Finite Simple Groups and Elliptic Curve Arithmetic*

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Abstract

Monstrous Moonshine emerged in the 1970s, from coincidences relating the largest sporadic simple group to moduli spaces of complex elliptic curves. More recently, manifestations of moonshine have materialized that relate sporadic simple groups to arithmetic invariants of elliptic curves over the rationals. In this talk I will describe forthcoming joint work with Cheng and Mertens that initiates a systematic approach to this phenomena. Our investigations yielded some unexpected results, including a connection between the congruent number problem of antiquity, and the smallest sporadic simple group.

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Contents

1 Some Background

1.1 Moonshine

- Moonshine is *hidden symmetry in geometric objects.*
- Moonshine is a situation in which a finite group organizes a family of discrete subgroups of some Lie group, and the discrete groups define representations of the finite group in a natural (no knowledge) way.

- Definition: *Moonshine* is a quadruple $(G, \mathcal{X}, \psi, P)$ where
  - G is a finite group,
  - $\mathcal{X} = \{\Gamma_g < \mathfrak{G} \mid [g] \subset G\}$ is the association of a discrete group Γ_g to each conjugacy class $[g]$ of G, for $\mathfrak{G}$ some fixed Lie group,
  - ψ is a representation of $\mathfrak{G}$ on some space $\mathfrak{F}$ of functions, and
  - P is a growth condition on functions in $\mathfrak{F}$,

  such that the optimal forms associated to $\mathcal{X}$, ψ and P define representations $\mathcal{W} = \mathcal{W}_{\mathcal{X}}^{(\psi,P)}$ of G.

- Definition: *Moonshine* is a quadruple $(G, \mathcal{X}, \psi, P)$ such that the optimal forms associated to $\mathcal{X}$, ψ and P define representations $\mathcal{W} = \mathcal{W}_{\mathcal{X}}^{(\psi,P)}$ of G.
- This is moonshine because it's lunacy to think that $\mathcal{X}$ can tell you about representations of G without further input from G itself.
- Ideally $\mathcal{W}_{\mathcal{X}}^{(\psi,P)}$ admits further structure (e.g. vertex operators, Virasoro action, geometric interpretation, &c.).

- In this talk
  - $\mathfrak{G}$ will be $SL_2(\mathbb{R})$ (or the Jacobi group $SL_2(\mathbb{R}) \ltimes (\mathbb{R}^2 \cdot S^1)$),
  - $\mathfrak{F}$ will be holomorphic (vector-valued) functions on the upper half-plane $\mathbb{H}$ (or $\mathbb{H} \times \mathbb{C}$ in case $\mathfrak{G}$ is the Jacobi group),
  - ψ will be the action of $\mathfrak{G}$ on (weakly holomorphic) mock modular forms (or mock Jacobi forms) of some fixed weight with some fixed shadow, and
  - P will be the condition that $f \in \mathfrak{F}$ satisfies $f(\tau) = P(e^{2\pi\Im(\tau)}) + O(1)$ as $\Im(\tau) \to \infty$ for some polynomial P.

- Definition: *Moonshine* is a quadruple $(G, \mathcal{X}, \psi, P)$ such that the optimal forms associated to $\mathcal{X}$, ψ and P define representations $\mathcal{W} = \mathcal{W}_{\mathcal{X}}^{(\psi,P)}$ of G.
- An *optimal* form for $\Gamma < \mathfrak{G}$ and (ψ, P) is a function in $\mathfrak{F}$ that has minimal growth in coefficients amongst those that are automorphic for Γ with respect to ψ and satisfy the growth condition P.
- An optimal form is uniquely determined up to something of negligible growth; e.g. a constant in the case of modular functions, a theta series in the case of (mock) modular forms of weight $\frac{1}{2}$, a cusp form in the case of weight $\frac{3}{2}$, &c.

- Definition: *Moonshine* is a quadruple $(G, \mathcal{X}, \psi, P)$ such that the optimal forms associated to $\mathcal{X}$, ψ and P define representations $\mathcal{W} = \mathcal{W}_{\mathcal{X}}^{(\psi,P)}$ of G.
- The passage from $(\mathcal{X}, \psi, P)$ to the G-modules $\mathcal{W}_{\mathcal{X}}^{(\psi,P)}$ is given by requiring

$$\mathrm{gtr}(g|\mathcal{W}_{\mathcal{X}}^{(\psi,P)}) = F_{\Gamma_g}^{(\psi,P)} \tag{1}$$

for $g \in G$, where $F_{\Gamma}^{(\psi,P)}$ is optimal for Γ and (ψ, P).

