Harmonic weak Maass forms and extensions

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Modularity in Quantum Systems (MOD20) -KITP
November 5, 2020

Introduction

- **Part 1** • A representation theoretic classification of harmonic weak Maass forms (joint work with Kathrin Bringmann).
  • Some examples of mock modular forms which are not harmonic.
- **Part 2** • What are Siegel mock modular forms?
  • A Siegel mock modular form of weight $3/2$ and genus 2 arising from arithmetic 0-cycles.

Harmonic weak Maass forms: review

Let (ρ, V) be a finite dimensional complex representation of $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$.
Harmonic weak Maass forms are smooth functions
$f : \mathfrak{H} \longrightarrow V$ satisfying the conditions: $\quad \tau = u + iv, \quad q = e(\tau) = e^{2\pi i \tau}$

$$f(\gamma\tau) = (c\tau + d)^k \rho(\gamma) f(\tau)$$

$$\Delta_k f = 0 \qquad \text{(harmonicity)}$$

$$f(\tau) = O(e^{Bv}), \quad (\textit{immoderate}\ \text{growth}).$$

Denote the space of these by $H_k^{\mathrm{img}}(\Gamma, \rho)$.
Note that the moderate growth condition is $f(\tau) = O(v^B)$.

Restrict to the scalar case, i.e., dim $V = 1$, and omit ρ.
The Fourier expansion has the form, $f = f^+ + f^-$,

$$f(\tau) = \sum_{n \gg -\infty} c_f^+(n)\, q^n + c_f^-(0)\, w_k(0, v) + \sum_{\substack{n \ll \infty \\ n \neq 0}} c_f^-(n)\, W_k(2\pi n v)\, q^n,$$

where

$$W_k(x) = \Gamma(1-k, -2x) + \begin{cases} \frac{(-1)^{1-k}\pi i}{(k-1)!} & \text{for } x > 0, \\ 0 & \text{for } x < 0, \end{cases}$$

$$w_k(0, v) = \begin{cases} v^{1-k} & \text{for } k \neq 1, \\ -\log v & \text{for } k = 1. \end{cases}$$

There are subspaces defined by conditions on the support of the coefficients:

$$
\begin{aligned}
H_k^\flat(\Gamma) &= \{f \in H^{\mathrm{img}}(\Gamma) \mid \ c_f^-(n) = 0 \text{ for } n > 0\},\\
H_k(\Gamma) &= \{f \in H^{\mathrm{img}}(\Gamma) \mid \ c_f^-(n) = 0 \text{ for } n \geq 0\}\\
H_k^\sharp(\Gamma) &= \{f \in H^{\mathrm{img}}(\Gamma) \mid \ c_f^+(n) = 0 \text{ for } n < 0\},\\
H_k^{\mathrm{mg}}(\Gamma) &= H_k^\flat(\Gamma) \cap H_k^\sharp(\Gamma) = \text{ functions of moderate growth}\\
&= \text{ automorphic forms.}
\end{aligned}
$$

This space includes the weakly holomorphic and classical modular forms

$$
M_k^!(\Gamma) = \{f = f^+ \in H^{\mathrm{img}}(\Gamma)\} \quad \supset \quad M_k(\Gamma) = M_k^!(\Gamma) \cap H_k^\sharp(\Gamma).
$$

Maass raising and lowering operators and the Laplacian are given by

$$R_k = 2i\frac{\partial}{\partial\tau} + k\,v^{-1}, \quad L_k = -2iv^2\frac{\partial}{\partial\bar{\tau}}, \quad \Delta_k = -R_{k-2}\circ L_k,$$

and the Bruinier-Funke ξ-operator is

$$\xi_k f(\tau) = v^{k-2}\,\overline{L_k f(\tau)}, \qquad f \mapsto \xi_k f =: \text{shadow of } f.$$

BF show that there is an exact sequence

$$0 \longrightarrow M_k^!(\Gamma) \longrightarrow H_k(\Gamma) \xrightarrow{\xi_k} M_{2-k}^!(\Gamma) \longrightarrow 0.$$

In particular, every form $g \in M_{2-k}^!(\Gamma)$ is a shadow, i.e., $g = \xi_k(f)$.

Classification using representation theory of $\mathrm{SL}_2(\mathbb{R})$

The first step is to pass to smooth functions on G.

Let $G = \mathrm{SL}_2(\mathbb{R})$, $\mathfrak{g} = \mathrm{Lie}\,(G)_{\mathbb{C}}$, $K = \mathrm{SO}(2)$.

Then $G \longrightarrow G/K \simeq \mathfrak{H}$, $g \mapsto g(i)$, and we can pullback and shift.

