3 July 2001
E. Diaconescu

Categories of D-branes

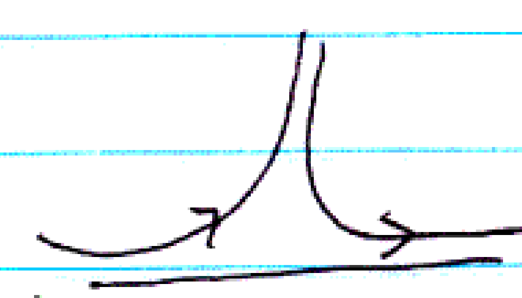
Douglas	A. Sen
Lazaroiu (SFT)	P. Aspinwall
Aspinwall - Lawrence	

String Field Theory

Open string field theory $(\mathcal{H}, *, Q, \int)$

$\mathcal{H} = \bigoplus_n \mathcal{H}_n$ $\mathbb{Z}$-graded string Hilbert space, n = ghost #

$*$ = associative product



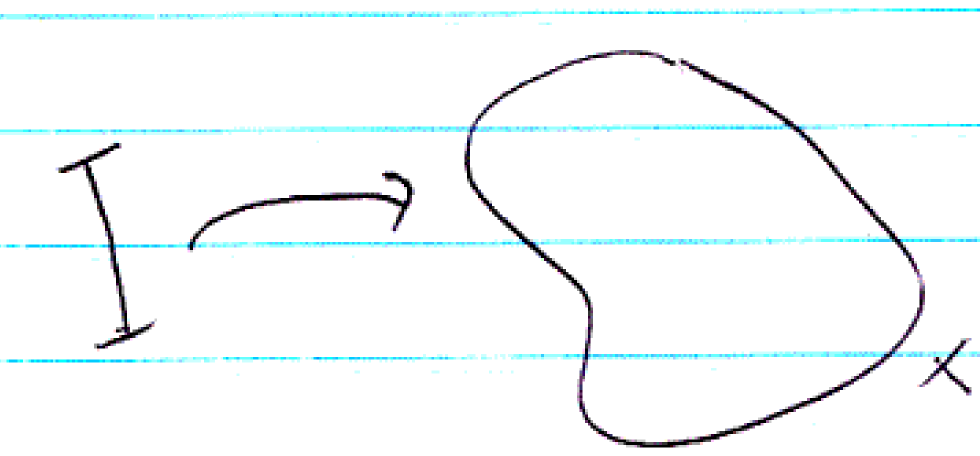
Q = degree 1 $\mathcal{H} \to \mathcal{H}$, $Q^2 = 0$

$$Q(a * b) = Qa * b + (-1)^{|a|} a * Q(b)$$

$\int : \mathcal{H} \to \mathbb{C}$ $\qquad \int a \neq 0 \Rightarrow \deg a = 3.$

