MARK GROSS GEOMETRICAL APPROACHES TO MIRROR SYMMETRY

C-Y (X, ω, Ω) ω Ricci-flat
Ω hol n-form

$$\frac{\omega^n}{n!} = c\,\Omega \wedge \bar{\Omega}$$

Def X CY $M \subseteq X$ is special Lagrangian if $\dim_{\mathbb{R}} M = \frac{1}{2}\dim_{\mathbb{R}} X$ and $\omega|_M = 0$, $\operatorname{Im} \Omega|_M = 0$

Conjecture: (SYZ) Let $X, \check{X}$ be a "mirror pair" of C-Y's. Then $\exists$ $f: X \to B$ $\check{f}: \check{X} \to B$ fibrations whose fibers are special Lagrangian, and generically fibers are n-tori and dual to one another, in the following sense:

$$\begin{array}{ccc} X_0 & \subseteq & X \\ \downarrow & & \downarrow \\ B_0 & \subseteq & B \end{array} \qquad \qquad \begin{array}{ccc} \check{X}_0 & \subseteq & \check{X} \\ \downarrow & & \downarrow \\ B_0 & \subseteq & B \end{array}$$

T^n bundle

$X_0 \to B_0$ & $\check{X}_0 \to B_0$ are dual torus bundles, i.e.
$\check{f}^{-1}(b)$ identified w/ $H^1(f^{-1}(b), \mathbb{R}/\mathbb{Z})$

This strong form of the conjecture should be revised, most likely.

Discriminant: $\Delta = B \setminus B_0$ should be codimension two in order to hope to compactify $\check{X}_0 \to B_0$ to $\check{X} \to \check{B}$.

Fact: If $f : X \to B$ is a C^∞ slag fibration, then Δ is codim 2.

Ex. (Joyce) $f : \mathbb{C}^3 \to \mathbb{R}^3$

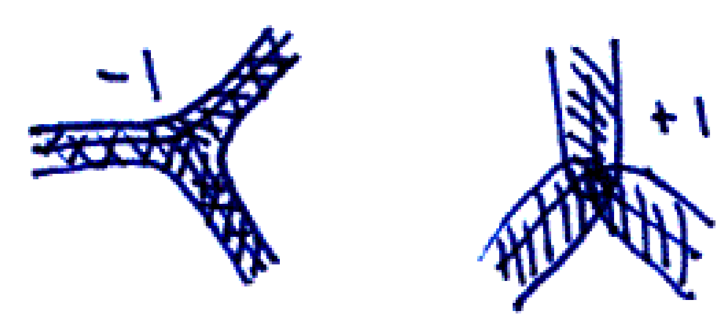
$$f(z_1, z_2, z_3) = (a, \operatorname{Re} c, \operatorname{Im} c), \quad a = |z_1|^2 - |z_2|^2$$

$$c = \begin{cases} z_3 - \bar{z}_1\bar{z}_2/|z_1|, & a<0 \text{ or } a=0,\ z_1,z_2\neq 0 \\ z_3, & a = z_1 = z_2 = 0 \\ z_3 - \dfrac{\bar{z}_1\bar{z}_2}{|z_2|}, & a>0 \text{ or } a=0,\ z_1,z_2\neq 0 \end{cases}$$

$f^{-1}(0, c_1, c_2)$ is singular $\forall\, c_1, c_2$

$\Delta \cong \mathbb{R}^2$ $(a=0)$ is codim 1

Joyce suggests Δ should be an amoeba and concludes that Δ must change under dualizing.



As a consequence, some smooth fibers have singular duals.

Moduli Spaces of Slags

Let $M \subseteq X$ be a slag.
Let B be the moduli space of deformations of $M \subseteq X$ as a slag.

McLean: B is a smooth manifold with a tangent space to $[M] \in B$ canonically identified with $\mathcal{H}^1(M;\mathbb{R})$ harmonic 1-forms $(\dim = b_1(M))$.

Look inside

$$\begin{array}{ccc} \mathcal{U} \subseteq & B \times X & \xrightarrow{p} X \\ \searrow & \downarrow f & \\ & B & \end{array}$$

ω, Ω forms on X.