For $f \in H_k^{\mathrm{img}}(\Gamma, \rho)$, define $\widetilde{f} \in C^\infty(G, V)$ by

$$\widetilde{f}(g) = j(g,i)^{-k}\, f(g(i)). \qquad j(g,\tau) = c\tau + d.$$

Then

$$\widetilde{f}(\gamma g\, \mathbf{k}_\theta) = \rho(\gamma)\, \widetilde{f}(g)\, e^{ik\theta}, \qquad \mathbf{k}_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

$$C\widetilde{f} = (k^2 - 2k)\,\widetilde{f}, \qquad C = \text{Casimir op.}$$

$$|\widetilde{f}(g)| = O(e^{B||g||}), \qquad \text{immoderate growth.}$$

Classification: 9 types

Let $C^\infty(G, V)^o = K$-finite vectors in $C^\infty(G, V)$.

The right action of G on $C^\infty(G, V)$ makes $C^\infty(G, V)^o$ into a $(\mathfrak{g}, K)$-module. Here $\mathfrak{g} = Lie(G)_{\mathbb{C}}$.

Problem: Given $f \in H_k^{\text{img}}(\Gamma)$, describe the $(\mathfrak{g}, K)$-submodule $\Pi(\widetilde{f})$ of $C^\infty(G, V)^o$ generated by $\widetilde{f}$.

This question was studied previously in

R. Schulze-Pillot, *Weak Maass forms and* $(\mathfrak{g}, K)$*-modules*, Ramanujan J. **26** (2011), 437–445.

Answer: (Bringmann-K.) There are 9 possibilities for the $(\mathfrak{g}, K)$-module $\Pi(\widetilde{f})$. and all 9 cases occur for vector valued Maass forms.

The most interesting cases are the *indecomposable* modules defined as *non-split extensions of irreducibles*.

Classification: dictionary

Recall that: on f $\iff$ on $\widetilde{f}$

$$\text{raising} = R_r \iff X_+ = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \in \mathfrak{p}_+, \qquad \mathfrak{g} = \mathfrak{k} + \mathfrak{p}_+ + \mathfrak{p}_-,$$

$$\text{lowering} = L_r \iff X_- = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \in \mathfrak{p}_-$$

$$\Delta_k f = -R_{k-2}L_k f = 0 \iff X_+X_-\widetilde{f} = 0 \iff C\tilde{f} = ((k-1)^2 - 1)\tilde{f}.$$

Writing $j = k \pm 2r$, let

$$\widetilde{f}_j = X_\pm^r \widetilde{f}.$$

A key point:

f harmonic $\implies$ the K-types in $\Pi(\widetilde{f})$ occur with multiplicity 1

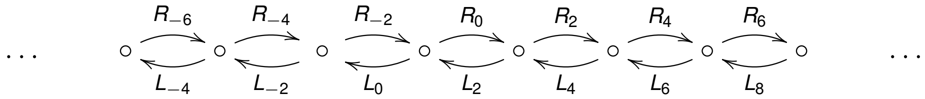
$\implies$ the $\widetilde{f}_j$'s span $\Pi(\widetilde{f})$.

Classification: principal series representations

Recall that the basic building blocks are the irreducible $(\mathfrak{g}, K)$-modules. These are all constituents of the principal series $I(s, \epsilon)$ of $\mathrm{SL}_2(\mathbb{R})$. Here for $s \in \mathbb{C}$ and $\epsilon = 0, 1$, ϕ is a smooth function on G with

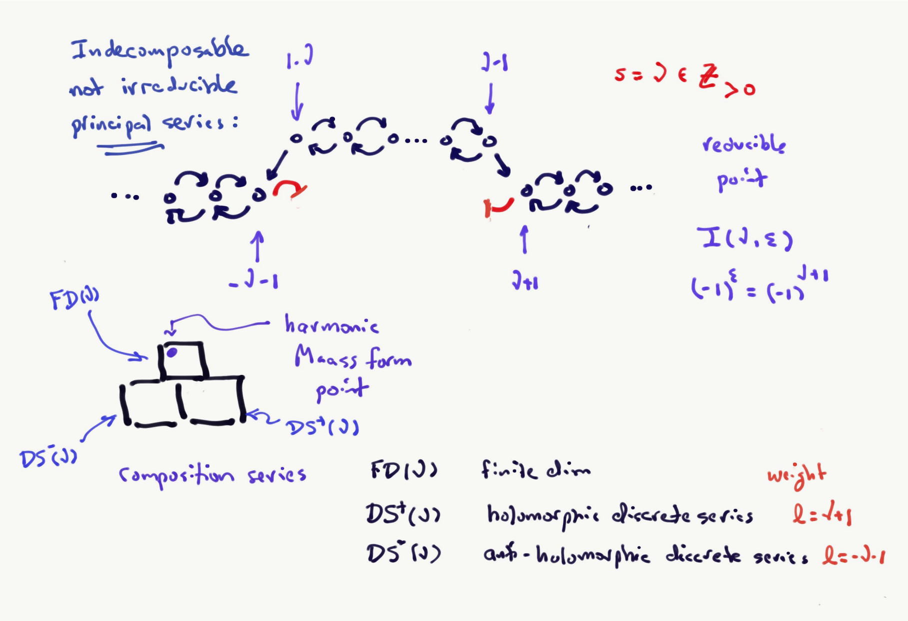
$$\phi \in I(s, \epsilon), \qquad \phi\left(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} g\right) = \mathrm{sgn}(a)^{\epsilon} |a|^{s+1} \phi(g).$$

