GROMOV - HAUSDORFF CONVERGENCE

Metric on set of metric spaces

Def (X, d_X) (Y, d_Y) two cpt metric spaces

Then $d_{G-H}(X,Y) \leq \epsilon$ if $\exists$ maps $f : X \to Y$ $g : Y \to X$ (not nec. continuous) s.t.

$$|d_X(x,y) - d_Y(f(x), f(y))| \leq \epsilon \quad \forall\, x,y \in X$$

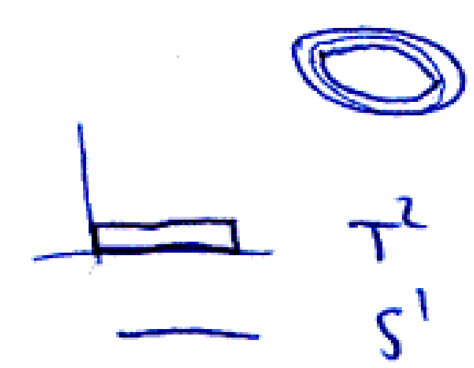
and $$d_X(x, g(f(x))) \leq \epsilon \quad \forall\, x \in X$$

and $$|d_Y(a,b) - d_X(g(x), g(y))| \leq \epsilon$$
$$|d_Y(a, f(g(a)))| \leq \epsilon$$

(f, g approx isometric inverses)

Ex: $T^2 = \mathbb{R}^2 / \langle (1,0), (0,\epsilon) \rangle$ with Euclidean metric

S^1



$f : S^1 \to T^2$ inclusion

$g : T^2 \to S^1$ projection

Then $d_{G-H}(T^2, S^1) \leq \epsilon$

So can have diff't dimensions. Also could have points

As $\epsilon \to 0$ $T^2 \to S^1$ in G-H sense.

Thm The space of cpt Riemannian mflds of dim n with $Ric \geq -k$ and $Diam \leq c$ is pre-compact in G-H topology. (In the space of cpt metric spaces.) (Every seq has a convergent subseq. — lim might not be in the set)

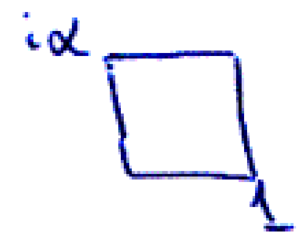
(OVER)

Let $\mathcal{X} \to \Delta$ (disk) be a degenerating family of CY n-folds, with $\mathcal{X}_0$ singular, w/ max. unipotent monodromy

$$(T - I)^n \neq 0, \quad (T-I)^{n+1} = 0$$

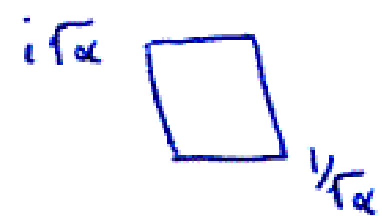
Fix a (monodromy inv.) Kähler class ω; let $t_i \in \Delta$, $t_i \to 0$ and consider seq. $(\mathcal{X}_{t_i}, \epsilon_i g_i)$, where g_i is Ricci-flat w/ Kähler class ω, and $\epsilon_i > 0$, chosen s.t. $\operatorname{diam} \mathcal{X}_{t_i} = C$.

Ex: Elliptic curve w/ periods $1, \frac{1}{2\pi i} \ln z$, z coord on Δ.



$\alpha \to \infty \Rightarrow$ Euclidean metric $\to$ noncpt inf. cylinder $\mathbb{R} \times S^1$

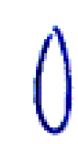
Rescale s.t. vol is const.



$\to \mathbb{R}$ as $\alpha \to \infty$

Rescale s.t. diameter is "const."

$\to$



S^1

Conjecture: (Gross-Wilson, Kontsevich-Soibelman, Todorov)

$(\mathcal{X}_{t_i}, \epsilon_i g_i)$ converges to an n-dim'l mfld B and $\exists$ dense open subset $B_0 \subseteq B$ with $B \setminus B_0$ Hausdorff codim 2, s.t. the metric on B_0 is induced by a Riemannian metric on B_0.

