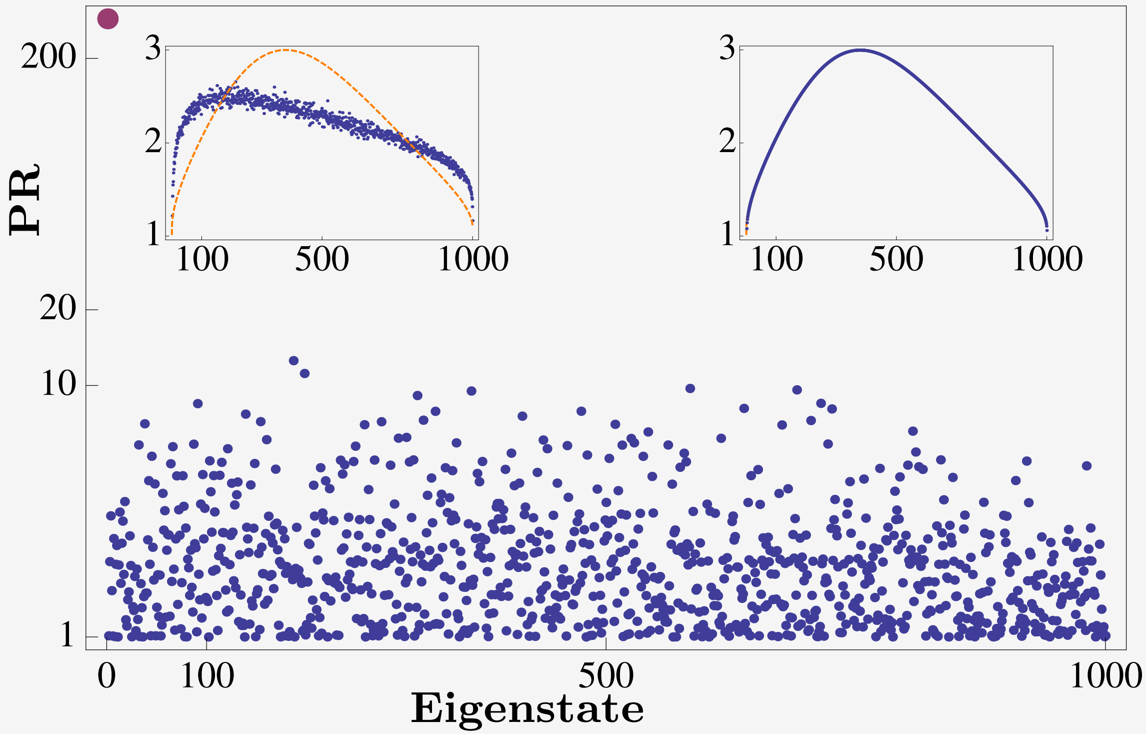
$$\hat{H}(u) = \sum_i \varepsilon_i \hat{n}_i + u \sum_{ij} \gamma_i \gamma_j c_i^\dagger c_j \quad u < 0$$

ε_i, γ_i - random (arbitrary)



Source: Wikipedia

Mesoscopic fluctuations:



Excited states:

$$\mathrm{PR}^{\max}_{\lambda_m} \approx \alpha \ln N$$

due to clustering in ε_i

PR for $u = -.004$, $N = 10^3$. ε_i, γ_i are independent random numbers uniformly distributed in interval $(-1, 1)$

What can we achieve with this notion of quantum integrability? – quite a lot!!

Definition: Quantum Hamiltonian H_0 is integrable if…

Consequences:

1. Exact Solution
2. Generate (ensembles of) integrable models
3. Commuting integrals $[H_i, H_j]=0;\ i, j=0,1\ldots$
4. Energy level crossings?
5. Poisson level statistics *and exceptions*
6. Generalized Gibbs Ensemble for dynamics?

Proof of Generalized Gibbs Ensemble for Type 1

$$\rho = Z^{-1} e^{-\sum_i \beta_i H_i} \qquad \langle O(t)\rangle_{t\to\infty} = \mathrm{Tr}\,\rho O\,?$$

$$\langle \mathrm{in}|H_i|\mathrm{in}\rangle = \mathrm{Tr}\,\rho H_i$$

Type 1 *H(u)*: # of integrals =*N-1* (max # – analog of classical integrability)

$$\langle O(t)\rangle_{t\to\infty} = \sum_{m=1}^{N} |c_m|^2 O_{mm} \qquad |\mathrm{in}\rangle = \sum_m c_m |\lambda_m\rangle$$

(diagonal ensemble)

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(diagonal ensemble)

$\#$ of integrals $= N-1 = \#$ of parameters $\beta_i = \#$ of independent $|c_m|$,
i.e. enough integrals to reproduce all $|c_m|$

Can determine β_i such that $\langle O(t) \rangle_{t\to\infty} = \mathrm{Tr}\, \rho O$

Specifically, $\beta_i = \dfrac{1}{u} \displaystyle\sum_m \frac{\ln |c_m|^2}{\mathcal{N}_m^2 (\lambda_m - \varepsilon_i)}$

As in Classical Mechanics integrals fully constrain the motion apart from linear in time phases (angle variables) that cancel out upon time-averaging. In both cases integrals completely fix infinite time averages.

Proof of Generalized Gibbs Ensemble for Type 1

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$$\langle O(t) \rangle_{t\to\infty} = \mathrm{Tr}\, \rho O\,?$$

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$H_{\mathrm{eff}}(u)$ **– a member of the commuting family**

General member of the commuting family: $H(u) = \sum_{i=1}^{N} d_i H_i(u) = T + uV$

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For quantum quenches, $u_i \to u_f$, in type 1 $H_{\mathrm{eff}}(u) \neq \beta H(u)$

The system effectively thermalizes with a different Hamiltonian (related to the localization of eigenstates *H*(u_f) in the eigenspace of *H*(u_i) seen above)

In a nonintegrable system expect $H_{\mathrm{eff}} = \beta H(u)$,
e.g. if we take T and V to be random matrices, $H_{\mathrm{eff}} = 0 \times H(u)$

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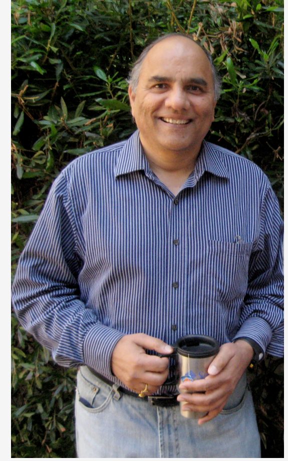

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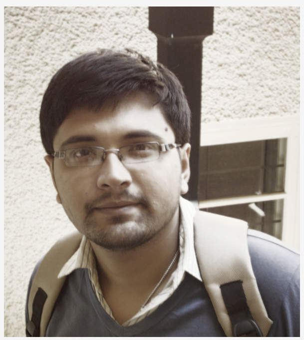

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