Scattering Amplitudes in Field Theory, Multiple Polylogarithms and the Coaction Principle

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Outline

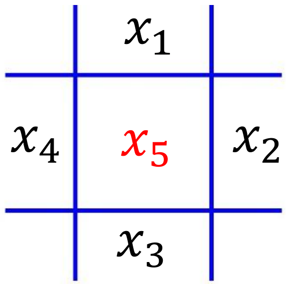
- Introduction
- Electron g-2 and LHC
- Euler sums and iterated integrals
- The co-action principle
- Electron g-2 redux
- Scattering in planar N=4 SYM
- ϕ^4 theory for ε expansion of $D = 3$ critical exponents
- Summary and outlook

Introduction

- Earlier we heard about loop-level scattering amplitudes in string theory from Eric d'Hoker.
- String theory has a scale $\alpha' \sim M_{\mathrm{Planck}}^{-2}$
- Most of the results Eric described were "low energy", kinematic variables $s, t, u \ll M_{\mathrm{Planck}}^{2}$, where the main results are polynomial in s, t, u. But because the world-sheet is a torus (at one loop), the modular parameter τ appears.
- In a field theory of massless particles, the only scales are kinematic.
- There is no "obvious" τ, but in complicated enough loop integrals, denominator singularities are parametrized by elliptic curves (or even Calabi-Yau n-folds).

One-loop 4-mass box integral

$L=1$



$n=8$

$$= \int d^4 x_5 \frac{x_{13}^2 x_{24}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

$$= \text{Bloch-Wigner dilogarithm}$$

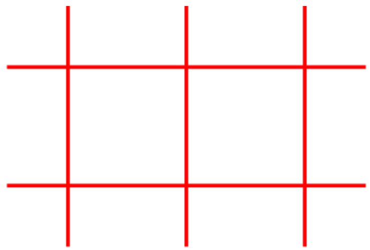
$$= \mathrm{Im}[\mathrm{Li}_2(z)] + \arg(1-z)\ln|z|$$

$$z\bar{z} = \frac{x_{12}^2\, x_{34}^2}{x_{13}^2\, x_{24}^2}$$

- Example of single-valued multiple polylogarithm.
- Real analytic function on $\mathbb{C}[z,\bar{z}] - \{0,1,\infty\}$
- Branch cuts in $z,\bar{z}$ cancel each other

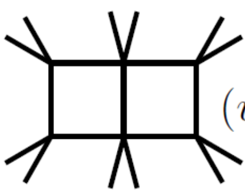
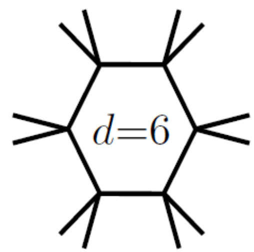
Two-loop train-track integral

Brown, Levin, 1110.6917;
Broedel et al., 1712.07089;
Eric's talk

$L=2$  = elliptic polylogarithm

$n = 10$

(analytic formula of this type not yet known)
(depends on 9 variables, generically)

 $(u,\ldots) = -\frac{1}{2}\int_u^{+\infty}\frac{du'}{u'}$ 

$(u',\ldots)$

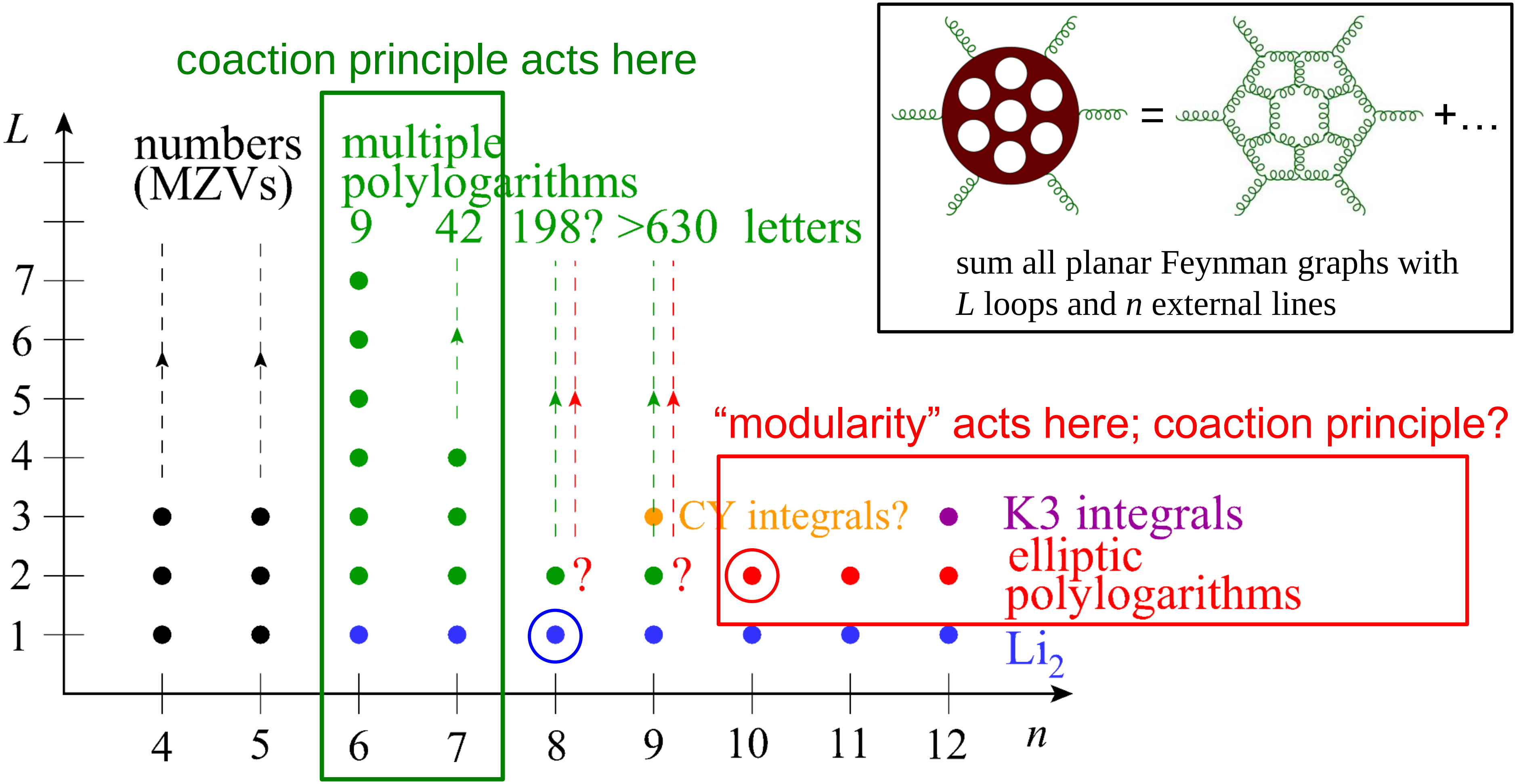
$$= \int_u^{\infty}\frac{du'}{\sqrt{\tilde{Q}(u')}} \times \left(\mathrm{Li}_3(\cdots) + \cdots\right)$$

Paulos, Spradlin, Volovich, 1203.6362

Caron-Huot, Larsen, 1205.0801

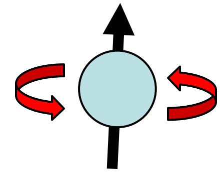
- $\tilde{Q}(u)$ a quartic polynomial from setting 7 propagators to zero, defines elliptic curve (with punctures from additional variables)
- Recent (formal?) series representation Ananthanarayan et al., 2007.08360

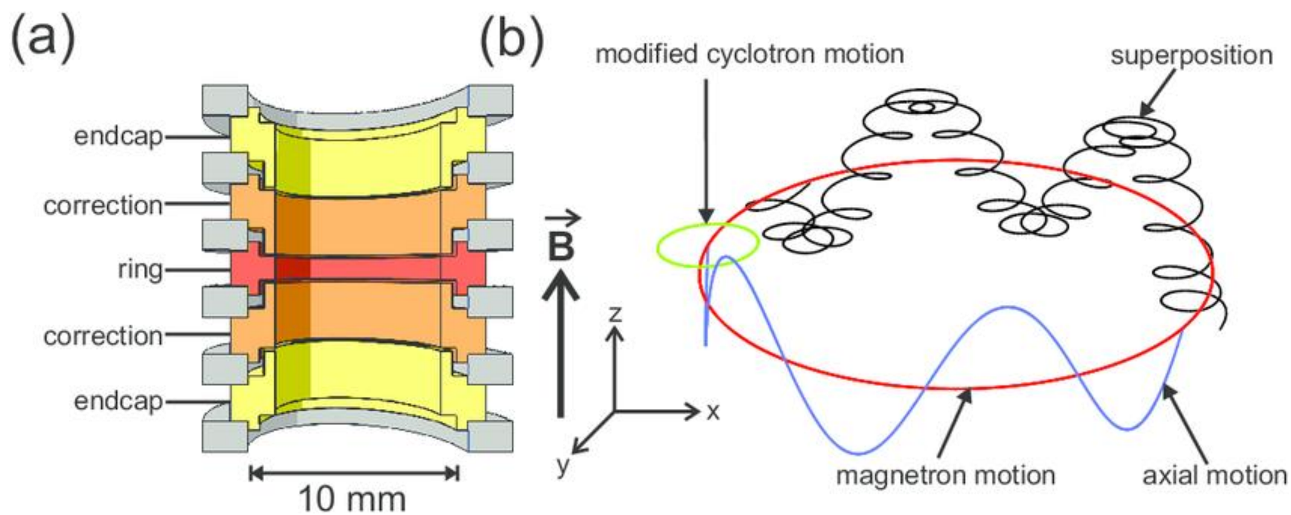
One context:
Loop amplitudes in planar N=4 SYM depend on $3(n\text{-}5)$ variables



But first:
the electron anomalous magnetic moment, a (precious) “baby” scattering amplitude

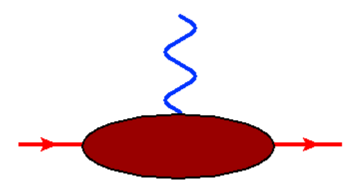
$$\vec{\mu}_e = g_e \frac{e\hbar}{2m_e c}\vec{S}_e$$





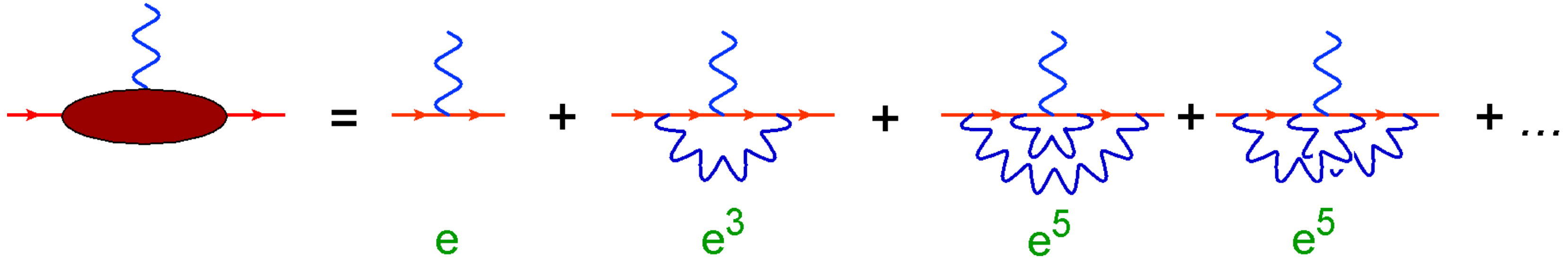
BASE, Eur. Phys. J. ST 224, 16, 3055 (2015)

Measurement doesn’t look much like particle scattering, but $a_e = (g_e - 2)/2$ can be computed from spin-flip part of $\gamma e \rightarrow e$ process as photon momentum → 0.