$\frac{\partial}{\partial y} \in T_{B,b}$. Then the corresponding elt of $\mathcal{H}^1(f^{-1}(b), \mathbb{R})$ is $\iota\left(\frac{\partial}{\partial y}\right) p^* \omega$, well defined (indep of lift of $\frac{\partial}{\partial y}$ to $\mathcal{U}$)

contraction

$$* \iota(\partial/\partial y)\omega = -\iota(\partial/\partial y) \operatorname{Im} \Omega$$

Let $\gamma_1, \dots \gamma_s$ be integ. basis for $H_1(f^{-1}(b), \mathbb{Z})$
Define 1-forms α_i on B by

$$\alpha_i(\partial/\partial y) = \int_{\gamma_i} \iota(\partial/\partial y) p^* \omega$$

Letting γ_i's vary continuously on fibres, get locally 1-forms $\alpha_1, \dots \alpha_s$ on B.
(up to monodromy)

Similarly, let $\Gamma_1 \dots \Gamma_s$ basis for $H_{n-1}(f^{-1}(b), \mathbb{Z})$

Define β_i (1-form) by

$$\beta_i(\partial/\partial y) = \int_{\Gamma_i} \iota(\partial/\partial y)\, p^* \operatorname{Im} \Omega$$

Prop (Hitchin)

Basic observation: α_i's, β_i's <u>closed</u> 1-forms,.

and $\alpha_1, \dots \alpha_s$ <u>or</u> $\beta_1, \dots \beta_s$ define a basis for T_B^*

Locally $\exists$ fn's $y_1, \dots y_n$, $\check{y}_1, \dots \check{y}_n$ on B s.t. $\alpha_i = dy_i$, $\beta_i = d\check{y}_i$

giving two different locally coord charts.

Choices we make: Basis of homology, constants for fn's
A diff't choice of $\gamma_1, \dots \gamma_s$ and a choice of consts of integration,
yield new coordinates

$$y_i' = \sum a_{ij} y_j + b_i \quad , \text{ with}$$

$$a_{ij} \in GL_n(\mathbb{Z})$$

(affine transformation)

This gives an <u>affine structure</u> on B, i.e. an atlas:
$\{\psi_i : U_i \to \mathbb{R}^s\}$ with

$$\psi_i \circ \psi_j^{-1} \in GL_n(\mathbb{Z}) \ltimes \mathbb{R}^n \subseteq \operatorname{Aff}(\mathbb{R}^n)$$

Nbhd of pt in base canonically identified with nbhd of H^1

Similarly, $\check{y}_1 \ldots \check{y}_n$ gives a different affine structure.

McLean Metric

$$g(v,w)|_b = -\int_{f^{-1}(b)} (\iota(v)\, p^*\omega) \wedge \underbrace{(\iota(w)\, p^*\,\mathrm{Im}\,\Omega)}_{= -*\,\iota(w)\, p^*\omega}$$

Hodge metric

This metric has a potential, i.e. locally $\exists$ fn K on B s.t. in coords $y_1, \ldots, y_n$

$$g_{ij} = \frac{\partial^2 K}{\partial y_i \partial y_j}$$

Also $\check{y}_i = \dfrac{\partial K}{\partial y_i}$ and in coords $\check{y}_1 \ldots \check{y}_n$

$$g_{\check{i}\check{j}} = \frac{\partial^2 \check{K}}{\partial \check{y}_i \partial \check{y}_j}$$ where $\check{K}$ is Legendre transform

$$\check{K} = \sum_{i=1}^{s} \check{y}_i y_i - K.$$

(If you know $\frac{2}{3}$ of $(K, \check{K}, g)$ you can get last $\frac{1}{3}$).

Let B be a manifold with affine structure, with pot'l K
$(GL_n(\mathbb{Z}) \ltimes \mathbb{R})$

2 diff torus bundles

① $$X = T_B^*/\langle dy_1, \ldots dy_n\rangle \quad \downarrow \quad B$$

$y_1, \ldots y_n$ local affine coords, so this makes sense

with canonical symp str.

② $$\check{X} = T_B/\langle \tfrac{\partial}{\partial y_1}, \ldots \tfrac{\partial}{\partial y_n}\rangle \quad \downarrow \quad B$$

dual torus bundle.

which has an (un.) canonical cpx structure.

If $x_1, \ldots x_n$ are fiber coords on T_B $\left(\sum x_i \frac{\partial}{\partial y_i}\right)$

set $z_j = e^{2\pi i(x_j + i y_j)}$ gives hol. coords.

In general, we apply these structures on a noncompact $B_0 \subseteq B$ and then want to compactify (X_0, ω) to (X, ω) ← can be done

or $(\check{X}_0, J)$ to $(\check{X}, J)$ ← can't expect to be able to do this

In addition, given K, $\check{f}: \check{X} \to B$, then $K \circ \check{f}$ is the Kahler potential of a "semi-flat" metric on $\check{X}$, i.e. one flat along fibers.

This metric is Ricci flat $\Leftrightarrow$ $\det\left(\frac{\partial^2 K}{\partial y_i \partial y_j}\right) = \text{const.}$ (*).

<u>Note:</u> $\exists$ examples of slag fibrations where K does <u>not</u> satisfy (*).