For generic s, $I(s, \epsilon)$ is irreducible. The picture for $\epsilon = 1$ is:

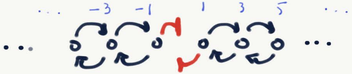


On $I(s, \epsilon)$, the Casimir acts by $C\phi = (s^2 - 1)\phi$.

Classification: the irreducible building blocks



One more case: $s = \lambda = 0$



$LDS^+(0)$ holomorphic limit DS

$LDS^-(0)$ antihol. limit DS.

So now we have the building blocks.

Since the Casimir eigenvalue is fixed on $\Pi(\widetilde{f})$, all of its constituents must be pieces of the same principal series.

Of course, the 9 cases include the irreducibles generated by a lowest weight vector:

$$FD(\nu), \qquad DS^+(\nu), \qquad LDS^+(0),$$

of weights

$$1-\nu, \qquad \nu+1, \qquad 1.$$

For reference, let us call these cases (1), (2) and (3).

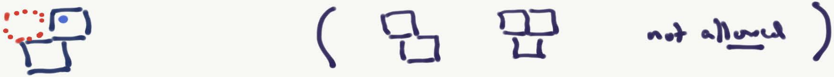
Some examples from $I(\nu, \epsilon)$, $\nu > 0$

harmonic vector

$X_+ X_- v_\ell = 0$

$\ell = 1 - \nu$

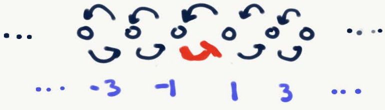
$I(-\nu, \epsilon)$ is dual to $I(\nu, \epsilon)$:



These examples are obtained by omitting part of $I(\nu, \epsilon)$ or $I(-\nu, \epsilon)$. For reference, we will call these cases (4), (5), (6) and (7).

There are two more cases:

Splicing $LDS^+(0)$ and $LDS^-(0)$!



weights.

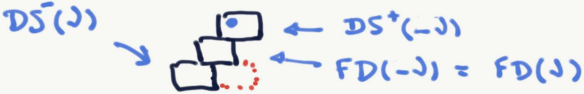
case (8)

An exotic case:

case (9)

◎ = harmonic generator

○ = harmonic, not generator

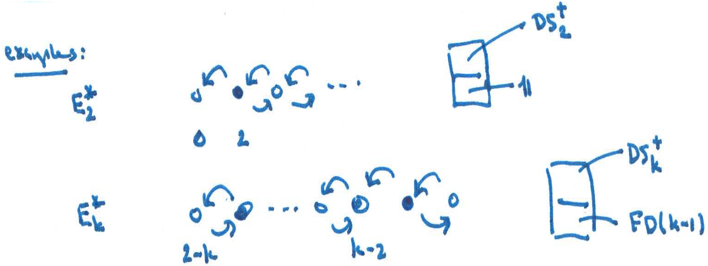


For reference, we will call these cases (8) and (9).

They are not pieces of principal series.

Classification: building examples

All of this is elementary Lie theory. The issue is to produce harmonic Maass form realizations. For example, here is **case (7)**:



These are pictures of the famous weight 2 Eisenstein series and its lesser known weight *k* generalization.

Example: $E_2^*(\tau)$

The familiar comutation

$$E_2^*(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)\, q^n - \frac{3}{\pi v}$$

$$L_2 E_2^* = -2iv^2 \frac{\partial}{\partial \bar{\tau}} E^*(\tau) = \frac{3}{\pi} \qquad \Longrightarrow \mathbb{C} \cdot 1 \subset M(\widetilde{E_2^*})$$

gives the trivial representation as a proper submodule of $M(\widetilde{E_2^*})$.
Observe that you cannot get back to the weight 2 form by applying $R_0 = 2i \frac{\partial}{\partial \tau}$. You are stuck in the trivial submodule.