1.2 First Wave

- Monstrous moonshine (McKay, Thompson [Tho79a, Tho79b], Conway–Norton [CN79]):

  Hidden symmetry in moduli spaces of complex elliptic curves

  $G = \mathbb{M}$ is the largest sporadic simple group,

  $$\#\mathbb{M} = 808017424794512875886459904961710757005754368000000000. \qquad (2)$$

  $\mathcal{X} = \{\Gamma_g \mid g \in \mathbb{M}\}$ where Γ_g is something that normalizes $\Gamma_0(N_g)$ for each $g \in \mathbb{M}$, and $o(g)|N_g$.

  ψ is the natural action of $SL_2(\mathbb{R})$ on holomorphic functions on $\mathbb{H}$.

  $P(x) = x$.

  $\mathcal{W}_{\mathcal{X}}^{(\psi,P)}$ is—according to work of Borcherds [Bor92]—the *moonshine module* vertex operator algebra $V^\natural$ of Frenkel–Lepowsky–Meurman [FLM84, FLM85, FLM88].

- Generalized monstrous moonshine (Norton [Nor87, Nor01]):

  Similar to monstrous moonshine but with $G = C_{\mathbb{M}}(h)$ the centralizer of any cyclic subgroup $\langle h \rangle$ of $\mathbb{M}$.

  $\mathcal{W}_{\mathcal{X}}^{(\psi,P)} = V_h^{\natural}$ is—according to work of Carnahan [Car12]—a twisted module for $V^{\natural}$.

- Conway moonshine (Conway–Norton [CN79]):

  Similar to monstrous moonshine but with $G = Co_0$ the Conway group.

  $\mathcal{W}_{\mathcal{X}}^{(\psi,P)} = V^{s\natural}$ is a vertex operator superalgebra counterpart to $V^{\natural}$ (see [DMC15]).

1.3 Second Wave

- Mathieu moonshine (Eguchi–Ooguri–Tachikawa [EOT11]):

  Hidden symmetry in complex K3 surfaces(?)

  $G = M_{24}$ is the largest sporadic simple Mathieu group, $\#M_{24} = 244823040$.

  $\mathcal{X} = \{\Gamma_0(o(g)) \mid g \in M_{24}\}$.

  ψ is the action of $SL_2(\mathbb{R})$ on mock modular forms of weight $\frac{1}{2}$ with shadow $\sum_{k\equiv 1 \bmod 4} kq^{\frac{k^2}{8}}$.

  $P(x) = -2x^{\frac{1}{8}}$.

  $\mathcal{W}_{\mathcal{X}}^{(\psi,P)}$ was shown to exist, as an M_{24}-module, by Gannon [Gan16].

- Umbral moonshine (Cheng–D–Harvey [CDH14a, CDH14b, CDH18]):

  More hidden symmetry in complex K3 surfaces?

  G is the outer automorphism group of a Niemeier lattice (e.g. M_{24}, $2.M_{12}$, &c.).

  $\mathcal{X} = \{\Gamma_0(o(g)) \mid g \in G\}$.

  ψ is the action of $SL_2(\mathbb{R})$ on vector-valued mock modular forms of weight $\frac{1}{2}$ with a certain fixed shadow.

  $P(x) = -2x^{\frac{1}{4m}}$ for m depending on ψ.

  $\mathcal{W}_{\mathcal{X}}^{(\psi,P)}$ is known to exist as a G-module (see D–Griffin–Ono [DGO15]).

- Thompson moonshine (Harvey–Rayhaun [HR16]):

  Hidden symmetry in ???

  $G = Th$ is the sporadic simple Thompson group, $\#Th = 90745943887872000$.

  $\mathcal{X} = \{\Gamma_0(o(g)) \mid g \in G\}$.

  ψ is the action of $SL_2(\mathbb{R})$ on (vector-valued) modular forms of weight $\frac{1}{2}$.

  $P(x) = 2x^{\frac{1}{4}}$

  $\mathcal{W}_{\mathcal{X}}^{(\psi,P)}$ is known to exist as a Th-module (see Griffin–Mertens [GM16]).

- Penumbral moonshine (D–Harvey–Rayhaun, coming soon!):

  Penumbral moonshine is to Thompson moonshine as umbral moonshine is to Mathieu moonshine.

