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P. Aspinwall

D-branes on Calabi-Yau varieties, I

1995 Polchinski : D-branes are important

A D-brane is where an open string ends.

– subspace of X ← target space (Calabi-Yau 3-fold)

The notion of a submanifold of X is poorly defined in string theory.
E.g. pick a 4-cycle on X – turns into a 2-cycle under monodromy

Douglas (Brunner, Douglas, Lawrence, Romelsberger)
Diaconescu, Fiol, Gomis, ...
Hori, Iqbal, Vafa
Ooguri, Oz, Yin
Johnson, hep-th/0007170

Building D-branes on a CY will take 7 steps

Talk I
1. Build CFT of closed strings
2. Twist to a T.Q.F.T.
3. Add boundaries to world sheet – 1 D-brane

Talk II
4. Add more than one D-brane
5. Remove physical grading dependence
6. Deform to derived category of coherent sheaves

Talks III + IV
7. Analyze non-TQFT aspects – notion of stability

A-Branes – 3-cycles in X ← exploit
B-Branes – even cycles in X ← study

Referencees for #7:
Monodromy – Kontsevich, Horja, Morrison, Seidel + Thomas

IIA/B Superstring – $N=2$ SUSY in 10 dimensions
Compactify on $\mathbb{R}^{1,3} \times X$
↑
Effective $N=2$ SUSY in 4 dimensions

<u>Soliton</u> is a background satisfying equations of motion (of 4 or 10 dim theory).

A <u>BPS soliton</u> is invariant under 1 of the 2 SUSY's.

A D-brane is such a BPS soliton.

In string theory we consider maps $\Phi : \Sigma \to X$
(Σ ↑ Riemann surface with no boundary)

$$S = \frac{1}{4\pi\alpha'}\int d^2x \left\{ (g_{I\bar{J}} - B_{I\bar{J}})\, \partial\Phi^I \bar{\partial}\Phi^{\bar{J}} + g_{I\bar{J}} \left(\psi_L^I \bar{D} \psi_L^{\bar{J}} + \psi_R^I D \psi_R^{\bar{J}} + R_{I\bar{J}K\bar{L}}\, \psi_L^I \psi_L^{\bar{J}} \psi_R^K \psi_R^{\bar{L}} \right) \right\}$$

$I, J = 1, \ldots, 6$ (real coords)

ψ_L^I, ψ_R^I are fermions: ψ_L is a section of $K_\Sigma^{1/2} \otimes \Phi^*(T_X)$

ψ_R . — — - $\bar{K}_\Sigma^{1/2} \otimes \Phi^*(T_X)$

$N = (2,2)$ supersymmetry on Σ

g_{IJ} is metric

B_{IJ} is an antisymmetric tensor
2-form - closed

$$\bar{D}\psi_L^J = \bar{\partial}\psi_L^J + \bar{\partial}\Phi^I \Gamma^J_{IK} \psi_L^K$$

Interesting operators

$$G_L^+ = g_{i\bar{j}} \psi_L^i \partial \Phi^{\bar{j}} \qquad G_L^- = g_{i\bar{j}} \psi_L^{\bar{j}} \partial \Phi^i$$

$$G_R^+ = g_{i\bar{j}} \psi_R^i \bar{\partial} \Phi^{\bar{j}} \qquad G_R^- = g_{i\bar{j}} \psi_R^{\bar{j}} \bar{\partial} \Phi^i$$

generate worldsheet supersymmetry

$g_{i\bar{j}}$ is the Kähler metric ($i, j = 1, \ldots, 3$)
required for $N = (2,2)$ supersymmetry

ψ_L^i lives in $\Phi^*(T_X^{1,0})$ etc.

Also "U(1)-currents"

$$\left.\begin{aligned} J_L &= g_{i\bar{j}}\, \psi_L^i \psi_L^{\bar{j}} \\ J_R &= g_{i\bar{j}}\, \psi_R^i \psi_R^{\bar{j}} \end{aligned}\right\}$$ Give a U(1)-charge to any operator in QFT.

Any operator has a conformal weight $(h, \bar{h})$

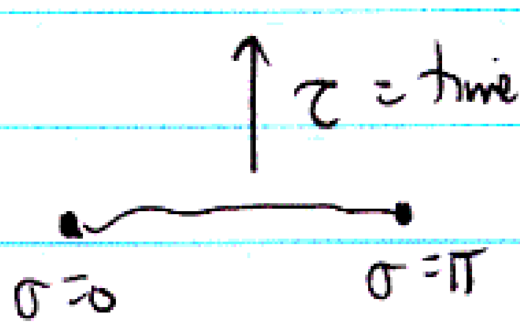
a U(1)-charge $(q, \bar{q})$

$$\boxed{S_L = \Omega_{ijk}\, \psi_L^i \psi_L^j \psi_L^k}$$ "spectral flow operator"

Ω_{ijk} is a (3,0)-form on X (Calabi-Yau $\Rightarrow$ Ω_{ijk} exists and non-vanishing)

Add a boundary to Σ

$\frac{\partial \Phi}{\partial \sigma} = 0$ (Neumann)



$\frac{\partial \Phi}{\partial \tau} = 0$ (Dirichlet)

$z = \sigma + i\tau$, $\bar{z} = \sigma - i\tau$

$\partial \Phi = -\bar{\partial} \Phi$ (Neumann)

$\partial \Phi = \bar{\partial} \Phi$ (Dirichlet)

In general, a single D-brane will impose

$$\partial\Phi^I = R^I{}_J\, \bar\partial\Phi^J$$

R is a matrix with eigenvalues $+1$ or -1

$\partial\Phi^I$ is related to ψ_L^I by SUSY

$\bar\partial\Phi$ is related to ψ_R^I

and so $\psi_L^I = R^I{}_J\, \psi_R^J$.