Example: $E_k^*(\tau)$

The weight k analogue E_k^* seems to be less well known, no doubt because it only occurs in a vector valued version.

For $m \in \mathbb{Z}_{\geq 0}$, let $\mathcal{P}_m$ be the space of polynomials in X of degree at most m with action of $\mathrm{SL}_2(\mathbb{R})$:

$$\rho_m(\gamma)p(X) = (-cX + a)^m p\left(\frac{dX - b}{-cX + a}\right), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

For $0 \leq r \leq m$, define $e_{r,m-r} : \mathfrak{H} \longrightarrow \mathcal{P}_m$ by

$$e_{r,m-r}(\tau)(X) = \frac{(-1)^{m-r}}{r!} v^{r-m} (X - \tau)^r (X - \bar{\tau})^{m-r}.$$

Example: $E_2^*(\tau)$

Then, for all $\gamma \in \mathrm{SL}_2(\mathbb{R})$,

$$e_{r,m-r}(\gamma(\tau)) = (c\tau + d)^{m-2r} \rho_m(\gamma) e_{r,m-r}(\tau),$$

and

$$e_{m,0} = \mathcal{P}_m\text{-valued holomorphic form of weight } -m.$$

It defines a vector-valued harmonic Maass form with $(\mathfrak{g}, K)$-module

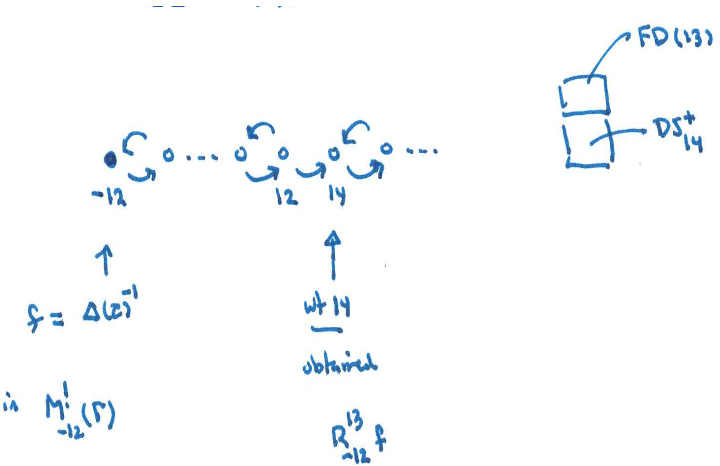
$$M(\tilde{e}_{m,0}) = \mathrm{span}\{\tilde{e}_{m,0}, \tilde{e}_{m-2,2}, \ldots, \tilde{e}_{2,m-2}, \tilde{e}_{0,m}\} \simeq FD(m+1).\ \textbf{case (1)}$$

Define:

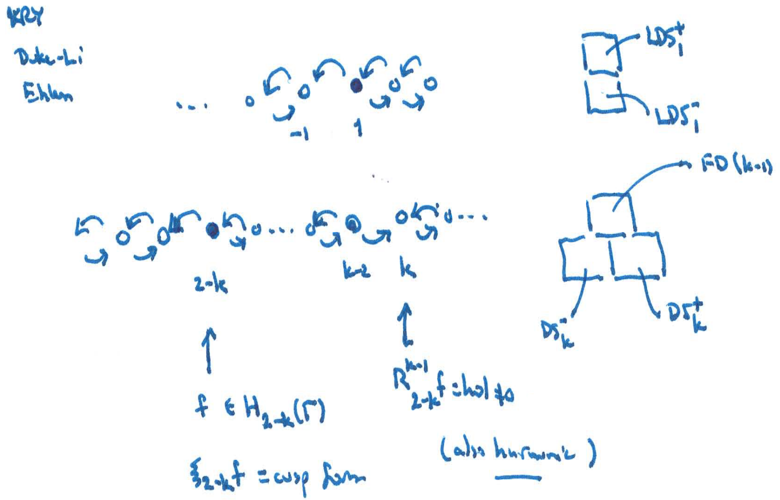
$$E^*_{m+2}(\tau) = \sum_{r=0}^{m} \frac{1}{r+1} \binom{m}{r} e_{r,m-r}(\tau)\, R^r E_2^*(\tau).$$

Fun Exercise: Check that this gives the claimed $(\mathfrak{g}, K)$-module!

Example: Weakly holomorphic forms



Here is a **case (5)** example, the weakly holomorphic form $\Delta(\tau)^{-1}$.



Here are cases (8) and (4).