The loop expansion

- **Feynman:** Draw all diagrams with specified incoming and outgoing particles, weight them by coupling factors at each vertex. For a given process, extra powers of coupling for each closed loop.



In quantum electrodynamics (QED), each additional loop suppressed by (Sommerfeld's) fine structure constant:

$$\frac{e^2}{4\pi\hbar c} \equiv \alpha = \frac{1}{137.035999\ldots}$$

QED state of numerical art today: 5 loops, 12,672 diagrams

30 gauge invariant sets

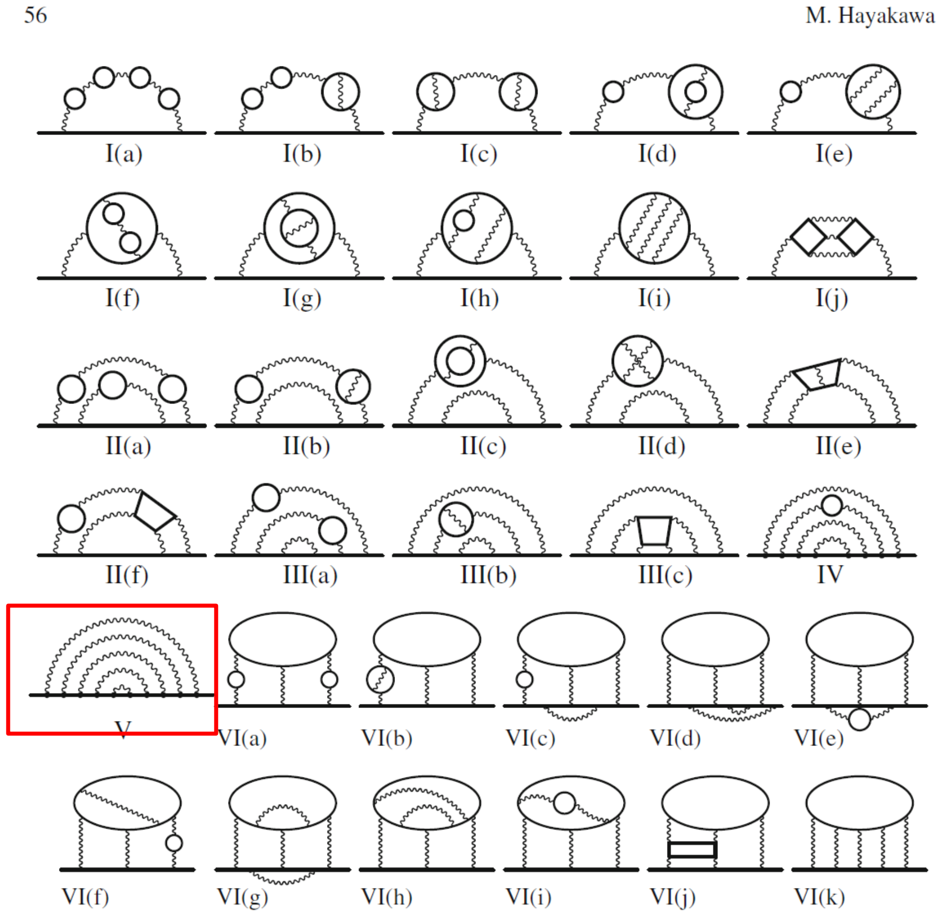


Fig. 2.7 Gauge-invariant subsets of self-energy-like diagrams at the tenth order

The most difficult set, 6354 diagrams, leading to 389 integrals. Evaluated numerically after Feynman Parameterization.

Aoyama, Hayakawa, Kinoshita, Nio, Watanabe, 2006-2017

Seven decades of g_e-2 theory

$$
\begin{aligned}
a_e &= \frac{\alpha}{\pi} \cdot \frac{1}{2} \\
&+ \left(\frac{\alpha}{\pi}\right)^2 \left[\frac{197}{144} + \frac{\pi^2}{12} - \frac{\pi^2}{2}\ln 2 + \frac{3}{4}\zeta_3\right] \\
&+ \left(\frac{\alpha}{\pi}\right)^3 \left[\frac{28259}{5184} + \frac{17101}{810}\pi^2 - \frac{298}{9}\pi^2 \ln 2 + \frac{139}{18}\zeta_3 \right. \\
&\qquad - \frac{239}{2160}\pi^4 + \frac{100}{3}\left\{\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{1}{24}(\ln^4 2 - \pi^2 \ln^2 2)\right\} \\
&\qquad \left. + \frac{83}{72}\pi^2\zeta_3 - \frac{215}{24}\zeta_5\right] + \ldots \\
&= 0.5\frac{\alpha}{\pi} \\
&\quad - 0.328478965579 1\ldots \left(\frac{\alpha}{\pi}\right)^2 \\
&\quad + 1.181241456587 2\ldots \left(\frac{\alpha}{\pi}\right)^3 \\
&\quad - 1.912245764926 4\ldots \left(\frac{\alpha}{\pi}\right)^4 \\
&\quad + 6.7(\pm 0.2) \left(\frac{\alpha}{\pi}\right)^5
\end{aligned}
$$

fully analytic

$$
\zeta_p = \sum_{k=1}^{\infty} \frac{1}{k^p} \qquad \mathrm{Li}_4\left(\frac{1}{2}\right) = \sum_{k=1}^{\infty} \frac{1}{2^k\, k^4}
$$

Schwinger 1948 (first term)

Karplus, Kroll 1950; Petermann 1957; Sommerfield 1957 (second term)

Kinoshita, Cvitanovic 1972; Laporta, Remiddi 1996 (third term)

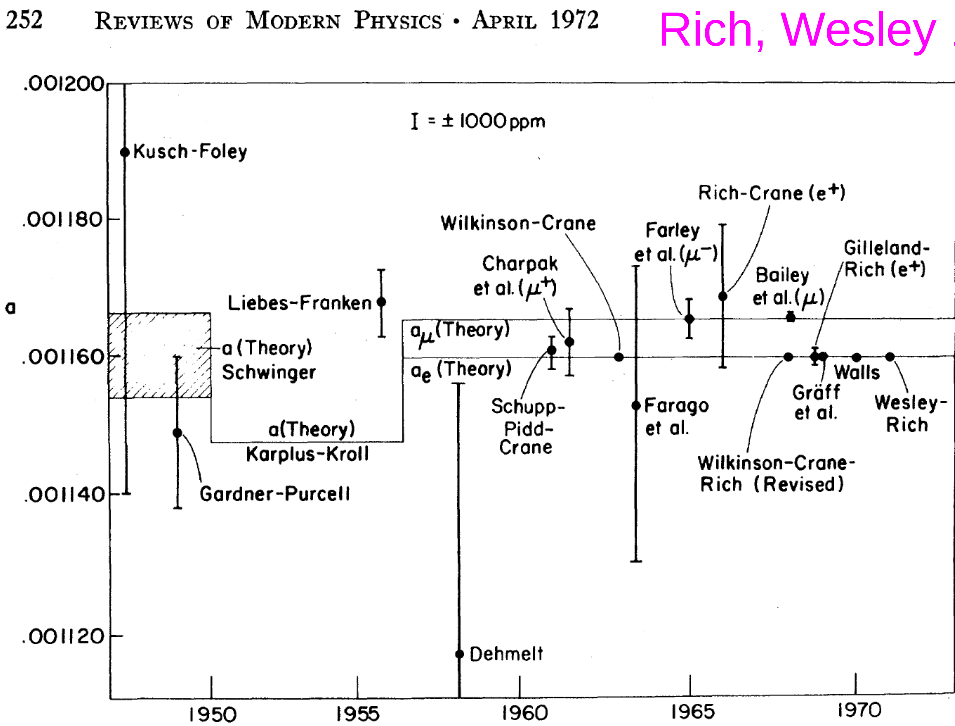
numerical

Aoyama, Hayakawa, Kinoshita, Nio, 2005-2007; Laporta arXiv:1704.06996 ($(\alpha/\pi)^4$ term)

(+ mass-dep.)

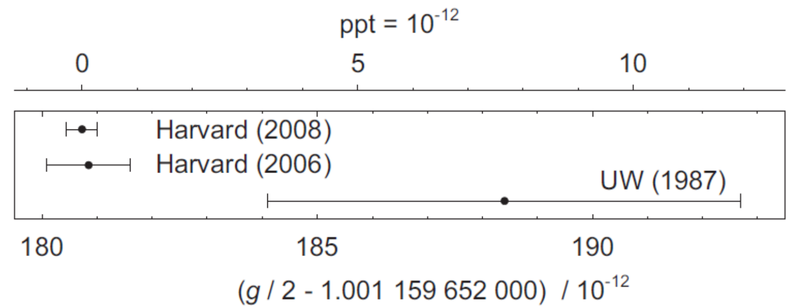
Aoyama, Hayakawa, Kinoshita, Nio, Watanabe, 2006-2017 ($(\alpha/\pi)^5$ term)

Matches incredible advances in experimental precision



Rich, Wesley 1972

Van Dyck, Schwinberg, Dehmelt, 1977-1987



Hanneke, Hoogerheide, Gabrielse, 2006-2010

Measuring Earth-Moon distance to width of human hair: 10^{-13}

$$g/2 = 1.001\ 159\ 652\ 180\ 73\ (28) \qquad [0.28\ \text{ppt}]$$

What numbers appear (or don't) in g_e-2?