1.4 Third Wave

- O'Nan moonshine (D–Mertens–Ono [DMO17a, DMO17b]):

  Hidden symmetry in elliptic curves over $\mathbb{Q}$*.*

  $G = ON$ is the (non-monstrous) sporadic simple group of O'Nan, $\#ON = 460815505920$.

  $\mathcal{X} = \{\Gamma_0(o(g)) \mid g \in ON\}$.

  ψ is the action of $SL_2(\mathbb{R})$ on (vector-valued) modular forms of weight $\frac{3}{2}$.

  $P(x) = -x^4$.

  $\mathcal{W}_{\mathcal{X}}^{(\psi,P)} = W^{ON}$ is known to exist as an ON-module (D–Mertens–Ono [DMO17a]).

- Application of O'Nan moonshine (see [DMO17a]):

  If $D < 0$ is the discriminant of $\mathbb{Q}(\sqrt{D})$, if 19 is inert in the ring of integers of $\mathbb{Q}(\sqrt{D})$, and if

$$\dim(W_D^{ON}) \not\equiv 14h(D) \bmod 19, \tag{3}$$

  where $h(D)$ is the class number of $\mathbb{Q}(\sqrt{D})$, then the elliptic curve defined by

$$y^2 = x^3 - 12096D^2x - 544752D^3 \tag{4}$$

  has finitely many rational points.

- Thompson moonshine in weight $\frac{3}{2}$ (Khaqan [Kha20]):

  More hidden symmetry in elliptic curves over $\mathbb{Q}$.

  $G = Th$.

  $\mathcal{X} = \{\Gamma_0(o(g)) \mid g \in Th\}$.

  ψ is the action of $SL_2(\mathbb{R})$ on (vector-valued) modular forms of weight $\frac{3}{2}$.

  $P(x) = 6x^5$.

  $\mathcal{W}_{\mathcal{X}}^{(\psi,P)} = W^{Th}$ is known to exist as a Th-module (Khaqan [Kha20]).

- Application of Thompson moonshine (see [Kha20]):
  If $D < 0$ is the discriminant of $\mathbb{Q}(\sqrt{D})$, if 19 is inert in the ring of integers of $\mathbb{Q}(\sqrt{D})$, and if

$$\dim(W_D^{Th}) \not\equiv 0 \bmod 19, \tag{5}$$

then the elliptic curve defined by

$$y^2 = x^3 - 12096D^2x - 544752D^3 \tag{6}$$

has finitely many rational points.

2 The Goal

- We see applications to arithmetic geometry in the third wave.
- Is this part of a more general story?
- If so, can we systematically develop a theory? Such a theory is our goal.

- One approach:
  1. Choose a family $\mathcal{X}$ and a pair (ψ, P).
  2. Choose a finite group G.
  3. Classify graded G-modules with traces defined by $\mathcal{X}$ and (ψ, P).
  4. Pursue consequences of this classification.

3 Our Method

3.1 Class Numbers

- With Cheng and Mertens we consider the case that $\mathcal{X} = \{\Gamma_0(o(g))\}$ and $F^{(\psi,P)}_{\Gamma_0(1)}$ is the weight $\frac{3}{2}$ (mock modular) Eisenstein series, a.k.a. the Hurwitz class number generating function.

$$F^{(\psi,P)}_{\Gamma_0(1)}(\tau) = \mathscr{H}(\tau) := -\frac{1}{12} + \sum_{D<0} H(D)q^{|D|} \tag{7}$$

- Here $H(D)$ is the *Hurwitz class number* of D, defined by

$$H(D) := \sum_{Q \in \mathcal{Q}(D)/\Gamma_0(1)} \frac{1}{\#\Gamma_0(1)_Q}, \tag{8}$$

where $\mathcal{Q}(D)$ is the set of integer coefficient binary quadratic forms $Ax^2 + Bxy + Cy^2$ of discriminant $D := B^2 - 4AC$.

- The coefficients of $\mathscr{H}$ are not generally integral, so we need to rescale it in order to obtain the graded dimension of a module for a group. This motivates the following.

- Question:

  For a given finite group G, what is the minimal positive integer m_G^{opt} such that there exists a non-trivial (virtual) graded G-module $W = \bigoplus_{D \leq 0} W_D$ with graded dimension given by

  $$12 m_G^{\text{opt}} \mathscr{H} = -m_G^{\text{opt}} + O(q^3) \tag{9}$$

  such that the graded trace of $g \in G$ on W is optimal for $\Gamma_0(o(g))$?