Example: A weight 1 incoherent Eisenstein series

Here is a favorite example for **case (8)** arising from a pair of Eisenstein series

$$E_D^{\pm}(\tau, s) = D^{\frac{1}{2}(s+1)}\Lambda(s, \chi_D) \cdot v^{\frac{s}{2}} \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \Phi_D^{\pm}(\gamma)\,(c\tau + d)^{-1}|c\tau + d|^{-s},$$

$$D \equiv 3 \bmod 4, \ \ \text{prime}, \ \ \mathcal{K} = \mathbb{Q}(\sqrt{-D}), \chi_D$$

$$\Phi_D^{\pm}(\gamma) = \begin{cases} \chi_D(a) & \text{if } D \mid c \\ \pm i\, D^{-\frac{1}{2}}\, \chi_D(c) & \text{if } (c, D) = 1 \end{cases}$$

$$\Lambda(s, \chi_D) = \pi^{-\frac{1}{2}(s+1)}\,\Gamma(\frac{1}{2}(s+1))\,L(s, \chi_D)$$

Functional equation: $\quad E_D^{\pm}(\tau, -s) = \pm E_D^{\pm}(\tau, s).$

Example: A weight 1 incoherent Eisenstein series

Then

$$E_D^-(\tau,0)=0, \qquad \frac{1}{2}E_D^+(\tau,0)=h_D+2\sum_{n=1}^{\infty}\rho_D(n)\,q^n.$$

where

$$\zeta_D(s)=\sum_{n=1}^{\infty}\rho_D(n)\,n^{-s}=\text{zeta function of }\mathbb{Q}(\sqrt{-D}).$$

There is a harmonic weak Maass form here:

$$\begin{aligned} f_D(\tau) &:= \frac{\partial}{\partial s}E_D^-(\tau,s)|_{s=0} \\ &= \sum_{n=1}^{\infty}a_D(n)\,q^n + a_D(0,v) + \sum_{n=1}^{\infty}a_D(-n,v)\,q^{-n}. \end{aligned}$$

Example: A weight 1 incoherent Eisenstein series

There are nice explicit formulas for the coefficients. For example, for $n > 0$,

$$a_D(n) = -2\log(D)(\mathrm{ord}_D(n)+1)\,\rho_D(n) - 2\sum_{p\neq D}\log(p)\,(\mathrm{ord}_p(n)+1)\,\rho_D(\frac{n}{p})$$

These are summarized in the Mellin transform:

$$\int_0^\infty (f_D(iv) - a_D(0,v))\, v^{s-1}\, dv$$

$$= \Lambda_D(s)\left(\log(D) + \frac{\Lambda'(s,\chi_D)}{\Lambda(s,\chi_D)} - \frac{\Lambda'(s)}{\Lambda(s)}\right).$$

What does this mean? Can other mock modular forms be constructed via Mellin inverse transforms?

A Siegel mock modular form

A detailed description of the mock modular form of genus 2 arises as a generating series for arithmetic 0-cycles is one of the main results of the book with Michael Rapoport and Tonghai Yang.

This is the analogue of the weight 1 'favorite ' example discussed earlier. Here is the setup:

$$\begin{aligned}
B &= \text{indefinite division quaternion algebra over } \mathbb{Q} \\
D = D(B) &= \text{prod of ramified primes} \\
\tau &= u + iv \in \mathfrak{H}_2 \\
\mathcal{E}(\tau, s; B) &= \text{weight } 3/2 \text{ incoherent Siegel-Eisenstein series} \\
\mathcal{E}(\tau, 0; B) &= 0 \qquad \text{incoherent} \\
\mathcal{E}'(\tau, 0; B) &= \phi_2^B(\tau) + \sum_{\substack{T \in \mathrm{Sym}_2(\mathbb{Z}) \\ T \not\geq 0}} a^B(T, v)\, q^T.
\end{aligned}$$

A Siegel mock modular form

Our Siegel mock modular form is then the holomorphic generating series

$$\phi_2^B(\tau) := \sum_{T \in \mathrm{Sym}_2(\mathbb{Z})_{>0}} \widehat{\deg}\, \mathcal{Z}(T)\, q^T, \quad \tau \in \mathfrak{H}_2,\ q^T = e(\mathrm{tr}(T\tau)).$$

Here the quantities $\widehat{\deg}\, \mathcal{Z}(T)$ are degrees of 0-cycles on an arithmetic surface, the Shimura curve associated to B.

They may be viewed as analogues of the Hurwitz class numbers that occur in the mock part of Zagier's weight $\frac{3}{2}$ Eisenstein series.