I need to use the boundary to "identify" the left-moving and right-moving SUSY's into one.

A: $J_L = -J_R$, $S_L = \text{const} \times \overline{S_R}$

B: $J_L = J_R$, $S_L = \text{const} \times S_R$

$\boxed{\text{Maintain } B_{IJ} = 0}$ $\left(\psi_L^I = R^I{}_J \psi_R^J,\ \cancel{J_L = g_{IJ}\psi_L^I\psi_L^J},\ J_L = g_{i\bar j}\psi_L^i\psi_L^{\bar j}\right)$

A: $J_L = -J_R \Rightarrow$ Symplectic (Kähler) form ω of block form $\begin{array}{c|cc} & - & + \\ - & 0 & 1 \\ + & -1 & 0 \end{array}$

$\Rightarrow$ D-brane is Lagrangian

$S_L = \text{Const} \times \overline{S_R} \Rightarrow$ D-brane is Special Lagrangian

$$\mathrm{Re}(e^{i\varphi}\Omega) = \text{Volume form}$$

$$\mathrm{Im}(e^{i\varphi}\Omega) = 0.$$

B: $J_L = J_R$, $S_L = \text{const} \cdot S_R$

$\Rightarrow$ D-brane is an even-dimensional subspace

= embedded as a complex submanifold

So far we ignored B-field.

Consider first closed strings.

For a map $\Phi : \Sigma \to X$

B gives a phase to the path integral $\exp\left\{2\pi i \int_\Sigma \Phi^*(B)\right\}$

Actually, $B \in H^2(X, U(1))$.

If we consider $B \in H^2_{DR}(X, \mathbb{R})$, $B \equiv B + \Gamma$ for $\Gamma \in H^2(X, \mathbb{Z})$

and $B \equiv B + d\Lambda$ for Λ = 1-form.

"gauge invariance"

B should be considered a flat connection on a gerbe.

If Σ has a boundary, this gauge invariance is broken

unless we include $\exp\left\{2\pi i \oint_{\partial\Sigma} \Phi^* A\right\}$

where $A \to A - \Lambda$

A is a 1-form on boundary.

A is a connection on a bundle over $\partial\Sigma$.

B is a flat connection on U(1)-gerbe
A is a flat connection on a U(1)-bundle?

Topological Field Theory

ψ_L^i was a section of $K_\Sigma^{1/2} \otimes \Phi^*(T_X^{1,0})$, etc.

Instead, consider "+ twist"

ψ_L^i is a section of $\Phi^*(T_X^{1,0})$
and $\psi_L^{\bar{i}}$ - - - - - - - of $K_\Sigma \otimes \Phi^*(T_X^{0,1})$

or "− twist"

ψ_L^i section of $K_\Sigma \otimes \Phi^*(T_X^{1,0})$
$\psi_L^{\bar{i}}$ - - - - - - $\Phi^*(T_X^{0,1})$

Similarly "twist" ψ_R to give 4 new theories
$(+,+), (+,-), (-,+), (-,-)$

$(+,+)$ theory is the target space complex conjugate of $(-,-)$
$(+,-)$ - - - - - - - $(-,+)$

A: $(+,-)$
B: $(-,-)$

Definition If the A-model of $\Phi : \Sigma \to X$ is equivalent to the B-model of $\Phi : \Sigma \to Y$ then X and Y are MIRROR Calabi-Yau's.

The original had supersymmetries on world-sheet ← BRST symmetry Q
– instead have 0-forms ← - - - - G
and 1-forms on Σ

e.g. A-model

$$\left.\begin{array}{l} \psi_L^i \to \chi^i \\ \psi_R^{\bar\imath} \to \chi^{\bar\imath} \end{array}\right\} \Sigma\text{-scalars}$$

$$\left.\begin{array}{l} \psi_L^{\bar\imath} \to \psi_z^{\bar\imath} \\ \psi_R^{i} \to \psi_{\bar z}^{i} \end{array}\right\} \text{1-forms} \quad \text{"} \psi_z^{\bar\imath}\, dz + \psi_{\bar z}^{i}\, d\bar z \text{"}$$

$$\delta \Phi^I = i\alpha \chi^I$$

$$\delta \chi^i = \delta \chi^{\bar\imath} = 0$$

$$\delta \psi_z^{\bar\imath} = -\alpha \partial \Phi^{\bar\imath} - i\alpha \chi^{\bar\jmath} \Gamma^{\bar\imath}_{\bar\jmath \bar k} \psi_z^{\bar k}$$

$$\delta \psi_{\bar z}^{i} = -\alpha \bar\partial \Phi^{i} - i\alpha \chi^{j} \Gamma^{i}_{jk} \psi_{\bar z}^{k}$$

where $\delta x = -i\alpha \{Q, x\}$

Suppose we consider only those operators for which $\{Q, \chi\} = 0$
Then any correlation function $\langle \chi_a \chi_b \cdots \rangle$ is invariant
under a shift $\chi \mapsto \chi + \{Q, -\}$ or a similar shift of S.

In A-model changing complex structure of X can shown to change
S by $\{Q, -\}$ — trivial.

Changing $B + iJ$ does matter!

$$B + iJ \in H^2(X, \mathbb{C}^*)$$

$\uparrow$ Kähler form

B-model Changing $B + iJ$ shifts action by $\{Q, -\}$ .. trivial
Changing cx. str. matters.