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D-branes on Calabi-Yau varieties, III

- TQFT of D-Branes = Derived Category of Coherent Sheaves
- Back to non-TQFT – Stability
  ↑
  4 key examples

Let $\mathcal{A} = \oplus \mathcal{A}^n$ be a collection of sheaves

Suppose $\mathcal{A}^n = \mathcal{E}^n \oplus \mathcal{F}^n$

$$\to \mathcal{E}^0 \oplus \mathcal{F}^0 \xrightarrow{\begin{pmatrix} d_{\mathcal{E}} & \varphi \\ 0 & d_{\mathcal{F}} \end{pmatrix}} \mathcal{E}^1 \oplus \mathcal{F}^1 \xrightarrow{\begin{pmatrix} d_{\mathcal{E}} & \\ & d_{\mathcal{F}} \end{pmatrix}} \mathcal{E}^2 \oplus \mathcal{F}^2 \to$$

$\mathcal{E}^\bullet, \mathcal{F}^\bullet$

<u>Category of Topological B-Branes</u>

Objects are B-Branes $\mathcal{E}^\bullet$

Morphisms are Open Strings $\mathrm{Hom}^P(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$

- Composition Yoneda pairing

$$\mathrm{Hom}^P(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \otimes \mathrm{Hom}^Q(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \to \mathrm{Hom}^{P+Q}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$$

– operator product for open strings

Let $T(X)$ be the category of all topological B-branes

- any TQFT is a full subcategory of $T(X)$

Clearly there is a functor $F : K^b_{LF}(X) \to T(X)$

Category of bounded complexes of locally-free sheaves

When are two D-branes different?

Suppose $\mathcal{E}_1^\bullet$ and $\mathcal{E}_2^\bullet$ are two complexes such that

$$\mathrm{Hom}^p(\mathcal{E}_1^\bullet, \mathcal{F}^\bullet) = \mathrm{Hom}^p(\mathcal{E}_2^\bullet, \mathcal{F}^\bullet)$$

and $\mathrm{Hom}^p(\mathcal{F}^\bullet, \mathcal{E}_1^\bullet) = \mathrm{Hom}^p(\mathcal{F}^\bullet, \mathcal{E}_2^\bullet)$ for all $\mathcal{F}^\bullet, p$.

Then $\mathcal{E}_1^\bullet$ and $\mathcal{E}_2^\bullet$ are isomorphic in $T(X)$.

Clearly there is a functor $F : K^b_{LF}(X) \to T(X)$

One can then show that F takes "quasi-isomorphisms" to isomorphisms

$$\begin{array}{ccccccccccc}
\mathcal{E}_1^{-1} & \to & \mathcal{E}_1^{0} & \to & \mathcal{E}_1^{1} & \to & \mathcal{E}_1^{2} & \to & \mathcal{E}_1^{3} & \to & \cdots \\
\downarrow \varphi & & \downarrow \varphi & & \downarrow \varphi & & \downarrow \varphi & & \downarrow \varphi & & \\
\mathcal{E}_2^{-1} & \to & \mathcal{E}_2^{0} & \to & \mathcal{E}_2^{1} & \to & \mathcal{E}_2^{2} & \to & \mathcal{E}_2^{3} & \to &
\end{array}$$

φ induces an isomorphism $\varphi^*: H^*(\mathcal{E}_1^\bullet) \to H^*(\mathcal{E}_2^\bullet)$

(If we ignore subtleties of locally free vs. coherent sheaves)
Then there exists a functor

$$G : D(X) \longrightarrow T(X)$$

$\uparrow$ derived category of X

Finally, let $T^0(X)$ be the subcategory of $T(X)$ consisting only of morphisms $\mathrm{Hom}^0(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$. Then it is not hard to show that

$$G^0 : D(X) \longrightarrow T^0(X) \quad \text{is full, faithful + dense}$$

i.e., an equivalence of categories

Restricting $T(X)$ to $T^0(X)$ actually gives no loss of info.
Let $[n]$ be an operator on complexes that shifts n places left.

$$(K[n])^i = K^{n+i}$$

Then $\mathrm{Hom}^p(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = \cancel{\mathrm{Hom}^0(\mathcal{E}^\bullet, \mathcal{F}^\bullet[p])}$
$\mathrm{Hom}^0(\mathcal{E}^\bullet, \mathcal{F}^\bullet[p])$

Actually, complexes of coherent sheaves are the same thing as complexes of locally free sheaves up to quasi-iso.