A Siegel mock modular form

Here is the underlying representation theory.

$$K \subset G = \mathrm{Mp}_2(\mathbb{R}) \longrightarrow \mathrm{Sp}_2(\mathbb{R}) \supset \underline{U(2)} \simeq U(2)$$
$$\Gamma \longrightarrow \Gamma_0(4D(B)^o) \subset \mathrm{Sp}_2(\mathbb{Z}).$$
$$G \longrightarrow G/K = \mathrm{Sp}_2(\mathbb{R})/U(2) = \mathfrak{H}_2.$$

Let $f(\tau) = \mathcal{E}'(\tau, 0; B)$ and

$$\widetilde{f}(g) = j(g,i)^{-3/2}\, f(g(i \cdot 1_2)), \qquad j(g,\tau) = \det(c\tau + d).$$

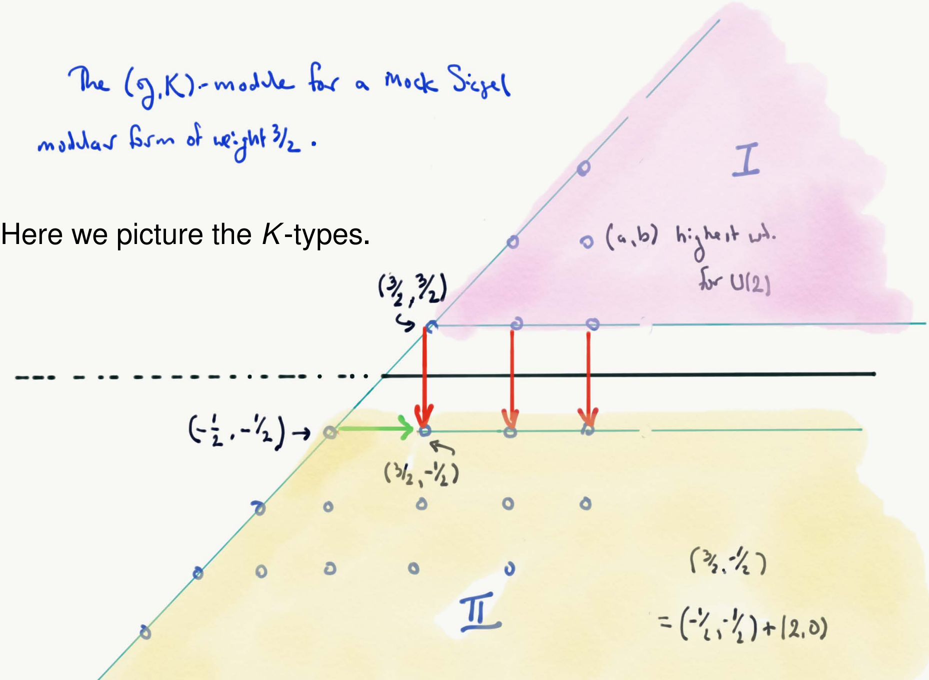
Let

$$\Pi(\widetilde{f}) \subset C^\infty(\Gamma\backslash G)^o$$

be the $(\mathfrak{g}, K)$-module generated by $\widetilde{f}$.

The $(\mathfrak{g}, K)$-module for a Mock Siegel modular form of weight $3/2$.

Here we picture the K-types.



A Siegel mock modular form

The picture shows the K types $\sigma_{(a,b)}$ for the reducible degenerate principal series $I(s,\epsilon)$ for G, indexed by their highest weight $(a,b) \in -(\frac{1}{2},\frac{1}{2}) + 2\mathbb{Z}^2$ with $a \geq b$.

They have multiplicity 1.

The K-types are linked by various raising and lowering operators coming from $\mathfrak{p}_+$ and $\mathfrak{p}_-$.

In the reducible degenerate principal series, transitions crossing the black line are 0, all others are nonzero.

The regions labeled I and II are the irreducible summands Π_1 and Π_2.

They are generated by the scalar K-types $\sigma_{(\frac{3}{2},\frac{3}{2})} = \det^{\frac{3}{2}}$ and $\sigma_{(-\frac{1}{2},-\frac{1}{2})} = \det^{-\frac{1}{2}}$ respectively.

Π_1 is a lowest weight representation (i.e., occurs for holomorphic Siegel modular forms of weight $\frac{3}{2}$).

A Siegel mock modular form

The $(\mathfrak{g}, K)$-module $\Pi(\widetilde{f})$ is a non-split extension with the same K-types as the principal series.