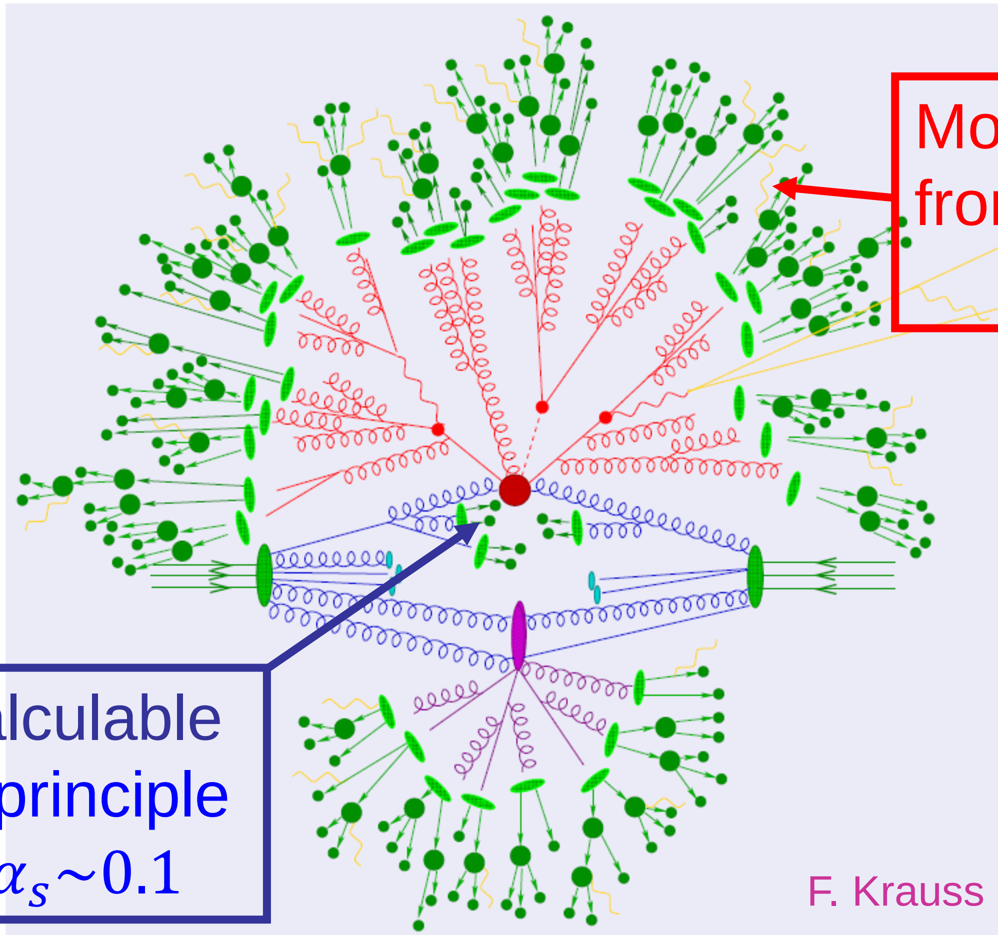
- $a_e^{(1)} = \frac{1}{2}$
- $a_e^{(2)} = \frac{197}{144} + \frac{\pi^2}{12} - \frac{\pi^2}{2}\ln 2 + \frac{3}{4}\zeta(3)$
- $a_e^{(3)} = \frac{28259}{5184} + \frac{17101}{810}\pi^2 - \frac{298}{9}\pi^2\ln 2 + \frac{139}{18}\zeta(3) - \frac{239}{2160}\pi^4 + $
  $+ \frac{100}{3}\left\{\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{1}{24}(\ln^4 2 - \pi^2\ln^2 2)\right\} + \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5)$
- Assign "transcendental weight" w to numbers in the formulas:
  $$w[\pi] = w[\ln(x)] = 1,$$
  $$w[\zeta(n)] = w[\mathrm{Li}_n(x)] = n$$
- Apparently $w \leq 2L - 1$ $(L = \text{loop order})$, but some terms are missing
- E.g. no $\ln 2, \ln^2 2$ or $\ln^3 2$ in $a_e^{(2)}$
- Do missing terms at lower loops imply missing terms at higher loops? **YES**, once we understood how to write them
- Do such patterns appear in other contexts? **YES**

Large Hadron Collider

Quantum chromodynamics at the LHC

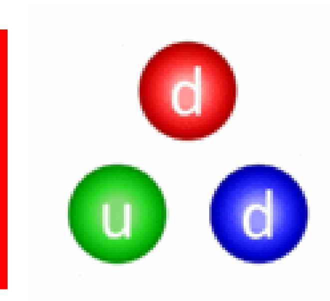
Energies enormous, many kinematic variables

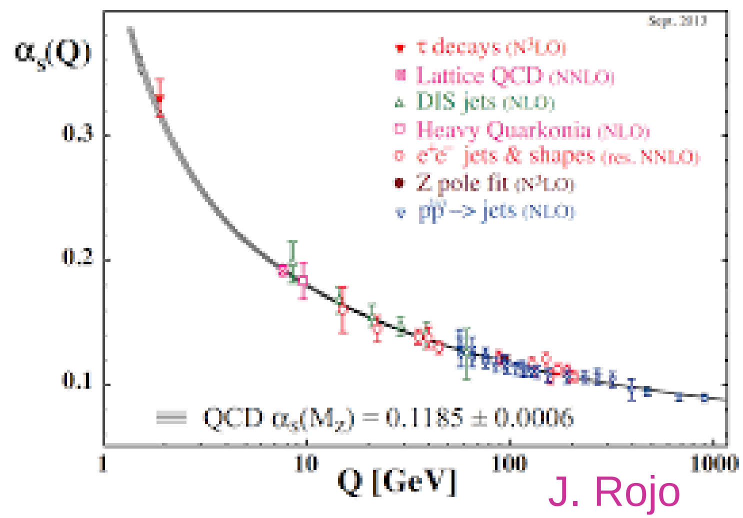
$$\alpha_s = \frac{g_s^2}{4\pi\hbar c}$$



Model or get from experiment $\alpha_s \sim 1$

Calculable in principle $\alpha_s \sim 0.1$

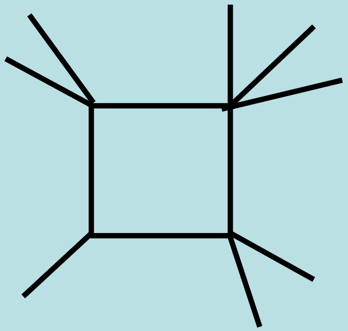




One loop amplitudes

- Numbers are very simple.
- At one loop all integrals are reducible to scalar box integrals + simpler

→ combinations of dilogarithms



$$\mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t}\ln(1-t)$$

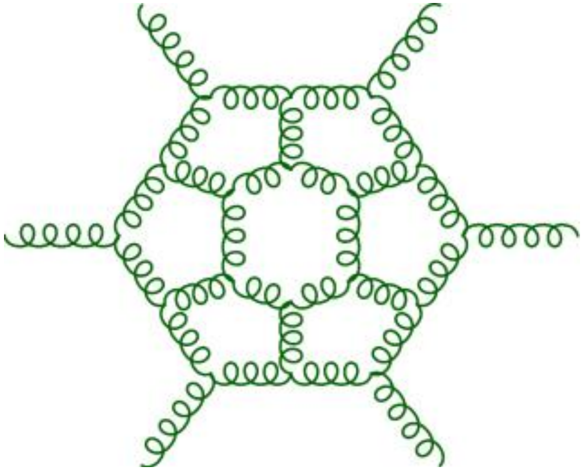
'‘t Hooft, Veltman (1974)

+ logarithms and rational terms

- Two-loop integrals are intricate, transcendental, multi-variate functions. Special values ~ those found in g_e-2

Number-theory patterns in real scattering?

- Some patterns visible in QCD
- However, we can see them easiest in a "toy theory", planar N=4 SYM, whose remarkable symmetries let us compute 6-point amplitudes up to 7 loops!



+ ~ 10^9 more Feynman diagrams

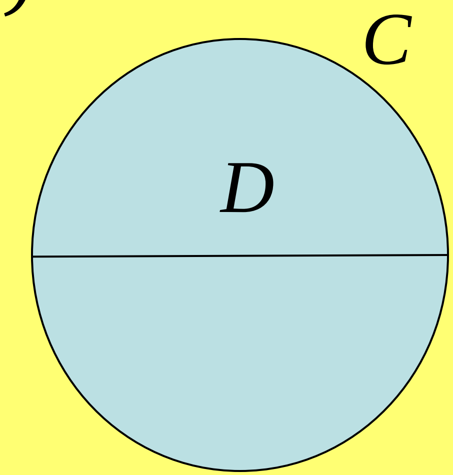
Transcendental numbers

- $\pi = \frac{C}{D} = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$

Madhava-Leibniz series

1300's

1676



- Special value of a special function:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Leonhard Euler

1707-1783

See I. Todorov, 1804.09553

- ~1726: Euler wins prize essay on ship-building, although he had never been on a ship before.
- Offer to join St. Petersburg Academy,

commissioned into Russian navy (not for long).

- In 1729, Euler began to play with values of infinite series.
- In particular, the "Basel problem":

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = ???$$

Euler sums

- Euler considered also the more general quantities, now called Riemann zeta values,

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$$

- Numerical convergence poor, important given computational tools of the day

- Euler realized that for faster convergence, one should embed $\zeta(n)$ into the alternating sums,

$$\phi(n) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^n} = (1 - 2^{1-n})\,\zeta(n)$$

Euler and the dilogarithm

- Euler also recognized $\zeta(2)$ and $\phi(2)$ as special values of a function, an iterated integral now called the dilogarithm [Leibniz → J. Bernoulli → Euler]:

$$\mathrm{Li}_2(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2} = -\int_0^x \frac{dt}{t} \ln(1-t) = \int_0^x \frac{dt}{t} \int_0^t \frac{dt'}{1-t'}$$

$$\mathrm{Li}_2(1) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \zeta(2),$$

$$\mathrm{Li}_2(-1) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -\phi(2)$$

Functional equation for better convergence

- Differentiating dilogarithm, $\frac{d}{dx}\mathrm{Li}_2(x) = -\frac{\ln(1-x)}{x}$

gives Euler's functional equation:

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) + \ln x \ln(1-x) = \mathrm{Li}_2(1)$$

- Setting $x = \frac{1}{2}$ to be well inside radius of convergence 1, Euler could get "high precision numerics", and **ascertained** that

  $\zeta(2) = \frac{\pi^2}{6}$, and later $\zeta(2n) = -\frac{B_{2n}}{2(2n)!}(2\pi i)^n$
- But $\zeta(3) = ???$

- "For n odd all my efforts have been useless until now" [Euler, 1749]

Euler's useless efforts not so useless

- While failing to find polynomial relations among $\zeta(n)$, Euler introduced nested sums, or multiple zeta values (MZV's):

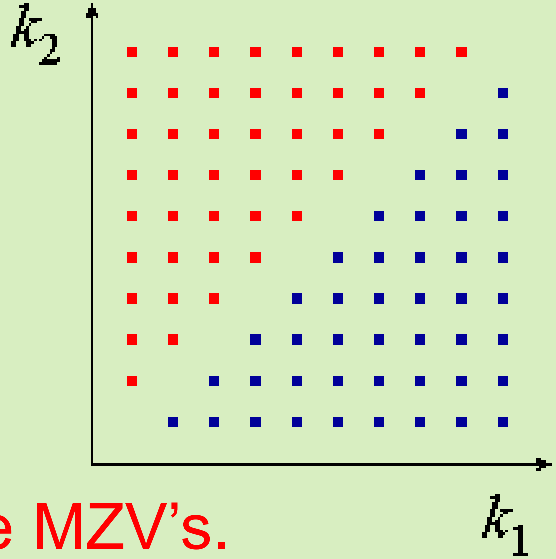
$$\zeta(n_1, \ldots, n_d) = \sum_{k_1 > \cdots k_d > 0} \frac{1}{{k_1}^{n_1} \ldots {k_d}^{n_d}}$$

- Weight = $n_1 + \cdots + n_d$, depth = d
- And similar alternating [Euler-Zagier] sums with minus signs in the numerator

MZVs obey many identities

- For example, $\zeta(n_1,n_2) = \sum_{k_1>k_2>0} \frac{1}{{k_1}^{n_1}{k_2}^{n_2}}$

obeys the "stuffle" identity,



$\zeta(n_1)\zeta(n_2) = \zeta(n_1,n_2) + \zeta(n_2,n_1) + \zeta(n_1+n_2)$

- The first irreducible MZV, that cannot be written in terms of $\zeta(n) \equiv \zeta_n$, is at weight 8,

$\zeta(5,3) \equiv \zeta_{5,3}$. → High loops needed to explore MZV's.