SFT action

$$S = \frac{1}{2g^2} \int \Psi * Q\Psi + \frac{2}{3} \Psi * \Psi * \Psi$$

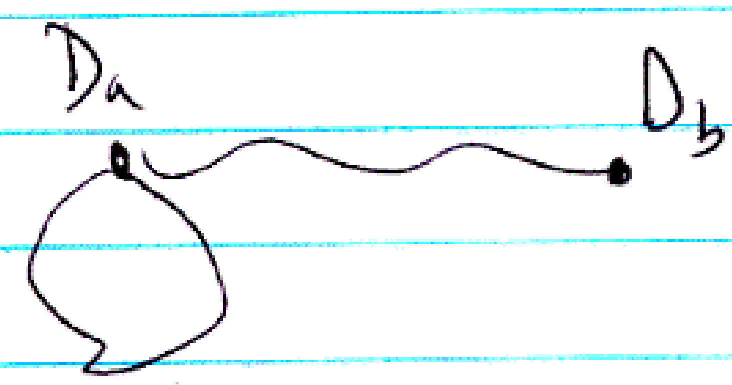


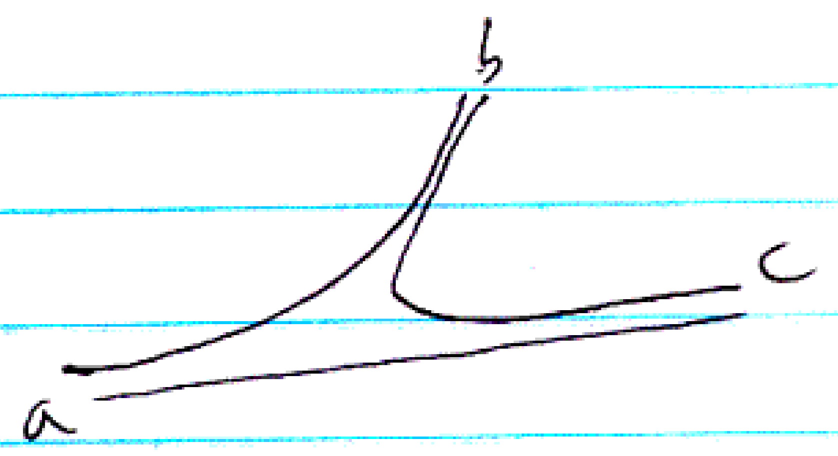
Specify boundary conditions for the endpoints of the string.

D-brane wrapped on X.

What happens if we allow multiple D-branes?

Finite collection of boundary conditions labelled by a





Lazaroiu: D-brane Category

Objects: D_a

$\mathrm{Hom}(D_a, D_b)$ = open string Hilbert space for boundary conditions a, b

$\uparrow$

$\mathbb{Z}$-graded, Q

$\Rightarrow$ The D-brane Category is a DG-category (differential graded category).

(Can get "new D-branes" as solutions to eqs. of motion)

Can construct the space of all string states

$$\mathcal{H} = \bigoplus_{a,b} \mathrm{Hom}(D_a, D_b)$$

Witten's topological B model

$X = CY$ 3-fold

$\Phi : \Sigma \to X$

Fermi fields $\quad \eta^{\bar{i}}, \theta^{\bar{i}} \in \Gamma(\phi^* T^{0,1}_X)$

$\rho^i \in \Gamma(\Omega^1_\Sigma \otimes \phi^* T^{1,0}_X)$.

$$\mathcal{L}_B = t \int d^2z \left(g_{IJ} \partial_z \Phi^I \partial_{\bar{z}} \Phi^J + i \eta^{\bar{i}} (D_z \rho^i_{\bar{z}} + D_{\bar{z}} \rho^i_z) g_{i\bar{i}} \right.$$
$$\left. + i \theta_i (D_{\bar{z}} \rho^i_z + D_z \rho^i_{\bar{z}}) + \text{4-fermi term} \right.$$

BRST Symmetry

$\delta \phi^i = 0$

$\delta \phi^{\bar{i}} = i \alpha \eta^{\bar{i}}$

$\delta \eta^{\bar{i}} = \delta \theta_i = 0$

$\delta \rho^i = -\alpha d\phi^i$

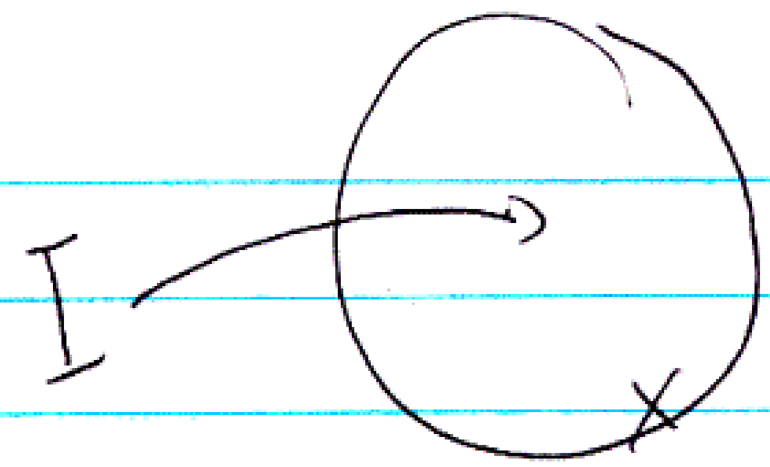
Boundary cond $\quad \partial_n \phi^I |_{\partial \Sigma_i} = 0$

$\theta|_{\partial \Sigma_i} = * \rho|_{\partial \Sigma_i} = 0$

Can couple this model to gauge fields on X

$E \to X$ vector bundle $\quad$ Preserve the fermionic symmetry $\Rightarrow F^{0,2} = 0$, $\bar{\partial}_A$.