It has Π_2 as a submodule and Π_1 as quotient:

$$0 \longrightarrow \Pi_2 \longrightarrow \Pi(\widetilde{f}) \longrightarrow \Pi_1 \longrightarrow 0.$$

The first downward red arrow is a <u>non-zero</u> lowering operator carrying

$$\sigma_{(\frac{3}{2},\frac{3}{2})} \quad \longrightarrow \quad \sigma_{(\frac{3}{2},-\frac{1}{2})} = (\det)^{-1/2}\mathrm{Sym}_2(\mathbb{C}^2), \qquad \text{not a scalar } K\text{-type.}$$

It corresponds to the classical lowering operator

$$L_{\frac{3}{2}} := \frac{\partial}{\partial\bar{\tau}} = \begin{pmatrix} \frac{\partial}{\partial\bar{\tau}_{11}} & \frac{1}{2}\frac{\partial}{\partial\bar{\tau}_{12}} \\ \frac{1}{2}\frac{\partial}{\partial\bar{\tau}_{12}} & \frac{\partial}{\partial\bar{\tau}_{22}} \end{pmatrix}.$$

It cannot be inverted.

A Siegel mock modular form

So the shadow $L_{\frac{3}{2}} f$ of $f = \mathcal{E}'(\tau, 0; B)$ corresponds to the 'not-holomorphic representation Π_{II}.

It can be linked via the horizontal red arrow to a Siegel-Eisenstein series of weight $-\frac{1}{2}$ associated to the space V^B of signature $(1, 2)$, hence the appearance of the indefinite representation numbers $\mathrm{Rep}(T; L^B)$ in the non-holomorphic Fourier coefficients of $\mathcal{E}'(\tau, 0; B)$.

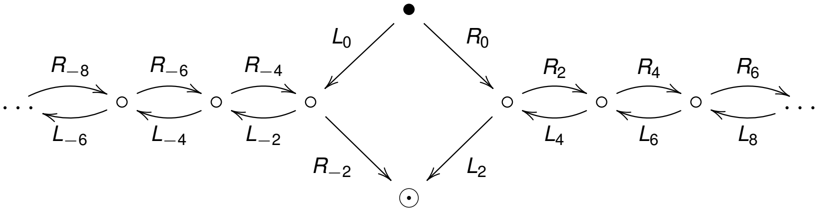
A Siegel mock modular form

I know of one other mock Siegel modular form of genus 2 constructed by Martin Westerholdt-Raum, *Harmonic weak Siegel Maaß forms I*, IMRN (2016) pp1442–1472.
The associated $(\mathfrak{g}, K)$-module is a subquotient of the degenerate principal series for $\mathrm{Sp}_2(\mathbb{R})$.

A final remark: non-harmonic examples

A related famous case:

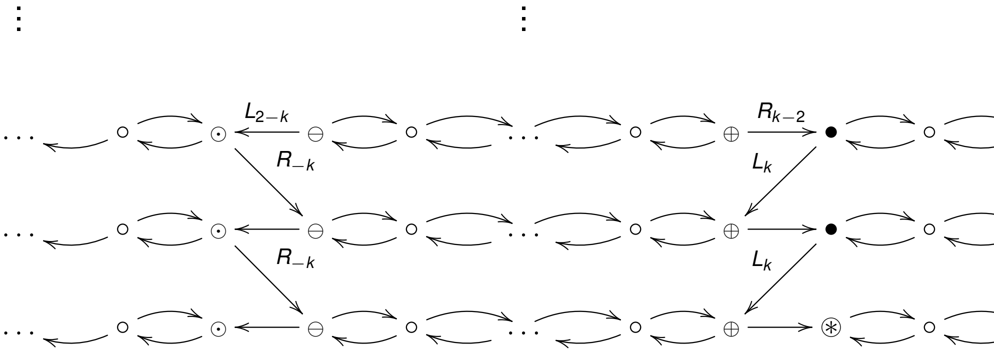
Figure 1. The $(\mathfrak{g}, K)$-module for the Kronecker limit formula



The black dot is the function on $G = \mathrm{SL}_2(\mathbb{R})$ pulled back from $\log ||\Delta(\tau)||$. The other corners of the square are the pullbacks of $v^2\overline{E_2^*(\tau)}$ (weight -2), $E_2^*(\tau)$, (weight 2) and the constants.

A final remark: without the harmonicity condition:

Figure 2. The $(\mathfrak{g}, K)$-module for the Taylor coefficients A_{r,ℓ,s_0} of $E_{\ell,s}(\tau)$ at $s_0 = k - 1$



A final remark: non-harmonic examples

Thanks!