3.2 An Infinite Family

- For N prime set

$$h_N := \begin{cases} \frac{N^2-1}{24} & \text{if } N \equiv 1 \bmod 4, \\ \frac{N^2-1}{12} & \text{if } N \equiv 3 \bmod 4. \end{cases} \tag{10}$$

- From properties of class numbers it follows that $m_G^{\text{opt}} | h_N$ for $G = \mathbb{Z}/N\mathbb{Z}$ for N prime. E.g. $h_{11} = 10$.
- So is it that $m_{\mathbb{Z}/11\mathbb{Z}}^{\text{opt}} = 10$, or can we do better?

- Define

$$m_N := \frac{h_N}{\#J_0(N)(\mathbb{Q})_{\text{tor}}} \tag{11}$$

where $J_0(N)$ is the Jacobian of the modular curve $X_0(N) := \Gamma_0(N)\backslash\widetilde{\mathbb{H}}$.

- Mazur proved [Maz77] that

$$\#J_0(N)(\mathbb{Q})_{\text{tor}} = \text{num}\left(\frac{N-1}{12}\right). \tag{12}$$

- It follows that m_N is an integer. E.g. $m_{11} = 2$.

Theorem 1 (Cheng–D–Mertens)**.** *For N prime and $G = \mathbb{Z}/N\mathbb{Z}$ we have $m_G^{\mathrm{opt}} | m_N$.*

- We can show that $m_G^{\text{opt}} = m_N$ for certain infinite families of primes N, such as those for which $N \equiv 5, 7 \bmod 12$.
- We will see momentarily that $m_{\mathbb{Z}/11\mathbb{Z}}^{\text{opt}} = m_{11} = 2$.
- We expect that $m_G^{\text{opt}} = m_N$ for all prime N but cannot prove this yet.

Theorem 2 (Cheng–D–Mertens)**.** *If $D < 0$ is the discriminant of $\mathbb{Q}(\sqrt{D})$, if 11 is inert in the ring of integers of $\mathbb{Q}(\sqrt{D})$, and if*

$$h(D) \not\equiv 0 \bmod 5, \tag{13}$$

where $h(D)$ is the class number of $\mathbb{Q}(\sqrt{D})$, then the elliptic curve defined by

$$y^2 = x^3 - 13392D^2x - 1080432D^3 \tag{14}$$

has finitely many rational points.

- Theorem 2 is an application of the $N = 11$ case of Theorem 1.
- The elliptic curve appearing (14) is the D-th quadratic twist of the Jacobian of $X_0(11)$.
- Theorem 1 also yields analogues of Theorem 2 for each prime $N \geq 17$, where the role of the Jacobian of $X_0(11)$ is replaced by an optimal quotient of the Jacobian of $X_0(N)$.

Theorem 3 (Cheng–D–Mertens)**.** *Let $N \geq 17$ be a prime. If $D < 0$ is the discriminant of $\mathbb{Q}(\sqrt{D})$, if N is inert in the ring of integers of $\mathbb{Q}(\sqrt{D})$, and if*

$$h(D) \not\equiv 0 \bmod \#J_0(N)(\mathbb{Q})_{\mathrm{tor}}, \tag{15}$$

where $h(D)$ is the class number of $\mathbb{Q}(\sqrt{D})$, then there exists an optimal quotient A of $J_0(N)$ for which the D-th quadratic twist of A has finitely many rational points.

- The Shimura correspondence [Shi73] suggests that there should be a weight two counterpart to Theorem 1.
- Such a result exists thanks to work of Beneish [Ben19b].
- The results of [Ben19b] relate representations of cyclic groups to mod p reductions of elliptic curves over $\mathbb{Q}$.
- Moreover, Beneish has given a vertex operator algebra realization of the weight two module for $G = \mathbb{Z}/11\mathbb{Z}$ [Ben19a].

- Theorem 3 is well-known (since before our work, cf. e.g. [AK87]).
- So it's natural to ask if the modules of Theorem 1 are manifestations of richer structure.

- Are the modules of Theorem 1 manifestations of richer structure?
- We will demonstrate that the answer is yes, at least for $N = 11$ and $N = 23$.

3.3 The Smallest Sporadic Simple Group

- $G = M_{11}$ is the smallest sporadic simple Mathieu group, $\#M_{11} = 7920$.