($\Rightarrow$ D-branes can be coherent sheaves)

Let's define again — ghost #1 ops are $\mathrm{Hom}^1(\mathcal{E}^\cdot, \mathcal{F}^\cdot)$
$= \mathrm{Hom}^0(\mathcal{E}^\cdot, \mathcal{F}^\cdot[1])$
or $\mathrm{Hom}^0(\mathcal{E}^\cdot[-1], \mathcal{F}^\cdot)$

$$
\begin{tikzcd}
{} \arrow[r] \arrow[dr] & \mathcal{E}^0 \arrow[r] \arrow[dr, "\varphi"] & \mathcal{E}^1 \arrow[r] \arrow[dr, "\varphi"] & \mathcal{E}^2 \arrow[r] \arrow[dr, "\varphi"] & \cdots \\
{} \arrow[r] & \mathcal{F}^0 \arrow[r] & \mathcal{F}^1 \arrow[r] & \mathcal{F}^2 \arrow[r] & {}
\end{tikzcd}
$$

Giving

$$\mathcal{E}^0 \oplus \mathcal{F}^0 \xrightarrow{\begin{pmatrix} d_E & 0 \\ \varphi & d_F \end{pmatrix}} \mathcal{E}^1 \oplus \mathcal{F}^1 \xrightarrow{\begin{pmatrix} d_E & 0 \\ \varphi & d_F \end{pmatrix}} \mathcal{E}^2 \oplus \mathcal{F}^2$$

which is another chain complex.

$$\mathrm{Cone}(\varphi : \mathcal{E}^\cdot[-1] \to \mathcal{F}^\cdot) \in D(X)$$

(This accounts for $\mathrm{Ext}^2(\mathcal{E}^n, \mathcal{E}^{n-1})$ etc)

(Diaconescu, Lazaroiu)

Stability of D-Branes back in non-TQFT

A BPS state has a central charge $Z \in \mathbb{C}$ where mass $= |Z|$.

If a BPS object decays into a set of other BPS objects then Z is conserved

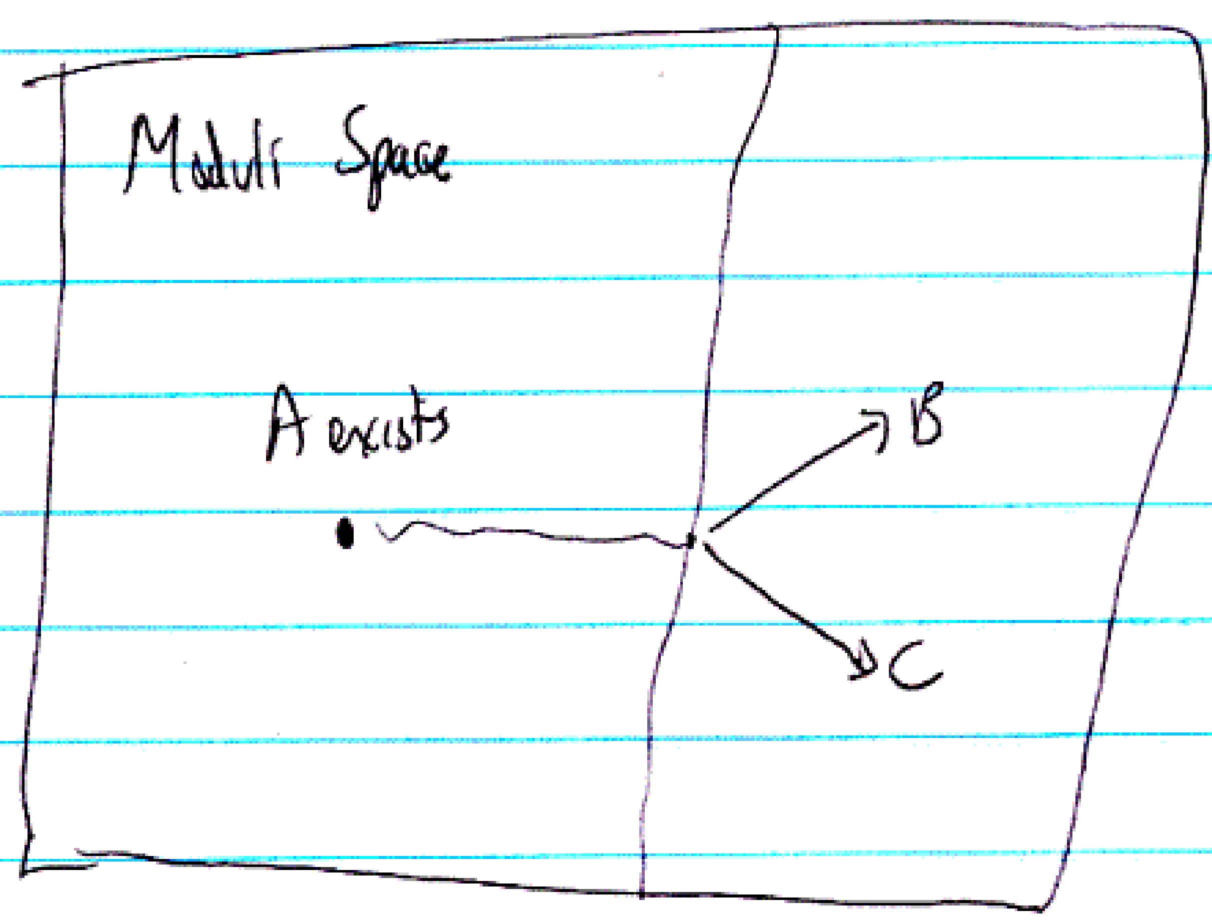
$$Z_{TOT} = \sum Z_i \Rightarrow |Z_{TOT}| \leq \sum_i |Z_i|.$$