- "MZV datamine", Blümlein, Broadhurst, Vermaseren, 0907.2557

solves all known relations to weight 24, also alternating (Euler) sums to at least weight 12

MZVs and Harmonic Polylogarithms (HPLs)

Remiddi, Vermaseren, hep-ph/9905237

- Classical polylogs $\mathrm{Li}_n(x) = \int_0^x \frac{dt}{t} \mathrm{Li}_{n-1}(t) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$

evaluate to Riemann zeta values $\mathrm{Li}_n(1) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \zeta_n$

- Define HPLs $H_{\vec{w}}(x)$, $w_i \in \{0,1\}$ by iterated integration:

$$H_{0,\vec{w}}(x) = \int_0^x \frac{dt}{t} H_{\vec{w}}(t), \quad H_{1,\vec{w}}(x) = \int_0^x \frac{dt}{1-t} H_{\vec{w}}(t)$$

- Then $H_{n_1,\ldots,n_d}(1) \equiv H_{\underbrace{0,\ldots,0,1}_{n_1},\ldots,\underbrace{0,\ldots,0,1}_{n_d}}(1) = \zeta_{n_1,\ldots,n_d}$

- Weight n = length of binary string; 2^n HPLs at weight n
- Derivatives of just two types:

$$dH_{0,\vec{w}}(x) = H_{\vec{w}}(x)\ d\ln x \qquad dH_{1,\vec{w}}(x) = -H_{\vec{w}}(x) d\ln(1-x)$$

HPLs and massless 2→2 scattering

$s+t+u=0$ → one dimensionless variable, $x=-\frac{t}{s}$

- Only interesting limits are

$$s\to 0, \qquad t\to 0, \qquad u\to 0$$

$$\to x\to\infty, \qquad x\to 0, \qquad x\to 1$$

- Match singular points of HPLs $H_{\vec{w}}(x)$.
- HPLs $H_{\vec{w}}(x)$ with weight ≤ 4 describe all massless QCD amplitudes through 2 loops

Anastasiou, Glover, Oleari, Tejeda-Yeomans; Bern, LD, de Freitas (~2000)

- weight ≤ 6 for planar N=4 SYM and later QCD amplitudes through 3 loops

Bern, LD, Smirnov, hep-th/0505205; Henn, Mistlberger, 1608.00850; Henn, Mistlberger, Smirnov, Wasser, 2002.09492

Generic iterated integrals

Chen; Goncharov; Brown

- Generalized polylogarithms of weight n are n-fold iterated integrals, defined (for $a_n \neq 0$) by

$$G(a_1, a_2, \ldots, a_n; z) = \int_0^z \frac{dt}{t-a_1} G(a_2, \ldots, a_n; t)$$

- Important property of space $\mathcal{G}$ of such functions: Hopf co-algebra Δ maps functions to products of "functions": $\Delta\mathcal{G} \subseteq \mathcal{G} \otimes \mathcal{G}'$

  Goncharov, math/0208144; Brown, 1102.1312

- Δ basically arises from chopping iterated integration contours into pieces.
- Weight is preserved, so $\Delta = \sum_{p,q=1}^{\infty} \Delta_{p,q}$ where $\Delta_{n-q,q} f^{(n)} = \sum_k f^{k,(n-q)} \otimes g^{k,(q)}$

Iterated integrals (cont.)

- Co-action $\Delta_{n-q,q} f^{(n)} = \sum_k f^{k,(n-q)} \otimes g^{k,(q)}$
- Special case $q = 1$ is just the derivative:

$$\Delta_{n-1,1} f = \sum_{s_k \in \mathcal{S}} f^{s_k} \otimes \ln s_k(x_a)$$

is equivalent to $\frac{\partial f}{\partial x_a} = \sum_{s_k \in \mathcal{S}} f^{s_k} \frac{\partial \ln s_k}{\partial x_a}$

- $\mathcal{S}$ = finite set of rational expressions, "symbol letters" s_k, depending on coordinates x_a
- f^{s_k} are pure functions, weight n-1
- Iterate the $\{n\text{-}1,1\}$ coproduct n times:

→ **Symbol** = {1,1,…,1} component of Δ

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

Symbol example

- $\frac{d}{dx}\mathrm{Li}_n(x) = \frac{\mathrm{Li}_{n-1}(x)}{x}$, $\frac{d}{dx}\mathrm{Li}_2(x) = -\frac{\ln(1-x)}{x}$

→ $\Delta_{1,\ldots,1}[\mathrm{Li}_n(x)] = -(1-x) \otimes x \otimes \cdots \otimes x$

- Also,

$$\Delta_{1,\ldots,1}[f \cdot g] = \Delta_{1,\ldots,1}[f] \sqcup\!\sqcup \Delta_{1,\ldots,1}[g]$$

shuffle product

→ $\Delta_{1,1}[\,\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) + \ln x \ln(1-x)\,]$

$= -(1-x) \otimes x - x \otimes (1-x) + x \sqcup\!\sqcup (1-x)$

$= 0$

(Symbol of Euler functional equation)

Symbols and co-actions

- Symbol trivializes all complicated polylogarithmic identities
- → incredibly useful for simplifying massively complicated expressions for two-loop QCD amplitudes Duhr, 1203.0454
- However, differentiating n times loses all information about constants, MZVs, etc.
- Components $\Delta_{n-3,3}$, $\Delta_{n-5,5}$, ... more useful for diagnosing structure of numbers like MZVs Brown, 1102.1310
- ∃ map between MZV's and non-abelian "f alphabet" $f_3, f_5, f_7, \ldots$ which makes the action of Δ manifest.
  $\zeta(2i+1) \rightarrow f_{2i+1}, \quad \zeta(5,3) \rightarrow -5 f_5 f_3 \equiv -5\, f_{5,3}$
- Similar alphabet for alternating sums, adding $f_1 \sim \ln 2$

Back to g_e-2

- What do two- and three-loop terms look like in f alphabet?
- O. Schnetz, 1711.05118, HyperlogProcedures MAPLE program
- $\frac{197}{144} + \frac{\zeta_2}{2} + 3\zeta_2 f_1 - f_3$
- $\frac{28259}{5184} + \frac{17101}{135}\zeta_2 + \frac{596}{3}\zeta_2 f_1 - \frac{278}{27} f_3 + \frac{511}{24}\zeta_4$
  $- \frac{350}{9} f_{1,3} - \frac{83}{9}\zeta_2 f_3 + \frac{86}{9} f_5$
- $\Delta_{n-q,q}$ for $q=2i+1$ means: "clip f_{2i+1} from the left"
- Operation always lands on something seen at lower loops
- Conversely: no naked f_1 at two loops
  → no f_1, $f_{1,1}$, $f_{1,1,1}$, $f_{3,1}$, … expected at higher loops

Co-action principle

Schnetz, 1302.6445; Brown, 1512.06409; Panzer, Schnetz, 1603.04289;…

- Suppose $\mathcal{H} \subset \mathcal{G}$ is some subspace of a space of generalized polylogs or MZVs which is picked out by "physics" in some way.
- Then the left factor in the co-action should be stable, i.e.

$$\Delta\mathcal{H} \subset \mathcal{H} \otimes \mathcal{K}$$

- **Note**: left ←→right here, versus f alphabet ordering
- This principle makes many predictions which can be tested in a variety of multi-loop settings.

Cosmic Galois Group

- There is a group action C dual to Δ
- The restriction $\Delta\mathcal{H} \subset \mathcal{H} \otimes \mathcal{K}$ corresponds to invariance under the group, $C \times \mathcal{H} \to \mathcal{H}$
- Group C is infinite dimensional analog of Galois group associated with roots of a polynomial equation
- Because this property appears "everywhere", termed "cosmic Galois group"
  Cartier (1996,2000); Andre (2008); Brown, 1512.06409, 1512.06410
- Precisely how the group acts (what numbers appear) depends on the physical problem

g_e-2 at four loops

- Computed "almost" analytically
  Laporta arXiv:1704.06996
- Contains non-polylog terms.
  Also, polylog terms require two different f alphabets, one associated with $G(a_1, \ldots, a_n; 1)$ where a_i are 4th roots of unity, f_i^4 another with 6th roots, f_i^6 + g_1^6
- **Co-action principle satisfied:**
  Clipping an f_i from left lands on a stable subspace, called the Galois conjugates.

$$
\begin{aligned}
a_e \cong\ & \frac{1}{2}\left(\frac{\alpha}{\pi}\right) \\
&+ \left(\frac{197}{144} + \frac{1}{12}\pi^2 + \frac{27}{32}f_3^6 - \frac{1}{4}g_1^6\pi^2\right)\left(\frac{\alpha}{\pi}\right)^2 \\
&+ \left(\frac{28259}{5184} + \frac{17101}{810}\pi^2 + \frac{139}{16}f_3^6 - \frac{149}{9}g_1^6\pi^2 - \frac{525}{32}g_1^6 f_3^6 + \frac{1969}{8640}\pi^4 - \frac{1161}{128}f_5^6\right. \\
&\quad \left. + \frac{83}{64}f_3^6\pi^2\right)\left(\frac{\alpha}{\pi}\right)^3 \\
&+ \left(\frac{1243127611}{130636800} + \frac{30180451}{155520}\pi^2 - \frac{255842141}{2419200}f_3^6 - \frac{8873}{36}g_1^6\pi^2 + \frac{126909}{2560}\frac{f_4^6}{\mathrm{i}\sqrt{3}}\right. \\
&\quad - \frac{84679}{1280}g_1^6 f_3^6 + \frac{169703}{3840}\frac{f_2^6\pi^2}{\mathrm{i}\sqrt{3}} + \frac{779}{108}g_1^6 g_1^6\pi^2 + \frac{112537679}{3110400}\pi^4 - \frac{2284263}{25600}f_5^6 \\
&\quad + \frac{8449}{96}g_1^6 g_1^6 f_3^6 - \frac{12720907}{345600}f_3^6\pi^2 - \frac{231919}{97200}g_1^6\pi^4 + \frac{150371}{256}\frac{f_6^6}{\mathrm{i}\sqrt{3}} + \frac{313131}{1280}g_1^6 f_5^6 \\
&\quad - \frac{121383}{1280}f_2^6 f_4^6 - \frac{14662107}{51200}f_3^6 f_3^6 + \frac{8645}{128}\frac{f_2^6 g_1^6 f_3^6}{\mathrm{i}\sqrt{3}} - \frac{231}{4}g_1^6 g_1^6 g_1^6 f_3^6 - \frac{16025}{48}\frac{f_4^6\pi^2}{\mathrm{i}\sqrt{3}} \\
&\quad + \frac{4403}{384}g_1^6 f_3^6\pi^2 - \frac{136781}{1920}f_2^6 f_2^6\pi^2 + \frac{7069}{75}f_2^4 f_2^4\pi^2 - \frac{1061123}{14400}f_3^6 g_1^6\pi^2 \\
&\quad + \frac{1115}{72}\frac{f_2^6 g_1^6 g_1^6\pi^2}{\mathrm{i}\sqrt{3}} + \frac{781181}{20736}\frac{f_2^6\pi^4}{\mathrm{i}\sqrt{3}} - \frac{4049}{1080}g_1^6 g_1^6\pi^4 + \frac{90514741}{54432000}\pi^6 \\
&\quad - \frac{95624828289}{2050048}f_7^6 - \frac{29295}{512}g_1^6 f_2^6 f_4^6 + \frac{107919}{512}g_1^6 f_3^6 f_3^6 + \frac{337365}{256}f_3^6 g_1^6 f_3^6 \\
&\quad - \frac{55618247}{409600}f_5^6\pi^2 - \frac{1055}{256}g_1^6 f_2^6 f_2^6\pi^2 + \frac{26}{3}f_1^4 f_2^4 f_2^4\pi^2 + \frac{553}{4}g_1^6 f_3^6 g_1^6\pi^2 \\
&\quad - \frac{35189}{1024}f_3^6 g_1^6 g_1^6\pi^2 + \frac{79147091}{2211840}f_3^6\pi^4 - \frac{3678803}{4354560}g_1^6\pi^6 \\
&\quad \left. + \boxed{\sqrt{3}(E_{4a} + E_{5a} + E_{6a} + E_{7a}) + E_{6b} + E_{7b} + U}\right)\left(\frac{\alpha}{\pi}\right)^4 .
\end{aligned}
$$

"Galois conjugates" through weight 5

wt.	dim.	words					
0	1	1					
1	0	—					
2	1	π^2					
3	2	f_3^6	$g_1^6\pi^2$				
4	6	f_4^6	$g_1^6 f_3^6$	$f_2^6\pi^2$	$f_2^4\pi^2$	$g_1^6 g_1^6\pi^2$	π^4
5	4	f_5^6	$g_1^6 g_1^6 f_3^6$	$f_3^6\pi^2$	$g_1^6\pi^4$		

- Weights 1 to 4 "expected to be stable"
- Weight 5 will undoubtedly have additions once next loop order is computed…

Co-action for QCD scattering amplitudes?

