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D-branes on Calabi-Yau varieties, II

A-model Depends on $B + iJ$
Independent of complex structure

"0-forms" χ^I

"Local operators" $\chi_W = W_{IJK} \chi^I \chi^J \chi^K \ldots$

Let $W = W_{IJK}\, d\Phi^I\, d\Phi^J\, d\Phi^K \ldots$ p-form

Then $\{Q, \chi_W\} = -\chi_{dW}$

So "Q-cohomology is de Rham cohomology"

For 3 2-forms we have "correlation functions"

$$\langle \chi_a \chi_b \chi_c \rangle = \int_X a \wedge b \wedge c + \text{instanton corrections (rational curve)}$$

H^2_{DR} generates the "quantum cohomology ring".

D-Branes – A-Branes
flat vector bundles $E \to M$, M is SLag submanifold.

Open string vertex operators $H^*_{DR}(M, \mathrm{End}\, E)$ (?)

B-model — Depends on cx structure
Independent of $B + J$

"0-forms" $\eta^{\bar{i}} = \psi_L^{\bar{i}} + \psi_R^{\bar{i}}$

$\theta_i = g_{i\bar{j}} (\psi_L^{\bar{j}} - \psi_R^{\bar{j}})$

Operators $\chi_V = V_{\bar{i}_1 \bar{i}_2 \cdots}{}^{j_1 j_2 \cdots} \eta^{\bar{i}_1} \eta^{\bar{i}_2} \cdots \theta_{j_1} \theta_{j_2} \cdots$ $\quad \begin{matrix} j_1, \ldots, j_p \\ i_1, \ldots, i_q \end{matrix}$

Let $V = V_{\bar{i}_1 \bar{i}_2 \cdots}{}^{j_1 j_2 \cdots} \, d\bar{\Phi}^{\bar{i}_1} d\bar{\Phi}^{\bar{i}_2} \cdots \frac{\partial}{\partial \Phi_{j_1}} \frac{\partial}{\partial \phi_{j_2}} \cdots \in C^q_{\bar{\partial}}(\Lambda^p T^{1,0})$

Then $\{Q, \chi_V\} = -\chi_{\bar{\partial} V}$.

So Q-cohomology is Dolbeault cohomology in $\Lambda^p T^{1,0}$.

For $a, b, c \in H^1_{\bar{\partial}}(T^{1,0})$

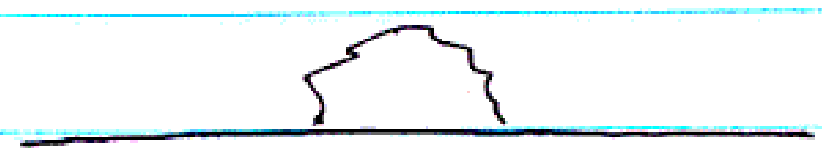
$$\langle \chi_a \chi_b \chi_c \rangle = \int a^i \wedge b^j \wedge c^k \wedge \Omega \wedge \Omega_{ijk}$$

$$\Omega_{ijk} = \bar{\Omega}^{\bar{i}\bar{j}\bar{k}} g_{i\bar{i}} g_{j\bar{j}} g_{k\bar{k}}$$

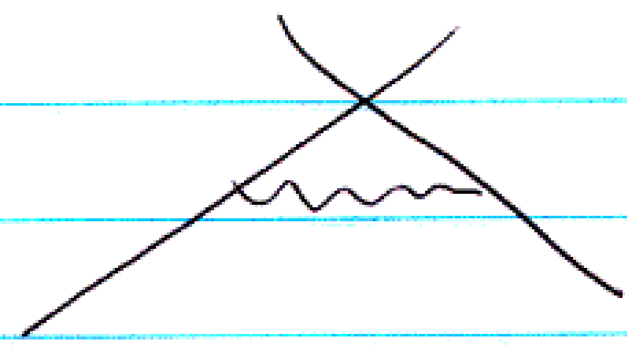
D-Branes — B-Branes
- flat vector bundles over complex submanifolds?
- Just vector bundles over the whole of X
  + modify TQFT so it can cope with holomorphic vector bundles (non-flat)
  i.e. Curvature is (1,1)

Open string vertex operators — $H^{0,q}(X, \operatorname{End} E)$.

A-Brane



What about



At one end of string ($\sigma = 0$) $\quad$ $\mathrm{Re}(e^{ia}\Omega) = \mathrm{Vol}$ form

$\mathrm{Im}(e^{ia}\Omega) = 0$

At other end $\quad$ $\mathrm{Re}(e^{ib}\Omega) = \mathrm{Vol}$

$\mathrm{Im}(e^{ib}\Omega) = 0$

If $e^{ia} = e^{ib}$ then we have integral oscillator modes on open string

$e^{ia} \neq e^{ib}$ - - - - - non-integral - - - - -

Introduce $\xi = \frac{1}{\pi}(a-b)$ $\quad$ $\xi \in [0,2)$

If $\xi \neq 0$ Spacetime SUSY is destroyed.