So the best we can do is have a "marginal decay"

where $\Rightarrow$ $\arg(z_1) = \arg(z_2) = \arg(z_3) = \cdots$

What typically happens is

marginal stability – real codim 1 subspace

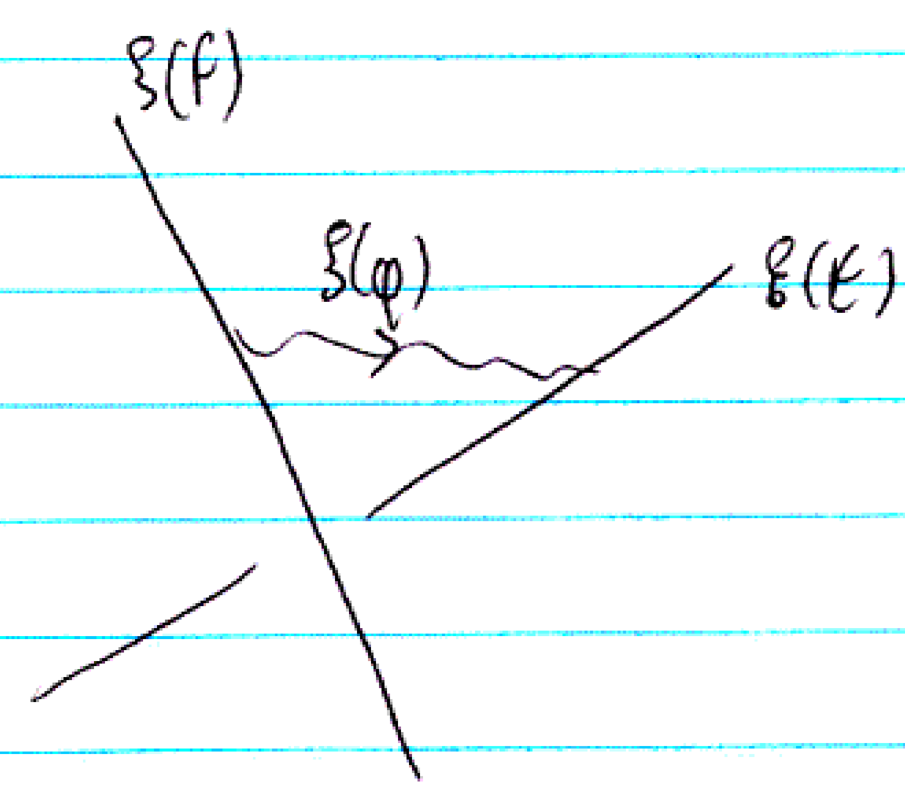


It can be shown that for an A-brane

$$Z = \int_\Gamma \Omega \qquad \Gamma \text{ is SLag}$$

So $\arg(Z)$ is exactly what we used earlier for grading

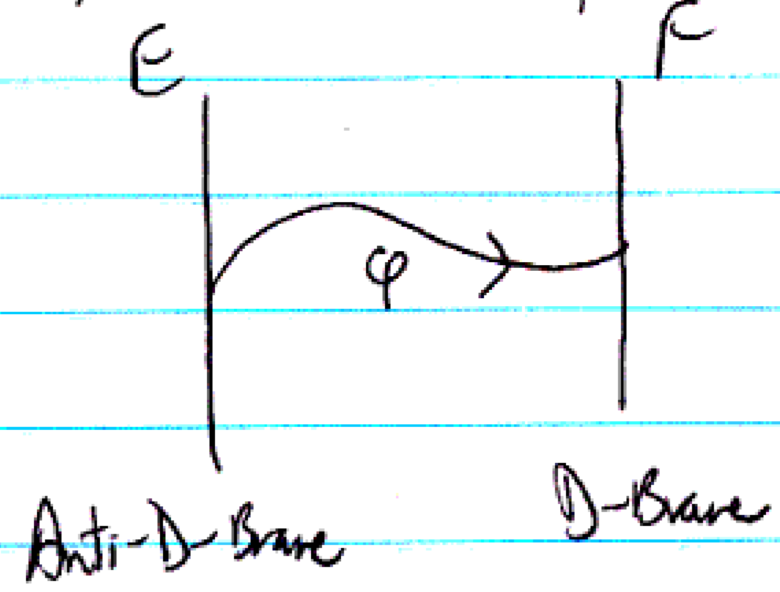