- Same Galois conjugates for g_e-2 appear in quark (chromo) magnetic moments through 3 loops, also q^2 dependence of form factors Bonciani, Mastrolia, Remiddi, hep-ph/0307295; Lee, Smirnov, Smirnov, Steinhauser, 1801.08151, 1804.07310; …
- Also evidence for interesting number theory in QCD β function, e.g. no $\pi's$ until 5 loops, when π^4 appears; predictions of π dependence at 6,7 loops Baikov, Chetyrkin, Kühn, 1606.08659; Baikov, Chetyrkin, 1804.10088, 1808.00237
- Unfortunately, know very few full QCD amplitudes beyond two loops, where co-action principle becomes more predictive.
- Can say a lot more for QCD's maximally supersymmetric cousin, N=4 supersymmetric Yang Mills theory (N=4 SYM), especially in (planar) limit of a large number of colors where it has many secret symmetries.

N=4 SYM particle content

Brink, Schwarz, Scherk; Gliozzi, Scherk, Olive (1977)

massless spin 1 gluon

4 massless spin 1/2 gluinos

6 massless spin 0 scalars

Gauge group:
$G = SU(N_c)$,
$N_c \to \infty$

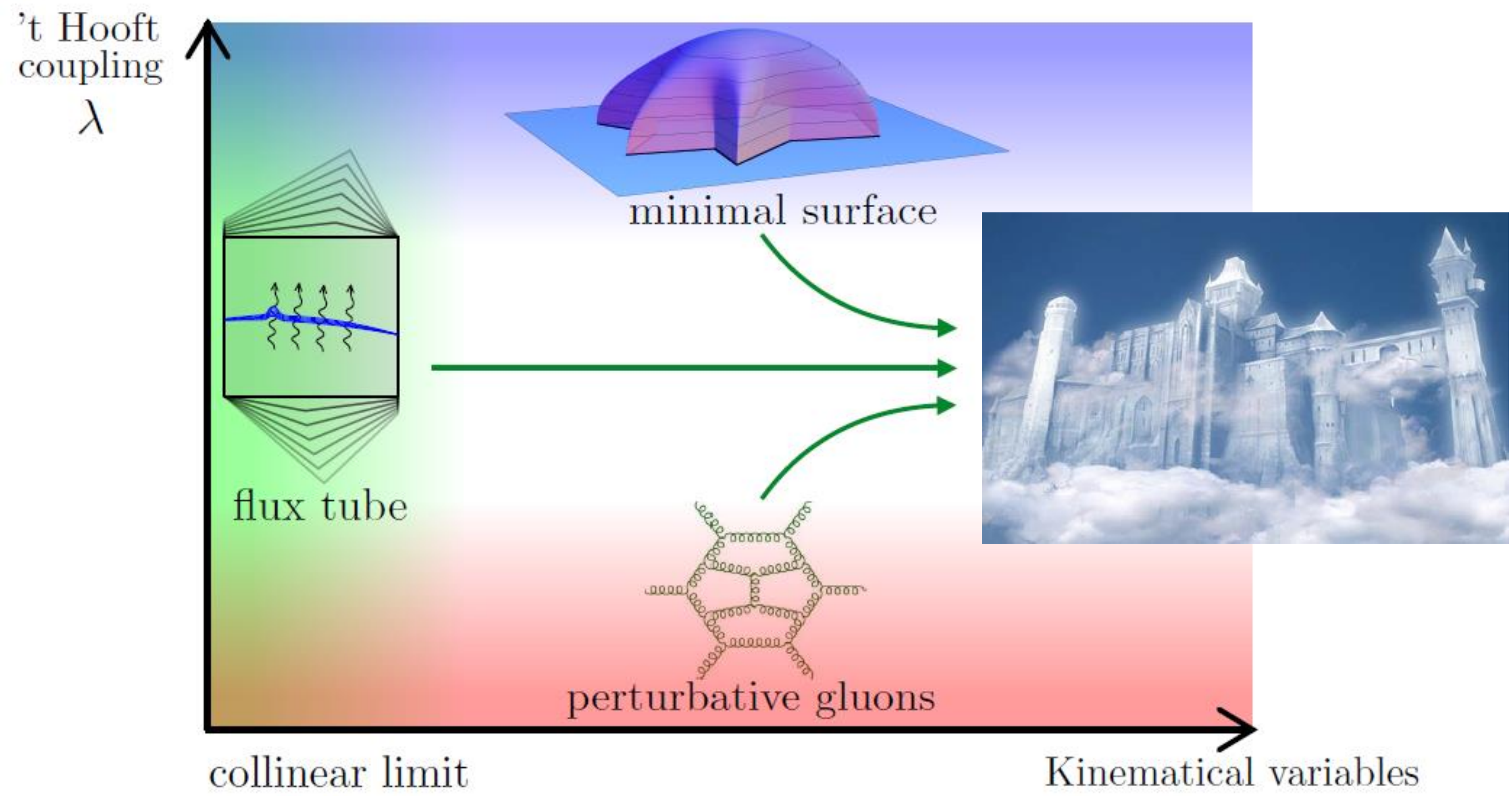
SUSY
Q_a, a=1,2,3,4
shifts helicity
by 1/2 $\longleftrightarrow$

$\mathcal{N}=4$	1	$\longleftrightarrow$	4	$\longleftrightarrow$	6	$\longleftrightarrow$	4	$\longleftrightarrow$	1
	g^-		$\lambda_{\bar{i}}^-$		$\bar{\phi}_{\bar{i}\bar{j}}, \phi_{ij}$		λ_i^+		g^+
helicity	-1		$-\frac{1}{2}$		0		$\frac{1}{2}$		1

all in adjoint representation of G

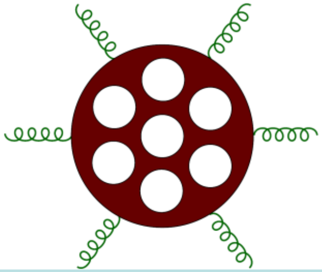
Solving for Planar N=4 SYM Amplitudes

Images: A. Sever, N. Arkani-Hamed



Bootstrapping amplitudes through 7 loops

S. Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou, 1903.10890 and 1906.07116; LD, Dulat, 20mm.nnnnn



- Six-gluon amplitude is first one not fixed by symmetries, depends on u, v, w (dual conformal cross ratios).
- Amplitude lives in remarkably small space of polylogarithmic **hexagon functions**, the weight $2L$ part at L loops.
- Space small enough that one can **bootstrap** the amplitude by writing a linear combination of functions and imposing constraints → unique solution.
- At $u = v = w = 1,$ the amplitudes, and all of their iterated $\{n\text{-}q,1,\ldots,1\}$ coproducts (derivatives) evaluate to MZVs.

f basis for $\mathcal{H}^{\mathrm{hex}}(1,1,1)$

#MZV	#	basis elements / Galois conjugates
12	6	$\zeta_{12}\,,\ \ 7f_{3,9}-6\zeta_4 f_{3,5}\,,\ \ 5f_{3,9}-3\zeta_6 f_{3,3}\,,\ \ \zeta_2 f_{3,7}-\zeta_6 f_{3,3}\,,\ \ 7f_{5,7}-\zeta_2 f_{5,5}-3\zeta_4 f_{5,3}\,,\ \ 5f_{7,5}-2\zeta_2 f_{7,3}$
9	5	$33f_{11}-20\zeta_8 f_3\,,\ \ \zeta_2 f_9-\zeta_8 f_3\,,\ \ 3\zeta_4 f_7-2\zeta_8 f_3\,,\ \ 3\zeta_6 f_5-2\zeta_8 f_3\,,\ 5f_{3,3,5}-2\zeta_2 f_{3,3,3}+\frac{5611}{132}\zeta_8 f_3$
7	3	$\zeta_{10}\,,\ \ 7f_{3,7}-\zeta_2 f_{3,5}-3\zeta_4 f_{3,3}\,,\ \ 5f_{5,5}-2\zeta_2 f_{5,3}$
5	3	$7f_9-6\zeta_4 f_5\,,\ \ 5f_9-3\zeta_6 f_3,\ \ \zeta_2 f_7-\zeta_6 f_3$
4	2	$\zeta_8\,,\ \ \zeta_{5,3}+5\zeta_3\zeta_5-\zeta_2(\zeta_3)^2=5f_{3,5}-2\zeta_2 f_{3,3}$
3	1	$7\zeta_7-\zeta_2\zeta_5-3\zeta_4\zeta_3=7f_7-\zeta_2 f_5-3\zeta_4 f_3$
2	1	ζ_6
2	1	$5\zeta_5-2\zeta_2\zeta_3=5f_5-2\zeta_2 f_3$
1	1	ζ_4
1	0	–
1	1	ζ_2
0	0	–
1	1	1

∂_3 (arrow from the weight-8 element $5f_{3,5}-2\zeta_2 f_{3,3}$ to $5f_5-2\zeta_2 f_3$)

∂_5 (arrow from the weight-8 element $5f_{3,5}-2\zeta_2 f_{3,3}$ to the empty entry –)

The values of the MHV amplitudes $\mathcal{E}^{(L)}(1,1,1)$ for $L = 1$ to 7 in the f-basis are:

$$\mathcal{E}^{(1)}(1,1,1) = 0\,,$$
$$\mathcal{E}^{(2)}(1,1,1) = -10\,\zeta_4\,,$$
$$\mathcal{E}^{(3)}(1,1,1) = \frac{413}{3}\,\zeta_6\,,$$
$$\mathcal{E}^{(4)}(1,1,1) = -\frac{5477}{3}\,\zeta_8 + 24\left[5\,f_{3,5} - 2\zeta_2 f_{3,3}\right],$$
$$\mathcal{E}^{(5)}(1,1,1) = \frac{379957}{15}\,\zeta_{10} - 384\left[7\,f_{3,7} - \zeta_2 f_{3,5} - 3\zeta_4 f_{3,3}\right] - 312\left[5\,f_{5,5} - 2\zeta_2 f_{5,3}\right],$$
$$\begin{aligned}\mathcal{E}^{(6)}(1,1,1) = &-\frac{2273108143}{6219}\zeta_{12} + 2264\left[7\,f_{3,9} - 6\zeta_4 f_{3,5}\right] + 6536\left[5\,f_{3,9} - 3\zeta_6 f_{3,3}\right] \\ &- 3072\left[\zeta_2 f_{3,7} - \zeta_6 f_{3,3}\right] + 5328\left[7\,f_{5,7} - \zeta_2 f_{5,5} - 3\zeta_4 f_{5,3}\right] \\ &+ 4224\left[5\,f_{7,5} - 2\zeta_2 f_{7,3}\right],\end{aligned}$$