Back to TQFT —

Only consider cases where SUSY is OK

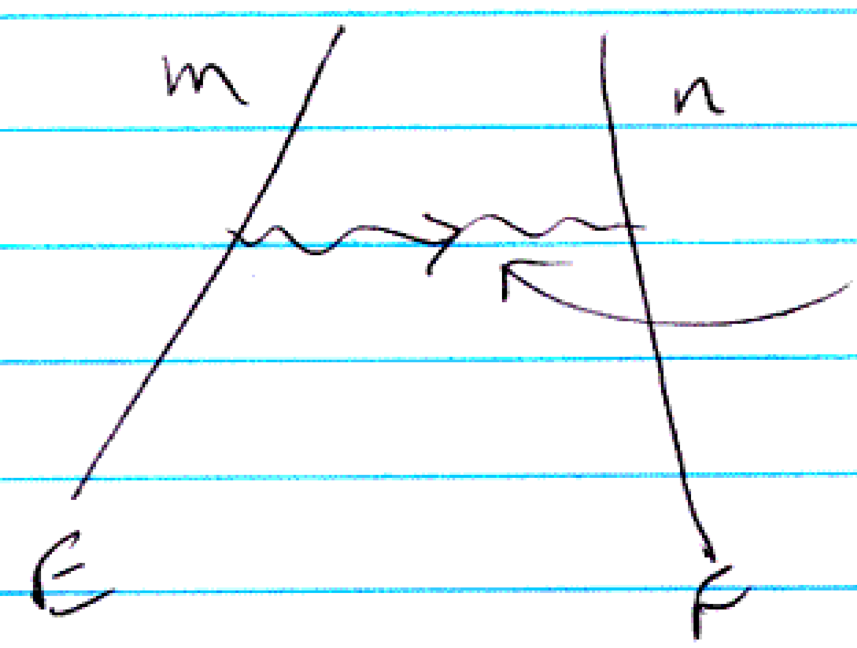
$\xi \in 2\mathbb{Z}$

Also introduce anti-branes — thought of as branes with $a \to a + \pi$.

— Extends ξ to $\mathbb{Z}$.

In TQFT I associate integers with D-Branes
- claim - these integers are associated with "ghost-number"

B-branes



$H^{0,q}_{\bar\partial}(X, E^\vee \otimes F)$

"ghost number" $= q + n - m$.

E, F are vector bundles over X.

B-model   Let $\mathcal{E} = \bigoplus_{n=-\infty}^{\infty} \mathcal{E}^n$ $\mathcal{E}^n$ is a locally free sheaf
(finite number of summands are nonzero)

Open string vertex op. is in $\mathrm{Ext}^q(\mathcal{E}^m, \mathcal{E}^n) \cong H^{0,q}_{\bar\partial}(X, (E^m)^\vee \otimes E^n)$

Deformations of TQFT $S \to \cancel{S + \int \theta \, d^2z}$ $S + \oint_C \{G, \varphi\} \, dz$

We may deform a TQFT by adding an operator to the action
Ghost number one operators maintain a relationship with untwisted theory.

$\mathrm{Ext}^0(\mathcal{E}^n, \mathcal{E}^{n+1})$ ← Focus here
$\mathrm{Ext}^1(\mathcal{E}^n, \mathcal{E}^n)$ ← already know these
$\mathrm{Ext}^2(\mathcal{E}^n, \mathcal{E}^{n-1})$
$\mathrm{Ext}^3(\mathcal{E}^n, \mathcal{E}^{n-2})$

Any can be used.

Use $\mathrm{Ext}^0(\mathcal{E}^n, \mathcal{E}^{n+1}) = \mathrm{Hom}(\mathcal{E}^n, \mathcal{E}^{n+1})$

i.e. $d_n : \mathcal{E}^n \to \mathcal{E}^{n+1}$

Applying this deformation deforms Q.

One can show that Q becomes

$$\underset{\substack{\uparrow \\ \bar\partial}}{Q^{\text{old}}} + d(\text{start}) + d(\text{end})$$

$Q^2 = 0 \Rightarrow d_{n+1} d_n = 0$ for all n.

So $\mathcal{E}^n \xrightarrow{d_n} \mathcal{E}^{n+1} \xrightarrow{d_{n+1}} \mathcal{E}^{n+2} \xrightarrow{d_{n+2}} \mathcal{E}^{n+3} \cdots$ is a complex.

Q is now associated to a triple complex

Let $0 \to \mathcal{F}^n \to \mathcal{F}^{n,0} \to \mathcal{F}^{n,1} \to \mathcal{F}^{n,2} \to$ be an injective resolution of $\mathcal{F}^n$

If $\varphi \in \mathrm{Hom}(\mathcal{E}^m, \mathcal{F}^{n,p})$

Then $Q^{\text{old}}\varphi = c_p \varphi \in \mathrm{Hom}(\mathcal{E}^m, \mathcal{F}^{n,p+1})$

$d(\text{start})\varphi = \varphi d_{m-1} \in \mathrm{Hom}(\mathcal{E}^{m-1}, \mathcal{F}^{n,p})$

$d(\text{end})\varphi = d_n \varphi \in \mathrm{Hom}(\mathcal{E}^m, \mathcal{F}^{n+1,p})$

To find total cohomology of Q apply a spectral sequence twice!

Total cohomology is given by "Hyper Ext" defined as follows:

Take $\mathcal{F}^\bullet = \mathcal{F}^{-1} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots$

Replace each $\mathcal{F}^n$ by an injective resolution and collapse to a single complex again:

$$\mathcal{F}^n_{inj} = \bigoplus_{k+p=n} \mathcal{F}^{k,p}$$

Then $\hom^p(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = \bigoplus_n \mathrm{Hom}(\mathcal{E}^n, \mathcal{F}^{n+p}_{inj})$

forms a complex of linear vector spaces.

$\mathrm{Hom}^p(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is cohomology of this complex

p: ghost number.

open string from $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$