modify to fix sign problem: $\xi = \frac{1}{\pi}\arg(\ )$



$$\xi = \frac{1}{\pi}\arg(\cdots)$$

$$\xi(\varphi) = \xi(E) - \xi(F) + p$$

Sen's tachyon condensate picture



$p=0$, so $\xi(\varphi) = \xi(E) - \xi(F)$

CFT arguments shows that $m^2(\varphi) = (\xi_F - \xi_E) - 1$

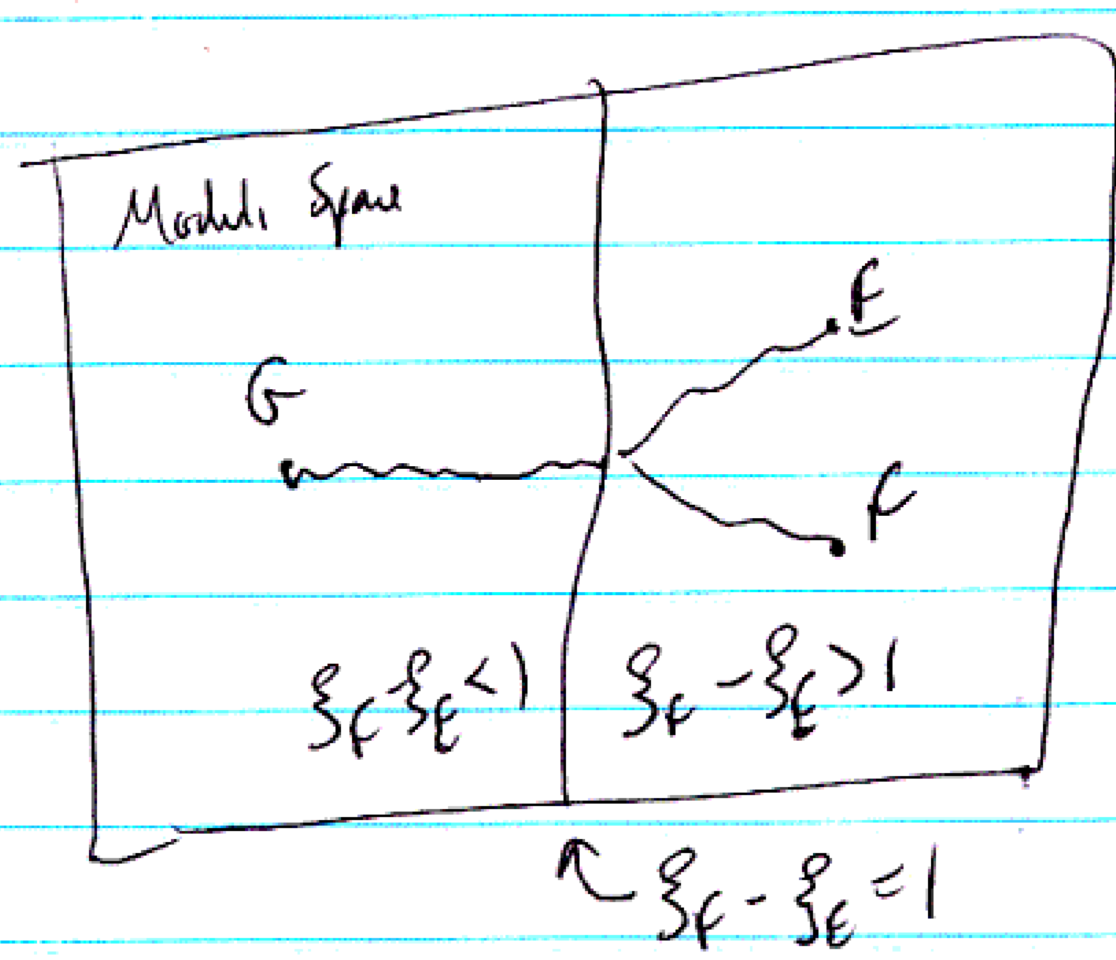
If $m^2 < 0$ then φ is a tachyon which will "stick" $E + F$ together.