The values of the NMHV amplitudes $E^{(L)}(1,1,1)$ for $L=1$ to 6 in the f-basis are

$$E^{(1)}(1,1,1) = -2\,\zeta_2\,,$$
$$E^{(2)}(1,1,1) = 26\,\zeta_4\,,$$
$$E^{(3)}(1,1,1) = -\frac{940}{3}\,\zeta_6\,,$$
$$E^{(4)}(1,1,1) = \frac{36271}{9}\,\zeta_8 - 24\left[5f_{3,5} - 2\zeta_2 f_{3,3}\right],$$
$$E^{(5)}(1,1,1) = -\frac{1666501}{30}\,\zeta_{10} + 528\left[7f_{3,7} - \zeta_2 f_{3,5} - 3\zeta_4 f_{3,3}\right] + 384\left[5f_{5,5} - 2\zeta_2 f_{5,3}\right],$$
$$E^{(6)}(1,1,1) = \frac{5066300219}{6219}\zeta_{12} - 4664\left[7f_{3,9} - 6\zeta_4 f_{3,5}\right] - 11384\left[5f_{3,9} - 3\zeta_6 f_{3,3}\right]$$
$$+ 5664\left[\zeta_2 f_{3,7} - \zeta_6 f_{3,3}\right] - 8928\left[7f_{5,7} - \zeta_2 f_{5,5} - 3\zeta_4 f_{5,3}\right]$$
$$- 6528\left[5f_{7,5} - 2\zeta_2 f_{7,3}\right].$$

Caveat

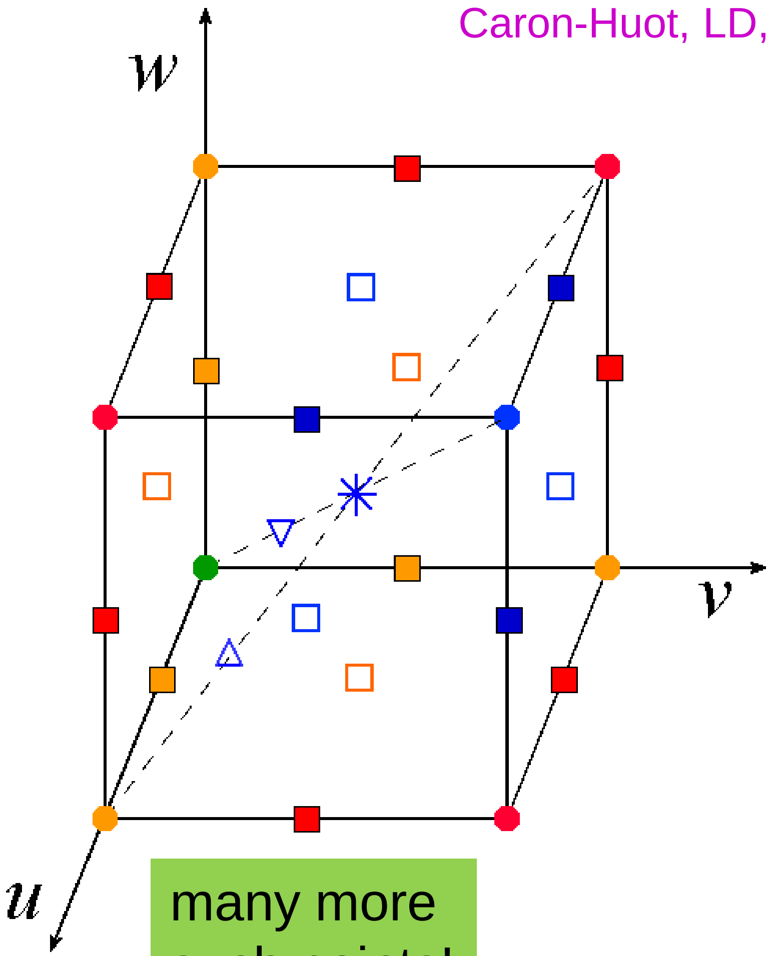
- To squeeze amplitudes into a space $\mathcal{H}^{\text{hex}}$ that obeys a co-action principle, we need to adjust their normalization slightly: $\mathcal{E} \to \frac{\mathcal{E}}{\rho}, \quad E \to \frac{E}{\rho}$

$$\rho(g^2) = 1 + 8(\zeta_3)^2 g^6 - 160\zeta_3\zeta_5 \, g^8 + \left[1680\zeta_3\zeta_7 + 912(\zeta_5)^2 - 32\zeta_4(\zeta_3)^2\right] g^{10} \\ - \left[18816\zeta_3\zeta_9 + 20832\zeta_5\zeta_7 - 448\zeta_4\zeta_3\zeta_5 - 400\zeta_6(\zeta_3)^2\right] g^{12} \\ + \left[221760\zeta_3\zeta_{11} + 247296\zeta_5\zeta_9 + 126240(\zeta_7)^2 - 3360\zeta_4\zeta_3\zeta_7 - 1824\zeta_4(\zeta_5)^2 \right. \\ \left. - 5440\zeta_6\zeta_3\zeta_5 - 4480\zeta_8(\zeta_3)^2\right] g^{14} + \mathcal{O}(g^{16}).$$

- We have ascertained what ρ is to all orders (related to determinant of BES kernel) Basso, LD, Papathanasiou, 2001.05460

6-gluon amplitude
→ many "cyclotomic" polylogs at unity

Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou, 1906.07116;
Ablinger, Blumlein, Schneider, 1105.6063, 1310.5645;
O. Schnetz, **HyperlogProcedures**



many more such points!

● MZVs
■, □ Alternating sums
✳ 4th roots of unity
▽, △ 6th roots of unity

finite
1 variable singular
2 variables singular
3 variables singular

Co-action principle applies to entire function space at every point where we've checked it!!

e.g.

$$u = v = w, \qquad y_u = y_v = y_w = y,$$
$$u = \frac{y}{(1+y)^2} \qquad 1 - u = \frac{1+y+y^2}{(1+y)^2}$$

Saturation

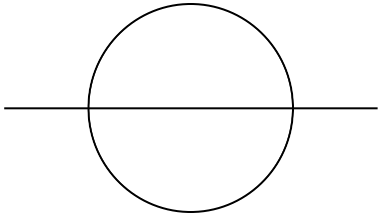
- Take iterated $\{n\text{-}1,1\}$ coproducts of these amplitudes → generate more and more lower weight functions until space is "saturated" and number declines again

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$L=1$	1	3	4												
$L=2$	1	3	6	10	6										
$L=3$	1	3	6	13	24	15	6								
$L=4$	1	3	6	13	27	53	50	24	6						
$L=5$	1	3	6	13	27	54	102	118	70	24	6				
$L=6$	1	3	6	13	27	54	105	199	269	181	78	24	6		
$L=7+$	1	3	6	13	27	54	105	200	338	331	210	85	27	6	1

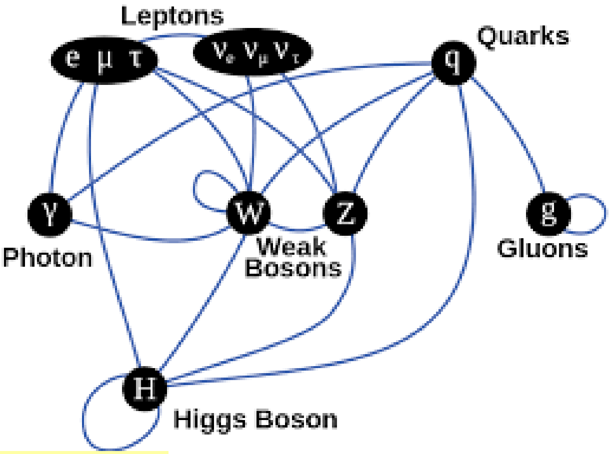
- Verifies that we have exactly the right function space (weight ≤ 7)

Bottom up: 1 3 6 13 27 54 105 200 372 679 1214 2136 …

ϕ^4 theory



Theory of Higgs boson, neglecting all other Standard Model couplings.



Pure $O(N)$ symmetric ϕ^4 theory in $D = 4 - 2\varepsilon$ experimentally relevant for ε expansion approach to critical exponents in $D = 3$

Wilson, Fisher (1972); Guillou, Zinn-Justin; Kleinert, Vasil'ev,…

High order computations required since $\varepsilon = 1/2$

- ε expansion recently completed to 6 loops

→ 3-4 digits accuracy for critical exponents after Borel resummation

Kompaniets, Panzer, 1705.06483

- Many primitive divergences known to much higher orders.

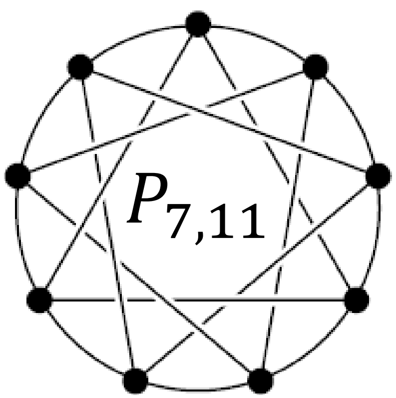
Co-action principle in ϕ^4 theory

- Earlier: Hopf algebra associated with nested structure of renormalization; knots and Feynman diagrams
  Broadhurst, Kreimer, hep-th/9504352, hep-th/9810087
- Co-action principle first formulated for ϕ^4 theory
- Much data now for primitive graphs, those with no subdivergences

  Schnetz, 1302.6445; Panzer, Schnetz, 1603.04289

Panzer, Schnetz, 1603.04289

- “Period” = UV divergence of ϕ^4 graph containing no subdivergences



- Here, co-action principle works “graph by graph”, i.e. result of clipping f_i on left is the period for a subgraph of original graph

Proof: Brown, 1512.06409

In the following table we demonstrate that the known ϕ^4 periods up to eight loops obey the coaction conjecture. For this we express the infinitesimal coaction in terms of ϕ^4 periods.