- $G = M_{11}$ is the smallest sporadic simple Mathieu group, $\#M_{11} = 7920$.
- Define $\alpha : [g] \mapsto \alpha_g$ by setting

$$\alpha_g \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) := \begin{cases} 1 & \text{if } o(g) \notin \{4, 8\}, \\ e\left(-\frac{2cd}{o(g)^2}\right) & \text{if } o(g) \in \{4, 8\}, \end{cases} \tag{16}$$

for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(o(g))$, where $e(x) := e^{-2\pi i x}$.

- Modified Question:

  For a given finite group G, and the assignment $\alpha : [g] \mapsto \alpha_g$ of a character of $\Gamma_0(o(g))$ to each conjugacy class $[g]$ of G, what is the minimal positive integer $m^{\text{opt}}_{(G,\alpha)}$ such that there exists a non-trivial (virtual) graded G-module $W = \bigoplus_{D \leq 0} W_D$ with graded dimension given by

$$12m^{\text{opt}}_{(G,\alpha)}\mathscr{H} = -m^{\text{opt}}_{(G,\alpha)} + O(q^3) \tag{17}$$

  such that the graded trace of $g \in G$ on W is optimal for $\Gamma_0(o(g))$ with character α_g?

Theorem 4 (Cheng–D–Mertens)**.** *For $G = M_{11}$ and α as in (16) we have $m^{\text{opt}}_{(G,\alpha)} = 2$.*

- Recall that a positive integer n is called a *congruent number* if it is the area of a right triangle with rational side lengths.

- Recall that a positive integer n is called a *congruent number* if it is the area of a right triangle with rational side lengths.
- For $m \in \mathbb{Z}$ let $\mathcal{W}(m)^{\mathrm{opt}}_{(G,\alpha)}$ be the set of (virtual) graded G-modules $W = \bigoplus_{D\leq 0} W_D$ with graded dimension given by

$$12m\mathscr{H} = -m + O(q^3) \tag{18}$$

  such that the graded trace of $g \in G$ on W is optimal for $\Gamma_0(o(g))$ with character α_g.
- Then $m = m^{\mathrm{opt}}_{(G,\alpha)}$ is the smallest positive integer such that $\mathcal{W}(m)^{\mathrm{opt}}_{(G,\alpha)}$ is not empty.

Theorem 5 (Cheng–D–Mertens)**.** *Set* $G = M_{11}$ *and let* α *be as in (16). Let* $D < 0$ *be square-free and satisfy* $D \equiv 21 \bmod 24$*. If there exists an* M_{11}*-module* $W = \bigoplus_D W_D$ *in* $\mathcal{W}(2)^{\rm opt}_{(G,\alpha)}$ *such that* W_D *contains an irreducible* 55*-dimensional* M_{11}*-submodule with non-zero multiplicity then* $|D|$ *is not a congruent number.*

- Richer structure for $N = 11$ has been established in a deeper sense in the weight two case.
- The action of $\mathbb{Z}/11\mathbb{Z}$ on the weight two module vertex operator algebra of Beneish [Ben19a] extends to an action of M_{11}.

3.4 The Second Largest Mathieu Group

- The forthcoming work with Cheng and Mertens also analyses the case that $G = M_{23}$ is the second largest sporadic simple group of Mathieu.
- $\#M_{23} = 10200960$.
- We formulate results that use the representation theory of M_{23} to tie together the arithmetic of elliptic curves with coprime conductors.

Theorem 6 (Cheng–D–Mertens)**.** *Set $G = M_{23}$. Let $D < 0$ be a fundamental discriminant that is coprime to $\#G$ and satisfies $(\frac{D}{23}) = 1$, and let p be a prime divisor of $h(D)$ that is coprime to both D and $\#G$. For such D and p, suppose that*

($\leftmoon$) $m_\chi(W_D) \equiv 0 \bmod p$ *for all irreducible characters χ of G, for all $W \in \mathcal{W}(4)^{\mathrm{qop}}_{(G,t_1)}$.*

Then for $N = 11$ we have that

($\star$) $\mathrm{Sel}_p(J_0(N) \otimes D) \neq \{0\}$*, and either* $Ш(J_0(N) \otimes D)[p] \neq \{0\}$ *or $J_0(N) \otimes D$ has positive rank.*

If also $(\frac{D}{2}) = 1$ or $(\frac{D}{7}) = -1$ then ($\star$) holds for $N = 14$, and if $(\frac{D}{3}) = 1$ or $(\frac{D}{5}) = -1$ then ($\star$) holds also for $N = 15$.

Thank you for your time.

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