Dr. Mark Gross, University Warwick (ITP 7/09/01) Geometrical Approaches to Mirror Symmetry

\[ \mathcal{CY} (X, \omega, \Omega) \text{ is a Calabi-Yau}\]
\[ \Omega \text{ has a volume form}\]
\[ \frac{\omega^n}{n!} = c \Omega \wedge \Omega \]

\[ \text{Definition: } X \in \mathcal{CY} \text{ is special Lagrangian if } \dim M = \frac{1}{2} \dim X \]
\[ \text{and } \omega|_M = 0, \quad \text{Im} \Omega|_M = 0\]

Conjecture: (Sie) Let \( X, \hat{X} \) be a "mirror pair" of \( \mathcal{CY} \)'s.

Then there exists a fibration whose fibers are special Lagrangian, and generically fibers are \( n \)-tori and dual to one another, in the following sense:

\[ X_0 \subseteq X \]
\[ B_0 \subseteq B \]

The bundle \( \hat{X} \to \hat{B} \) is dually torus bundles, i.e.

\[ \hat{B}_0 \subseteq \hat{B} \]

This strong form of the conjecture should be revised, most likely.

Discriminant: \( \Delta = B \setminus \hat{B}_0 \) should be codimension two in order to hope to compactify \( X_0 \to \hat{B}_0 \) to \( X \to \hat{B} \).
Fact: If $f : X \to \mathbb{B}$ is a $C^\infty$ algebra fibration, then $\Delta$ is codim 2.

**Example (Gongyo)** $f : \mathbb{C}^2 \to \mathbb{R}^2$

$$f(z_1, z_2, z_3) = (a_1, \text{Re} c, \text{Im} c), a = 1z_1^2 - 1z_2z_3$$

$$c = \begin{cases} 
\frac{z_3 - z_2 \bar{z}_3}{1z_2}, & a < 0 \text{ or } a = 0, z_2 \neq 0 \\
\frac{z_3 - z_1 \bar{z}_3}{1z_1}, & a > 0 \text{ or } a = 0, z_1 \neq 0
\end{cases}$$

$f^{-1}(0, c_1, c_2)$ is singular at $c_1, c_2$

$\Delta \cong \mathbb{R}^2$ ($a = 0$) is codim 1

Gongyo suggests $\Delta$ should be an amoeba and concludes that $\Delta$ must change under dualizing.

As a consequence, some smooth fibers have singular duals.
Moduli Space of Degenerates

Let \( M \times X \) be a degenerate.
Let \( B \) be the moduli space of deformations of \( M \times X \) as a degenerate.

\[ \text{Remark: } B \text{ is a smooth manifold with a tangent space } \mathcal{M} \in B \text{ canonically identified with } H^1(M; \mathbb{R}) \]
harmonic 1-forms \( \dim \mathcal{M} = b_1(M) \).

Look inside:
\[ U \rightarrow B \times X \xrightarrow{f} X \]
\( \omega, \Omega \) forms on \( X \).

\[ \frac{\partial}{\partial y} \in T_{B \times x} \]
Then the corresponding elt of \( H'((\cdot, x), \mathbb{R}) \)
is \( -i(\frac{\partial}{\partial y}) p^*\omega \), well defined (index of lift of \( \frac{\partial}{\partial y} \) to \( \mathbb{R} \))

\[ * \left( (\frac{\partial}{\partial y}) \omega \right) = -i(\frac{\partial}{\partial y}) \text{ Im } \omega \]

Let \( \{e_i\} \) be an integer basis for \( H((\cdot, x), \mathbb{R}) \).
Define 1-forms \( \alpha_i \) on \( B \) by
\[ \alpha_i \left( \frac{\partial}{\partial y} \right) = \int_{y_i} (\frac{\partial}{\partial y}) p^*\omega \]

Letting \( \alpha_i \)'s vary continuously on fibres, get locally 1-forms \( \alpha_1, \ldots, \alpha_s \) on \( B \).
(up to monodromy)
Similarly, let $\Gamma_1, \ldots, \Gamma_s$ basis for $H_{n-1}(\mathcal{X}(b), \mathbb{Z})$.