(volume of fibers goes to zero)
must periods

Remark/Conjecture: Since $\mathcal{X} \to \Delta$ is maximally unipotent $\exists$ a point x_0 in $\mathcal{X}_0$ where $\mathcal{X}_0$ locally looks like $x_0 \cdots x_n = 0$.

Near this point in $\mathcal{X}$, $\mathcal{X} \to \Delta$ is given by

Indep of which such point in $\mathcal{X}$ you pick.

$$z = x_0 \cdots x_n$$

This has a T^n action

$$(x_0, \dots x_n) \mapsto (e^{i\theta_0} x_0, \dots, e^{i\theta_n} x_n), \quad \sum \theta_i = 0$$

This gives a T^n inside $\mathcal{X}_z$ for $z \neq 0$.

<u>Conj.</u> · This T^n can be deformed to a sLag T^n on $\mathcal{X}_{t_i}$, t_i close to 0.

· Let $B_{0,i}$ be the moduli space of deformations of this sLag T^n

· Compactify $B_{0,i}$ to a metric space B_i (By taking closure in space of rectifiable currents)

Then if metric on B_i scaled to keep $\mathrm{Diam}\, B_i$ const, then B_i converges to B (same B as before).

Q: Unique, indep of sequence?
A: Hopefully

In partic., $B_0 \subseteq B$ will have two affine structures as well as a metric.

<u>Advantages</u> (to this viewpoint)

1) In the limit, we expect $\Delta_i = B_i \setminus B_{0,i}$ to have area $\to 0$ so should converge to something of codim 2

2) Since only talking about moduli spaces, not fibrations, don't need to worry about singular fibers or deformations intersecting each other, or T^n's filling out a dense subset.

3) These conjectures may be accessible.

(Could be T^n's don't fill out, but more and more so as $t_i \to 0$)

Problem: How do we reconstruct a CY from B.

Thm (Gross, Wilson)

These conjectures hold for K3, generically.

Idea of proof:

$X = (X, \omega, \Omega)$

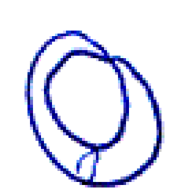
Use hyperkähler rotation. A K3 surface X with a sLag T^2 fibration $f: X \to S^2$ is elliptic in a diff't cpx structure: $(X_K,\ \Omega_K = \operatorname{Im}\Omega + i\omega,\ \omega_K = \operatorname{Re}\Omega)$

$f: X_K \to \mathbb{P}^1$ elliptic.

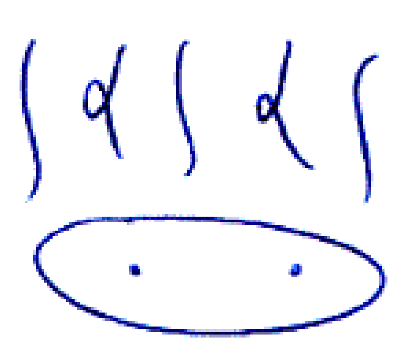
Approaching a l.c.s.l essentially means keeping X_K fixed and letting $\int_{f^{-1}(b)} \omega_K \to 0$, with $\operatorname{Vol}(X_K)$ fixed.

Problem: $f: X \to \mathbb{P}^1$ a fixed elliptic fibration

Describe behavior of a seq. of Ricci-flat metrics ω_ϵ with $\int_{X_b} \omega_\epsilon = \epsilon \to 0$ and $\int_X \omega_\epsilon^2 = C$.

Assume f has 24 I_1 fibers $\propto$  (genericity)

Write down an approximately Ricci-flat metric.



$f^{-1}(B_0)$

$$\begin{array}{c} X \\ \downarrow \\ B_0 \subseteq B \end{array}$$

$B \setminus B_0 = \{p_1, \dots, p_{24}\}$

Can write down an explicit Ricci-flat semi-flat metric on $f^{-1}(B_0)$.