period	$\sum_m f_m^N \delta_m(P_\bullet)$
P_1	0
P_3	$6 f_3 P_1$
P_4	$20 f_5 P_1$
P_5	$\frac{441}{8} f_7 P_1$
$P_{6,1}$	$168 f_9 P_1$
$P_{6,2}$	$\frac{2}{3} f_3 P_3^2 + \frac{1\,063}{9} f_9 P_1$
$P_{6,3}$	$\frac{63}{5} f_3 P_4 - 30 f_5 P_3$
$P_{6,4}$	$-\frac{648}{5} f_3 P_4 + 720 f_5 P_3$
$P_{7,1}$	$\frac{33\,759}{64} f_{11} P_1$
$P_{7,2}$	$\frac{7}{12} f_3 P_3 P_4 - \frac{5}{18} f_5 P_3^2 - \frac{195\,379}{192} f_{11} P_1$
$P_{7,3}$	$\frac{1}{3} f_3 P_3 P_4 - \frac{31}{9} f_5 P_3^2 - \frac{960\,211}{240} f_{11} P_1$
$P_{7,4}, P_{7,7}$	$\frac{160}{21} f_3 P_5 - 20 f_5 P_4 + 70 f_7 P_3$
$P_{7,5}, P_{7,10}$	$-\frac{24}{7} f_3 P_5 + 45 f_5 P_4 - \frac{63}{2} f_7 P_3$
$P_{7,6}$	$\frac{7}{12} f_3 P_3 P_4 + \frac{145}{18} f_5 P_3^2 + \frac{502\,247}{64} f_{11} P_1$
$P_{7,8}$	$f_3(7 P_{6,3} - \frac{161}{30} P_3 P_4) + \frac{527}{9} f_5 P_3^2 + \frac{2\,756\,439}{20} f_{11} P_1$
$P_{7,9}$	$f_3(\frac{7}{2} P_{6,3} - \frac{133}{80} P_3 P_4) - \frac{217}{24} f_5 P_3^2 + \frac{4\,136\,619}{160} f_{11} P_1$
$P_{7,11}$	$f_2^6(-\frac{2\,755}{864} P_{6,1} + \frac{35}{27} P_3^3) + \frac{14}{9} f_4^6 P_5 + \frac{1\,017}{22} f_6^6 P_4 - \frac{36\,918}{43} f_8^6 P_3$
$P_{8,1}$	$1716 f_{13} P_1$
$P_{8,2}$	$f_3(\frac{145}{147} P_3 P_5 - \frac{27}{80} P_4^2) + \frac{29}{40} f_5 P_3 P_4 + \frac{47}{16} f_7 P_3^2 + \frac{94\,871\,691}{22\,400} f_{13} P_1$
$P_{8,3}$	$f_3(2 P_4^2 - \frac{320}{189} P_3 P_5) - 13466 f_{13} P_1$
$P_{8,4}$	$f_3(\frac{27}{80} P_4^2 + \frac{1}{147} P_3 P_5) + \frac{11}{40} f_5 P_3 P_4 - \frac{97}{16} f_7 P_3^2 - \frac{76\,207\,221}{22\,400} f_{13} P_1$
$P_{8,5}$	$\frac{789}{112} f_3 P_{6,1} - \frac{2\,930}{147} f_5 P_5 + \frac{3\,549}{40} f_7 P_4 - 180 f_9 P_3$
$P_{8,6}, P_{8,9}$	$\frac{488}{441} f_3 P_3 P_5 - \frac{29}{2} f_7 P_3^2 - \frac{1\,717\,423}{336} f_{13} P_1$
$P_{8,7}, P_{8,8}$	$-\frac{81}{10} f_5 P_3 P_4 + \frac{75}{4} f_7 P_3^2 - \frac{9\,819\,147}{2\,800} f_{13} P_1$

Summary

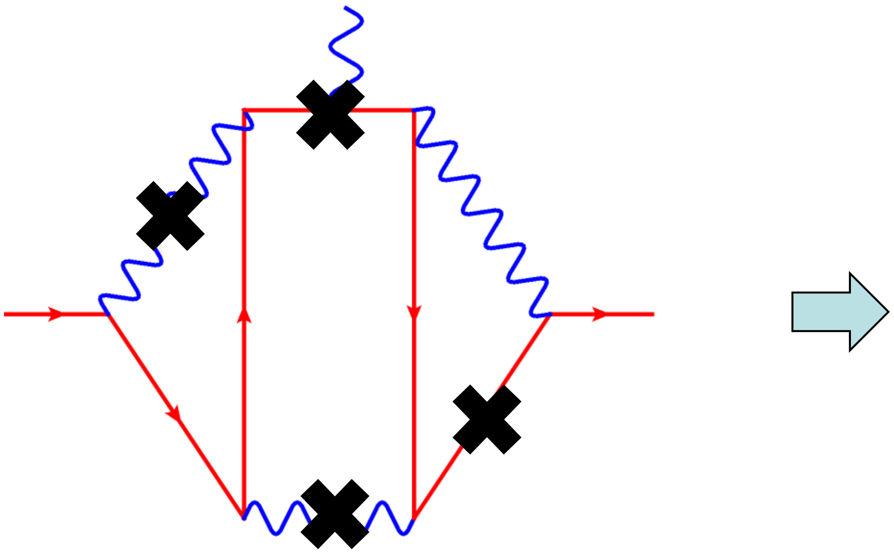
- Many important physical quantities expressed in terms of the (conjecturally) transcendental MZVs, and related generalizations.
- Properties of numbers unveiled by embedding them into (polylogarithmic) functions with an associated Hopf co-algebra
- Whenever there is a lot of theoretical data – g_e-2, planar N=4 SYM amplitudes, ϕ^4 theory – the relevant numbers appear to obey a co-action principle.

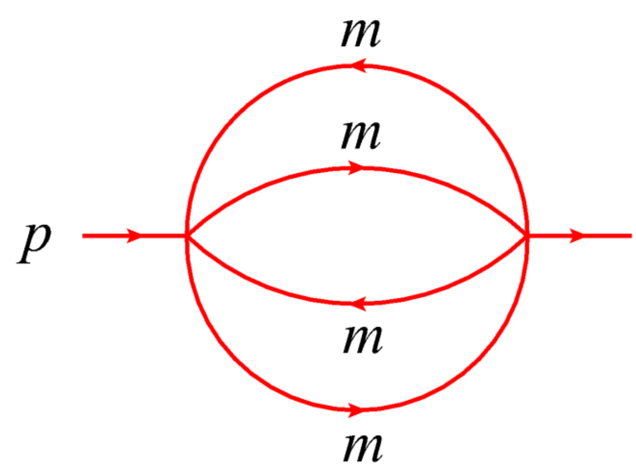
Outlook

- In many cases, polylogarithms and MZVs do not suffice for multi-loop Feynman integrals
  – need elliptic polylogarithms or "worse".
- How exactly co-action works there is still in infancy
- To how many arenas of QFT can these ideas be applied?
- One slightly negative result comes from 7-point planar N=4 SYM amplitudes: ζ values recently fixed [LD, Liu, 2007.12966]; few "missing ζ values"
- Does any general principle lurk behind what is there (including the rational numbers??)
  as well as what is not there?

Extra Slides

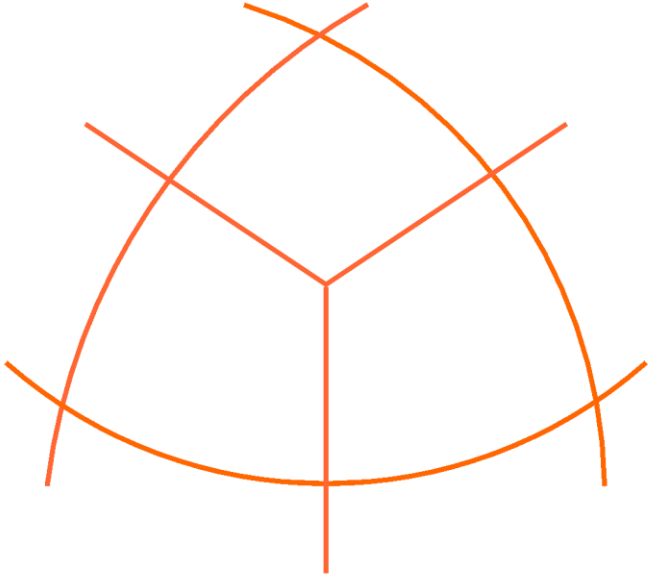
3 loop g-2 goes bananas





General-mass banana integral has K3 singularities,
but equal-mass case (for $p^2 \neq m^2$) is elliptic. No punctures.
Iterated integrals of modular forms for $\Gamma_1(6)$
Broedel, Duhr, Dulat, Marzucca, Penante, Tancredi, 1907.03787
See also Bloch, Kerr, Vanhove, 1406.2664 [unequal mass 3-loop banana]
and Bloch, Vanhove, 1309.5865 [elliptic dilog for 2-loop sunset]

“Calabi-Yau” Polylogarithms



Bourjaily, McLeod, Vergu, Volk, von Hippel, Wilhelm, 1910.01534

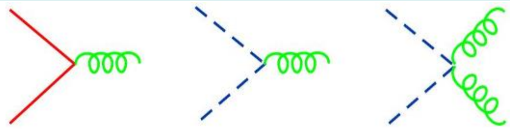
- Singularity is a Calabi-Yau hypersurface in $\mathbb{WP}^{1,1,1,1,4}$
- Has $L = 3$, $n = 9$
- However, in contrast to train-track integrals, it can’t be identified directly with any particular planar N=4 SYM amplitude, so the CY polylogarithmic part could cancel out of the amplitude.

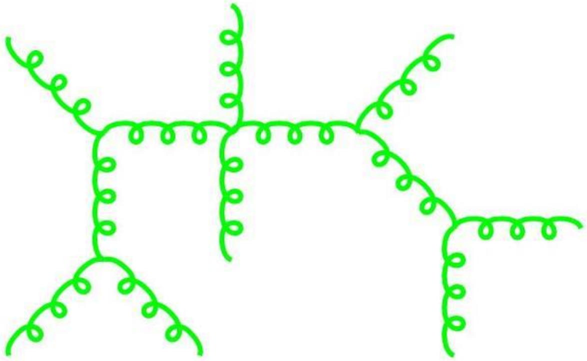
How are QCD and N=4 SYM related?

At tree level they are essentially identical

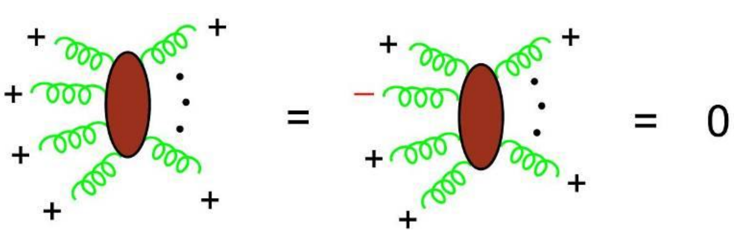
Consider a tree amplitude for *n* gluons.