Define $\beta_i$ ($1$-form) by

$$\beta_i : (\frac{\partial}{\partial y_j}) \rightarrow \int_{\Gamma_i} f \cdot \text{Im} \alpha$$

Keep (stated)

Basic observation: $\alpha_i$'s $\beta_i$'s closed $1$-forms,

and $\alpha_1, \ldots, \alpha_s$ define a basis for $T_{\mathcal{X}}^\vee$

or $\beta_1, \ldots, \beta_s$

Locally, fix $y_1, \ldots, y_n$ as $B$ st.

$$\alpha_i = dy_i, \quad \beta_i = d\alpha_i,$$

giving two different local coordinate charts.

Choices we make: Basis of homology, contents for $\alpha_i$'s

A shift choice of $\gamma_1, \ldots, \gamma_s$ and a choice of contours of integration,

yield new coordinates

$$y'_i = \sum a_{ij} y_j + b_i, \quad \text{with}$$

$$a_{ij} \in \text{GL}_n(\mathbb{Z})$$

(affine transformation)

This gives an affine structure on $B$, i.e. an atlas.

$$\{ \psi_i : U_i \rightarrow \mathbb{R}^n \} \quad \text{with}$$

$$\psi_i \circ \psi_j^{-1} \in \text{GL}_n(\mathbb{Z}) \times \mathbb{R}^n \leq \text{Aff}(\mathbb{R}^n)$$

Note $\psi_i$ in basis canonically identified with $\alpha_i$ of $\mathcal{X}'$.
Similarly, $\tilde{y}_1, \ldots, \tilde{y}_n$ give a different affine structures.

Hodge metric

$$g(v, w) = \int_{f^{-1}(b)} (f^*(\omega))^m (f^*(\omega)) \wedge (f^*(\omega))$$

Hodge metric

This metric has a potential, i.e. locally $\exists f^* K$ on $B$

s.t. in coords $y_1, \ldots, y_n$

$$g_{ij} = \frac{\delta_{ij}}{\partial y_i \partial y_j}$$

Also

$$\tilde{y}_i = \frac{dK}{\partial y_i} \quad \text{and in coords } \tilde{y}_1, \ldots, \tilde{y}_n$$

$$g_{\tilde{y}_i \tilde{y}_j} = \frac{d\tilde{y}_i}{\partial y_j} \wedge (\partial y_j)$$ where $\tilde{K}$ is Legendre transform

$$\tilde{K} = \sum_{i=1}^n \tilde{y}_i \tilde{y}_i - K$$
(If you know \( \frac{2}{3} \) of \( (k, \bar{k}, g) \) you can get last \( \frac{1}{3} \)).

Let \( B \) be a manifold with affine structure with \( \text{pol}(L) K \) \( (GL_n(Z) \cdot K \cdot R) \)

2 diff toy bundles

1. \( X = T_B / \langle \text{deg}, \ldots, \text{deg} \rangle \)

   \[ \downarrow \]

   \[ B \]

   with canonical symp str.

2. \( \bar{X} = T_{\bar{B}} / \langle \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n} \rangle \)

   \[ \downarrow \]

   \[ \bar{B} \]

   which has an (un-)canonical cpx structure.

If \( x_1, \ldots, x_n \) are fiber coords on \( T_B \) \( (\Sigma z_i \bar{z}_j) \)

set \( \bar{z}_j = e^{2i\pi (x_j + iy_j)} \), give hol. coords.
In general, we apply these structures to a
noncompact \( B_0 \in \mathcal{B} \) and then want to
compactlyfy \( (x_0, w) \) to \( (x, w) \) \( \Leftarrow \) can be done
or \( (\tilde{x}_0, \tilde{J}) \) to \( (\tilde{x}, \tilde{J}) \) \( \Leftarrow \) can't expect
to be able to
do this

In addition, given \( K, \tilde{\epsilon}: \tilde{x} \rightarrow B \), the

\( K_{\tilde{x}} \) is the Kähler potential

of a "semi-flat" metric on \( \tilde{x} \), i.e., one

that are along fibers.

This metric is Ricci flat \( \Rightarrow \) det \( \left( \frac{\delta K}{\delta y^i \delta y^j} \right) = \text{const.} \). \( \Leftarrow \)

Note: 3 examples of slay fibrations where

\( K \) does not satisfy \( \Leftarrow \).