$$\text{"}\ 0 \to E \xrightarrow{\varphi} F \to G \to 0\ \text{"}$$

make G — G is "stable"

If $m^2 = 0$ marginal case

$m^2 > 0$ unstable.



Claim: for a B-brane $\mathcal{E}^\bullet \in D(X)$,

$$Z = \int_X e^{B+iJ}\, ch(\mathcal{E}^\bullet)\sqrt{td(X)} \; + \text{quantum corrections}$$

α'

$$ch(\mathcal{E}^\bullet) = \sum_{n=-\infty}^{\infty} (-1)^n\, ch(\mathcal{E}^n)$$

"Anomaly Inflow" Green, Harvey, Moore
Minasian, Moore
Freed, Witten.

Π-stability (Douglas, Fiol, Römelsberger)

We want the derived category version of A + B makes C

We saw that an open string $\varphi : A[-1] \to B$

makes a D-brane $\mathrm{Cone}(\varphi)$.

Cones in the derived category are closely related to "distinguished triangles"

$$\begin{tikzcd} & A^\bullet[-1] \arrow[dr] & \\ \mathrm{Cone}(\varphi)^\bullet \arrow[ur, "{[1]}"] & & B^\bullet \arrow[ll] \end{tikzcd} \quad = \quad \begin{tikzcd} & A^\bullet \arrow[dr, "{[1]}"] & \\ \mathrm{Cone}(\varphi)^\bullet \arrow[ur] & & B^\bullet \arrow[ll] \end{tikzcd}$$

$$\Rightarrow ch(\mathrm{Cone}(\varphi)^\bullet) = ch(A^\bullet) + ch(B^\bullet)$$

$$\boxed{\xi = -\frac{1}{\pi} \arg Z}$$

If $\xi(B)-\xi(A)=1$ marginal

$\xi(B)-\xi(A)<1$ Cone(φ) stable

$\xi(B)-\xi(A)>1$ Cone(φ) is unstable

E.g. Boring old vector bundles at large radius

$$Z = \int e^{B+iJ}\, ch(F) \sqrt{td(X)} \qquad J \gg \text{ generators of } H^2(X,\mathbb{Z}),\quad B=0$$

$$= \int e^{iJ}(r+c_1)\cdot 1 + (\text{lower order in } J)$$

$$= \int_X \left(-\tfrac{1}{6} i r J^3 - \tfrac{1}{2} J^2 c_1\right)$$

$$= -irV - \tfrac{1}{2} d \qquad V = \mathrm{Vol}(X),\quad d = \int J^2 c_1$$

$$\Rightarrow \xi = -\tfrac{1}{\pi} \arg Z$$

$$= -\tfrac{1}{\pi} \tan^{-1}\left(\tfrac{2V}{\mu}\right) \qquad \mu = d/r.$$

Now consider $0 \to A \xrightarrow{\varphi} B \to C \to 0$ vector bundles

$\Rightarrow ch(B) = ch(A) + ch(C)$

$\Rightarrow Z(B) = Z(A) + Z(C) \Rightarrow \xi(B)$ lies between $\xi(A)$ and $\xi(C)$

Now suppose $\mu(A) > \mu(B)$. Then $\xi(A) > \xi(B) \Rightarrow \xi(\varphi) < 0$

In any triangle

$$\begin{tikzcd} & A \arrow[dr, "\varphi"] & \\ C \arrow[ur, "\gamma", "{[+1]}"'] & & B \arrow[ll, "\beta"] \end{tikzcd}$$

Since going around a triangle is shift-left,

$$\xi(\varphi) + \xi(\beta) + \xi(\gamma) = 1.$$

$$\text{But } \xi(\varphi) < 0 \Rightarrow \xi(\beta) < 0 \Rightarrow \xi(\gamma) > 1$$

$$\Rightarrow B \text{ is unstable.}$$