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M. Westerholt-Raum, *Harmonic weak Siegel Maaß forms I*,

A final remark: non-harmonic examples

There is a classical Eisenstein series of weight $\frac{3}{2}$ associated to B with value at $s = \frac{1}{2}$:

$$\mathcal{E}(\tau, \frac{1}{2}; D) = c(D)\Lambda_D(2) + \sum_{m>0} 2\,\delta(d;D)\,H_0(m;D)\,q^m.$$

Here $D = D(B)$ and

$$c(D)\Lambda_D(2) = -\frac{1}{12}\prod_{p|D}(p-1)$$

$$\delta(d;D) = \prod_{p|D}(1-\chi_d(p))$$

$$H_0(m;D) = \frac{h(d)}{w(d)} \sum_{\substack{c|n \\ (c,D)=1}} c \prod_{\ell|c}(1-\chi_d(\ell)\,\ell^{-1}).$$

A final remark: non-harmonic examples

The second term in the Laurent expansion at $s = \frac{1}{2}$ is a generating series for heights of arithmetic curves on the arithmetic surface

$$\mathcal{E}'(\tau, \frac{1}{2}; D) = \sum_m \langle\, \hat{Z}(m, v), \hat{\omega} \,\rangle\, q^m.$$

(i) If $m > 0$ and there is no prime $p \mid D$ for which $\chi_d(p) = 1$, then

$$\begin{aligned}
&\mathcal{E}'_m(\tau, \frac{1}{2}; D) \\
&= 2\,\delta(d; D)\, H_0(m; D) \cdot q^m \cdot \Bigg[\frac{1}{2} \log(d) + \frac{L'(1, \chi_d)}{L(1, \chi_d)} - \frac{1}{2} \log(\pi) - \frac{1}{2}\gamma \\
&+ \frac{1}{2} J(4\pi m v) + \sum_{\substack{p \\ p \nmid D}} \left(\log |n|_p - \frac{b'_p(n, 0; D)}{b_p(n, 0; D)} \right) + \sum_{\substack{p \\ p \mid D}} K_p \log(p) \Bigg].
\end{aligned}$$

A final remark: non-harmonic examples

The non-holomorphic part here involves the special function

$$J(t) = \int_0^\infty e^{-tr}\,[(1+r)^{\frac{1}{2}} - 1]\,r^{-1}\,dr.$$

Mathematica gives the expression

$$J(t) = -2 + \gamma + \pi^{\frac{1}{2}}\,t^{-\frac{1}{2}}\,e^{t} - i\pi\,\mathrm{erf}(it^{\frac{1}{2}}) + \frac{2}{3}\,t\,{}_2F_2(1,1;2,\frac{5}{2};t) + \log(4t).$$

Here $\mathrm{Erfi}(z) = -i\mathrm{erf}(iz)$.

The rest of the Fourier expansion is:

A final remark: non-harmonic examples

(ii) If there is a unique prime $p \mid D$ such that $\chi_d(p) = 1$, then with $K = \mathrm{ord}_p(n)$,

$$\mathcal{E}'_m(\tau, \frac{1}{2}; D) = 2\,\delta(d; D/p)\, H_0(m; D) \cdot (p^k - 1)\, \log(p) \cdot q^m.$$

(iii) If $m < 0$, then

$$\mathcal{E}'_m(\tau, \frac{1}{2}; D) = 2\,\delta(d; D)\, H_0(m; D) \cdot q^m \cdot \frac{1}{4\pi}\, |m|^{-\frac{1}{2}}\, v^{-\frac{1}{2}} \int_1^\infty e^{-4\pi|m|vr} r^{-\frac{3}{2}}\, dr,$$

where, for $m < 0$, $H_0(m; D)$

(iv)

$$\mathcal{E}'_0(\tau, \frac{1}{2}; D) = c(D)\, \Lambda_D(2) \left[\frac{1}{2} \log(v) - 2\frac{\zeta'(-1)}{\zeta(-1)} - 1 + 2C + \sum_{p|D} \frac{p \log(p)}{p-1} \right].$$

(v) All other Fourier coefficients of $\mathcal{E}'(\tau, \frac{1}{2}; D)$ vanish.

A final remark: non-harmonic examples

Next we compute the ‘shadow’:

$$\xi_{\frac{3}{2}}\{\, \mathcal{E}'(\tau, \frac{1}{2}; D)\,\} = -\,\frac{1}{4}\sum_{m=1}^{\infty} \delta(d;D)\, H_0(m;D)\cdot (4\pi m)^{-\frac{1}{2}}\, q^{-m}\,\Gamma(\frac{1}{2}, 4\pi m v)$$

$$+\, c(D)\,\Lambda_D(2)\,\frac{1}{2}\, v^{\frac{1}{2}}$$

$$-\,\frac{1}{2\pi}\sum_{m=1}^{\infty} \delta(-d;D)\, H_0(-m;D)\, m^{-\frac{1}{2}}\, q^{m}.$$

Here, in the first term, $4m = n^2 d$ where $-d$ is a fundamental discriminant, while, in the third term, $4m = n^2 d$ with $d > 0$ a fundamental discriminant.

Note that hence $\mathcal{E}'(\tau, \frac{1}{2}, B)$ is *not harmonic!*