A = connection on E



$\Psi(\phi^i(\sigma), \eta(\sigma), -)$

String field theory (Witten) $t \to \infty$

$\Rightarrow$ localization of Ψ on constant maps.

$$\Psi(\phi^I, \eta^{\bar I}) = a^0 + a^1_{\bar i}\eta^{\bar i} + a^2_{\bar i \bar j}\eta^{\bar i}\eta^{\bar j} + a^3_{\bar i\bar j\bar k}\eta^{\bar i}\eta^{\bar j}\eta^{\bar k} + \ldots$$

$$a^q_{\bar i_1 \ldots \bar i_q}\, d\bar z_1^{\bar i_1}\ldots d\bar z_q^{\bar i_q} \Leftrightarrow a^q \in \Omega^{0,q}(\mathrm{End}\, E)$$

$\mathcal{H} = \oplus\, \Omega^{0,q}(\mathrm{End}\, E)$ q = ghost #

$Q = \bar\partial$

$* = \wedge$

$\int = \int_X \Omega \wedge \mathrm{Tr}_E$

SFT action = holomorphic Chern-Simons theory

$$S = \int \Omega \wedge \mathrm{Tr}\left(a^1 \bar\partial a^1 + \frac{2}{3}(a^1)^3\right)$$

$\boxed{\bar\partial a^1 + a^1 \wedge a^1 = 0}$

Gauge symmetry $a^1 \to a^1 + \bar\partial\varepsilon + [a^1, \varepsilon]$

Douglas add a $\mathbb{Z}$-grading of D-branes.

$$\text{D-branes} = (E, n) \quad n \in \mathbb{Z} \text{ grading}$$

$$\Big\updownarrow$$

$$(F, p)$$

Collection of D-branes E_n

$$\begin{tikzcd} E_m \arrow[r, squiggly] & E_n \end{tikzcd}$$

$$\mathcal{H} = \oplus\, \Omega^{0,q}(E_m^\vee \otimes E_n)$$

$$\text{ghost } \# = q + n - m.$$

$\star$: Define a super-vector bundle

$$E_+ = \bigoplus_{n \text{ even}} E_n$$

$$E_- = \bigoplus_{n \text{ odd}} E_n$$

$$\mathcal{E} = (E_+, E_-)$$

$$\mathcal{H} = (\oplus\, \Omega^{0,q})\, \hat{\otimes}\, End(\mathcal{E})$$

$\psi \in \mathcal{H}$, $\deg \psi = 1$

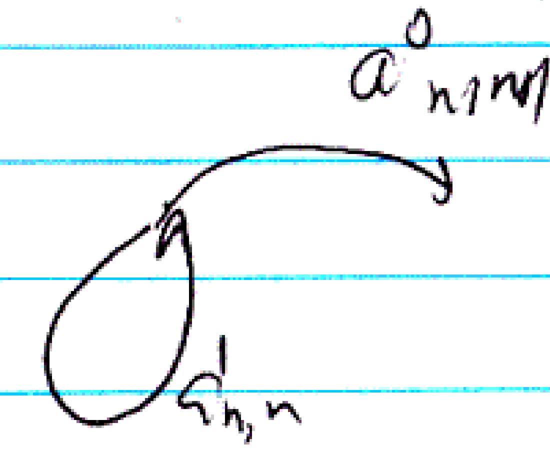
$$\begin{tikzcd} E_{n-1} \arrow[r, "a^0_{n-1,n}"] \arrow[loop below, "a^1_{n,n}"] & E_n \arrow[r, "a^0_{n,n+1}"] \arrow[l, bend right=60, "a^2_{n,n-1}"'] & E_{n+1} \arrow[ll, bend right=70, "a^3_{n+1,n-1}"'] \end{tikzcd}$$

$$S[\psi] = \int \Omega \wedge \mathrm{Tr}_s\left(\psi \bar{\partial} \psi + \psi^3\right)$$

$$\bar\partial a^0_{n,n+1} + a^0_{n,n+1} a^1_{n,n} + a^1_{n+1,n} a^0_{n,n} = 0$$

$$\bar\partial a^1_{n,n} + a^1_{n,n} a^1_{n,n} + a^2_{n+1,n} a^0_{n,n+1} + a^0_{n-1,n} a^2_{n,n-1} = 0$$

$$\bar\partial a^2 = \cdots$$



$$\left.\begin{array}{l} C = (E_n, a^q_{n,n+1-q}) \\ C' = (E'_n, a'^q_{n,n+1-q}) \end{array}\right\} \mathcal{C}$$

$$\mathrm{Hom}^\bullet(C, C') = \bigoplus_{m,n} \bigoplus_q \Omega^{0,q}(E_m^\vee \otimes E'_n)$$

$k = q + n - m.$

$H^0(\mathcal{C})$. Can show: if Q is full subcategory of $\mathcal{C}$ generated by complexes.
Then $H^0(Q) = T^0(X)$

$$\mathcal{A} = \bigoplus_{n,m} \bigoplus_q \Omega^{0,q}(E_m^\vee \otimes E_n)$$