Locally, if $U \subseteq B_0$, y hol coord on U

$$f^{-1}(U) = \underset{y}{\underset{\psi}{U}} \times \underset{x}{\underset{\psi}{\mathbb{C}}} \Big/ \langle \tau_1(y), \tau_2(y) \rangle$$

Then
$$\omega_{SF,\epsilon} = \frac{i}{2}\Big(W(dx + b\,dy) \wedge (d\bar{x} + \bar{b}\,d\bar{y}) + W^{-1}\, dy \wedge d\bar{y} \Big)$$

$$W = \frac{\epsilon}{\mathrm{Im}(\bar{\tau}_1 \tau_2)}$$

$$b = -\frac{W}{\epsilon}\Big[\mathrm{Im}(\tau_2 \bar{x})\, \partial_y \tau_1 + \mathrm{Im}(\bar{\tau}_1 x)\, \partial_y \tau_2 \Big]$$

(comes from affine structures yesterday, after writing down Ω explicitly).

Ooguri-Vafa metric near singular fibers

Ricci-flat metric near I_1 fiber.

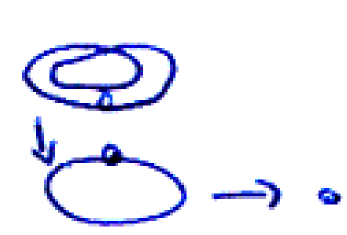
Let Δ be the unit disk with coord $y = y_1 + i y_2$.

Let $\bar{Y}$ be the space $\underset{y_1 y_2}{\Delta} \times \underset{\omega}{\mathbb{R}} / \epsilon\mathbb{Z}$, $Y = (\Delta \times \mathbb{R}/\epsilon\mathbb{Z}) - \{0\} \times (\mathbb{Z}/\epsilon\mathbb{Z})$

Let $X \to Y$ be an S^1 bundle w/ chern class $1 \in H^2(Y, \mathbb{Z}) \cong \mathbb{Z}$

$$\begin{array}{ccccc} X & \hookrightarrow & \bar{X} = X \cup \{pt\} & & \\ \downarrow & & \downarrow & \searrow & \\ Y & \hookrightarrow & \bar{Y} & \longrightarrow & \Delta \end{array}$$

$\bar{X}$ = mfld



} fibers over origin

Let θ be a connection 1-form on S^1-bundle $X \to Y$ with curvature form $\alpha = d\theta$. Suppose V is harmonic fn on Y s.t.

$$* dV = \frac{\alpha}{2\pi i}$$

Then if we set

$$\mathrm{Re}\,\Omega = dy_1 \wedge \theta/2\pi i + V\, dy_2 \wedge du$$

$$\mathrm{Im}\,\Omega = dy_2 \wedge \theta/2\pi i + V\, du \wedge dy_1$$

$$\omega = du \wedge \theta/2\pi i + V\, dy_1 \wedge dy_2$$

These define a hyperkahler structure on X. (not complete)

(Gibbons-Hawking ansatz)
for any S^1-inv't hyperkahler
4-mfld

Let

$$V = \frac{1}{4\pi} \sum_{n=-\infty}^{\infty} \left(\frac{1}{\sqrt{(u+n\epsilon)^2 + y_1^2 + y_2^2}} - a_{|n|} \right)$$

with

$$a_{|n|} = \begin{cases} 1/n\epsilon & n \neq 0 \\ 2(-\gamma + \ln 2\epsilon)/\epsilon & n = 0 \end{cases}$$

This gives a metric on X which extends to $\bar{X}$.

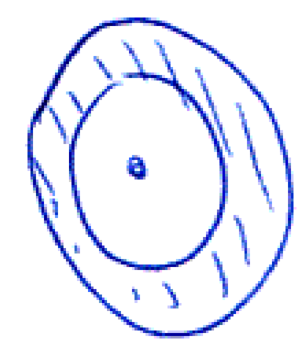
Fourier expansion

$$V = -\frac{1}{4\pi\epsilon} \ln|y|^2 + \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{1}{2\pi\epsilon} e^{2\pi i m u/\epsilon} K_0(2\pi|my|/\epsilon)$$

↑ Bessel function $O(e^{-y})$

Though $O(e^{-c/\epsilon})$ contribution is large for

Glue in to semiflat metric



To within $O(e^{-c/\epsilon})$ of actual
Ricci-flat metric.

With precise description of metric, can
prove conjectures.