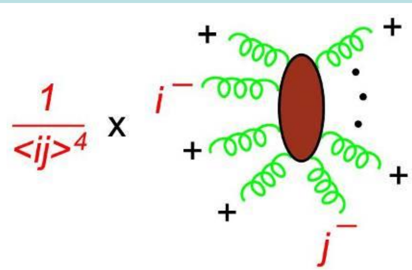
Fermions and scalars cannot appear because they are produced in pairs





Hence the amplitude is the **same** in QCD and N=4 SYM.
So the QCD tree amplitude “secretly” obeys all identities of N=4 supersymmetry:





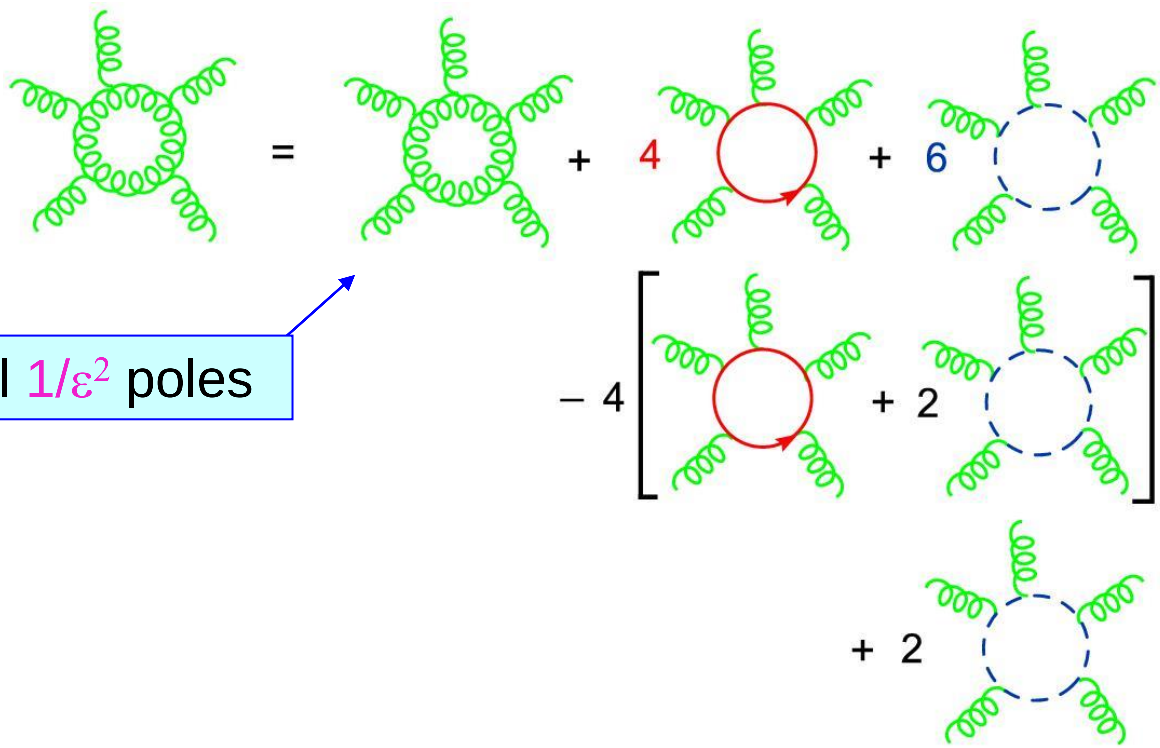
$$\frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle\cdots\langle n\, 1\rangle}$$

independent of *i,j*

At loop level, QCD and N=4 SYM differ

However, it is profitable to rearrange the QCD computation to exploit supersymmetry

gluon loop



N=4 SYM

contains all $1/\varepsilon^2$ poles

N=1 multiplet

scalar loop --- no SUSY, but also no spin tangles

Strong coupling and soap bubbles

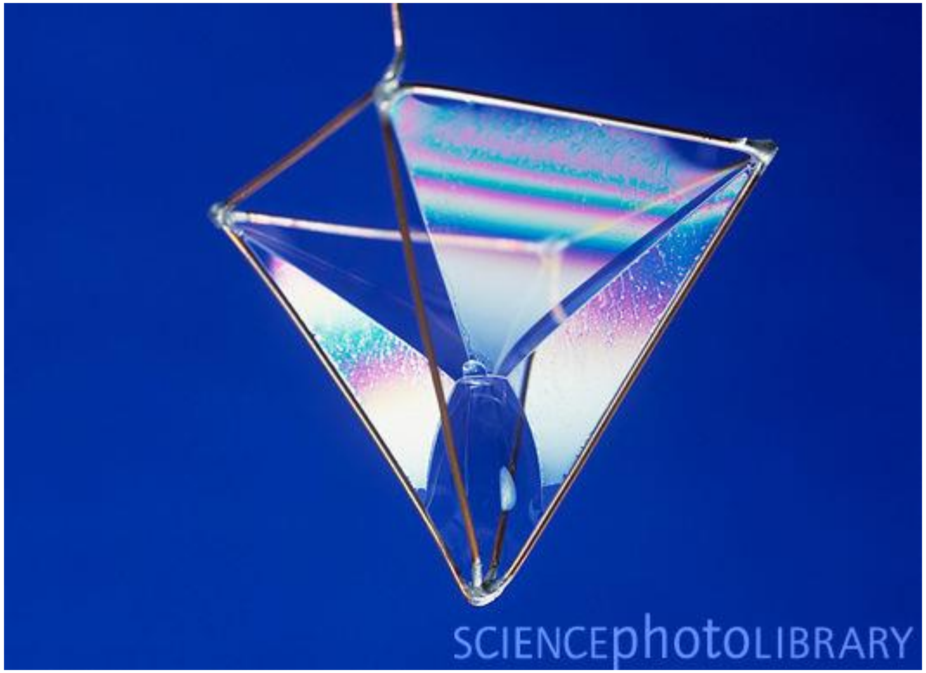
Alday, Maldacena, 0705.0303

- Use AdS/CFT to compute scattering amplitude
- High energy scattering in string theory semi-classical: two-dimensional string world-sheet stretches a long distance, classical solution minimizes area

Gross, Mende (1987,1988)

Classical action imaginary
→ exponentially suppressed tunnelling configuration

$$A_n \sim \exp[-\sqrt{\lambda} S_{\rm cl}^{\rm E}]$$



Is $\zeta(3)$ Transcendental?

- Still not known!
- $\zeta(3)$ is proven to be irrational Apéry, 1973
- Also proven: For any $\varepsilon > 0$, at least $2^{(1-\varepsilon)\frac{\ln s}{\ln \ln s}}$ of the odd Riemann ζ values between 3 and s are irrational.
  Fischler, Sprang, Zudilin, 1803.08905
- It is a "folklore conjecture" (i.e. all physicists believe it) that $\pi, \zeta(3), \zeta(5)$, ... are algebraically independent over $\mathbb{Q}$
- Follows from Grothendieck's period conjecture for mixed Tate motives, but this seems impossible to prove
- To make formal mathematical progress, usually define motivic multiple zeta values, $\zeta \rightarrow \zeta^{\mathfrak{M}}$
- We won't worry about the distinction here.

$$
\begin{aligned}
\mathcal{E}^{(7)}(1,1,1) = {} & \frac{2519177639}{1260}\zeta_{14} - 63968\Big[5f_{9,5} - 2\zeta_2 f_{9,3}\Big] - 77952\Big[7f_{7,7} - \zeta_2 f_{7,5} - 3\zeta_4 f_{7,3}\Big] \\
& - 34976\Big[7f_{5,9} - 6\zeta_4 f_{5,5}\Big] - 95552\Big[5f_{5,9} - 3\zeta_6 f_{5,3}\Big] + 44640\Big[\zeta_2 f_{5,7} - \zeta_6 f_{5,3}\Big] \\
& - \frac{413920}{11}\Big[33f_{3,11} - 20\zeta_8 f_{3,3}\Big] + 28000\Big[\zeta_2 f_{3,9} - \zeta_8 f_{3,3}\Big] \\
& + 62720\Big[3\zeta_4 f_{3,7} - 2\zeta_8 f_{3,3}\Big] + \frac{218696}{3}\Big[3\zeta_6 f_{3,5} - 2\zeta_8 f_{3,3}\Big] \\
& - 4992\Big[5f_{3,3,3,5} - 2\zeta_2 f_{3,3,3,3} + \frac{5611}{132}\zeta_8 f_{3,3}\Big] .
\end{aligned}
$$

Amplitude values at (1,1,1) through 5 loops

$$\mathcal{E}^{(1)}(1,1,1) = 0\,,$$
$$\mathcal{E}^{(2)}(1,1,1) = -10\,\zeta_4\,,$$
$$\mathcal{E}^{(3)}(1,1,1) = \frac{413}{3}\,\zeta_6\,,$$
$$\mathcal{E}^{(4)}(1,1,1) = -\frac{5477}{3}\,\zeta_8 + 24\left[\zeta_{5,3} + 5\,\zeta_3\,\zeta_5 - \zeta_2\,(\zeta_3)^2\right],$$
$$\mathcal{E}^{(5)}(1,1,1) = \frac{379957}{15}\,\zeta_{10} - 12\left[4\,\zeta_2\,\zeta_{5,3} + 25\,(\zeta_5)^2\right] - 96\left[2\,\zeta_{7,3} + 28\,\zeta_3\,\zeta_7 + 11\,(\zeta_5)^2 - 4\,\zeta_2\,\zeta_3\,\zeta_5 - 6\,\zeta_4\,(\zeta_3)^2\right]$$

MHV

$$E^{(1)}(1,1,1) = -2\,\zeta_2\,,$$
$$E^{(2)}(1,1,1) = 26\,\zeta_4\,,$$
$$E^{(3)}(1,1,1) = -\frac{940}{3}\,\zeta_6\,,$$
$$E^{(4)}(1,1,1) = \frac{36271}{9}\,\zeta_8 - 24\left[\zeta_{5,3} + 5\,\zeta_3\,\zeta_5 - \zeta_2\,(\zeta_3)^2\right],$$
$$E^{(5)}(1,1,1) = -\frac{1666501}{30}\,\zeta_{10} + 12\left[4\,\zeta_2\,\zeta_{5,3} + 25\,(\zeta_5)^2\right] + 132\left[2\,\zeta_{7,3} + 28\,\zeta_3\,\zeta_7 + 11\,(\zeta_5)^2 - 4\,\zeta_2\,\zeta_3\,\zeta_5 - 6\,\zeta_4\,(\zeta_3)^2\right]$$

NMHV

Six loops

$$\mathcal{E}^{(6)}(1,1,1) = -\frac{2273108143}{6219}\zeta_{12}$$
$$+ \frac{260}{3}\left[140\zeta_5\zeta_7 - 56\zeta_2\zeta_3\zeta_7 - 10\zeta_2(\zeta_5)^2 - 60\zeta_4\zeta_3\zeta_5 + 49\zeta_6(\zeta_3)^2\right]$$
$$+ 384\left[\zeta_2\zeta_{7,3} + 14\zeta_2\zeta_3\zeta_7 + 3\zeta_2(\zeta_5)^2 - 7\zeta_6(\zeta_3)^2\right]$$
$$+ 120\left[4\zeta_4\zeta_{5,3} + 20\zeta_4\zeta_3\zeta_5 - 7\zeta_6(\zeta_3)^2\right]$$
$$+ \frac{5392}{3}\left[\zeta_{9,3} + 27\zeta_3\zeta_9 + 20\zeta_5\zeta_7 - 2\zeta_2\zeta_3\zeta_7 - \zeta_2(\zeta_5)^2 - 6\zeta_4\zeta_3\zeta_5 - 5\zeta_6(\zeta_3)^2\right]$$

MHV

$$E^{(6)}(1,1,1) = \frac{5066300219}{6219}\zeta_{12}$$
$$- \frac{344}{3}\left[140\zeta_5\zeta_7 - 56\zeta_2\zeta_3\zeta_7 - 10\zeta_2(\zeta_5)^2 - 60\zeta_4\zeta_3\zeta_5 + 49\zeta_6(\zeta_3)^2\right]$$
$$- 528\left[\zeta_2\zeta_{7,3} + 14\zeta_2\zeta_3\zeta_7 + 3\zeta_2(\zeta_5)^2 - 7\zeta_6(\zeta_3)^2\right]$$
$$+ 60\left[4\zeta_4\zeta_{5,3} + 20\zeta_4\zeta_3\zeta_5 - 7\zeta_6(\zeta_3)^2\right]$$
$$- \frac{9952}{3}\left[\zeta_{9,3} + 27\zeta_3\zeta_9 + 20\zeta_5\zeta_7 - 2\zeta_2\zeta_3\zeta_7 - \zeta_2(\zeta_5)^2 - 6\zeta_4\zeta_3\zeta_5 - 5\zeta_6(\zeta_3)^2\right]$$

NMHV