## Topological version of SYZ

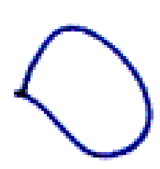
Let's study continuous maps $f: X^6 \to B^3$, proper connected fibers, general fiber a torus. (Too general - want to mimic slag fibs in top. cat.)

Say $f$ is well-behaved if it satisfies

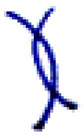
1) $\Delta \subseteq B$ (discriminant locus) is a graph
2) $\exists$ a set $\mathrm{Crit}(f) \subseteq X$ s.t. $f(\mathrm{Crit}(f)) = \Delta$ s.t. $X_b \setminus \mathrm{Crit}(f)$ is dense in $X_b$
3) Locally on $B$ (i.e. over open $U \subseteq B$) $\exists$ rk 3 v.b. $\mathcal{F}$ and a (degenerating) family of lattices $\Lambda \subseteq \mathcal{F}$ s.t. $f^{-1}(U) \setminus \mathrm{Crit}(f) \to U$ coincides with $\mathcal{F}/\Lambda \to U$

Ex

$X_b \setminus \mathrm{Crit}(f) = \mathbb{R} \times S^1$ lattice $\to$ rk 1



$X_b \setminus \mathrm{Crit}(f) = \mathbb{R}^2$ lattice $\to$ rk 0

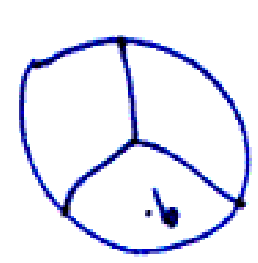
Nothing of the form  $\square$

4) Some other boring conditions

Ex: E.g. lattice $1, \frac{1}{2\pi i} \ln z$ on $\mathbb{C}$ degenerates to rk 1

## Local Monodromy Group

$b_0 \in \Delta$ $\quad U$ small nbhd



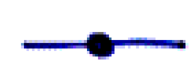
Pick: $b \in U \setminus \Delta$

Get rep $\rho: \pi_1(U \setminus \Delta, b) \to \mathrm{Aut}(H_1(X_b, \mathbb{Z}))$

Def We say $X_{b_0}$ semi-stable if $\mathrm{Im}\,\rho$ is a unipotent subgrp

$$\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

Classification of Monodromy Groups (semistable) (in well-behaved case):

1)  $\begin{pmatrix} 1 & 1 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$ $I_1 \times S^1$

2) 

$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

"Negative Vertex"

3) 

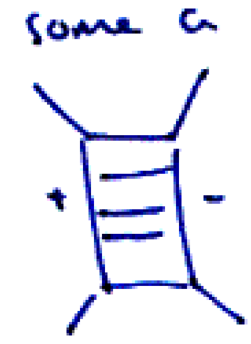
$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$, $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}^{-1}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1}$

Dual "Positive Vertex"

4) 

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & -1 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & -a-1 \\ 0 & 1 & a-1 \\ 0 & 0 & 1 \end{pmatrix}$

$a \in \mathbb{Z}$

For some $a$ can deform to  



Examples:

1) $X^4 \to D^2$ $T^2$ fibration w/ $I_1$ fiber

$X^4 \times (0,1) \times S^1 \to D \times (0,1)$

bad curve of $I_1 \times S^1$

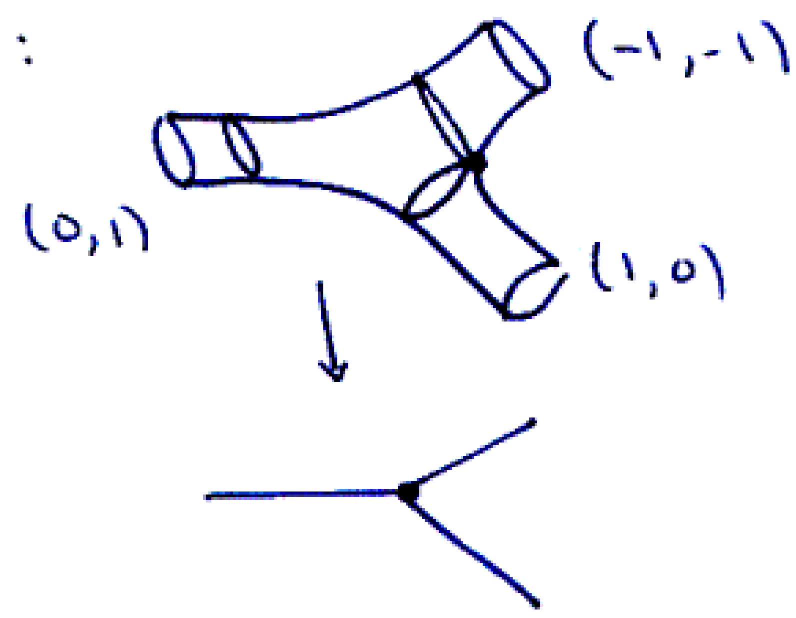
2) Construction: Let $B^3$ be 3-ball and consider $U$ 



Let $\bar{Y} = B \times T^2$

Let $S \subset B \times T^2$ be pair of pants  mapping to $\Delta$

S:




Let $Y = \bar{Y} \setminus S$

$H^2(Y;\mathbb{Z}) = H^2(T^2;\mathbb{Z}) \oplus \mathbb{Z}$

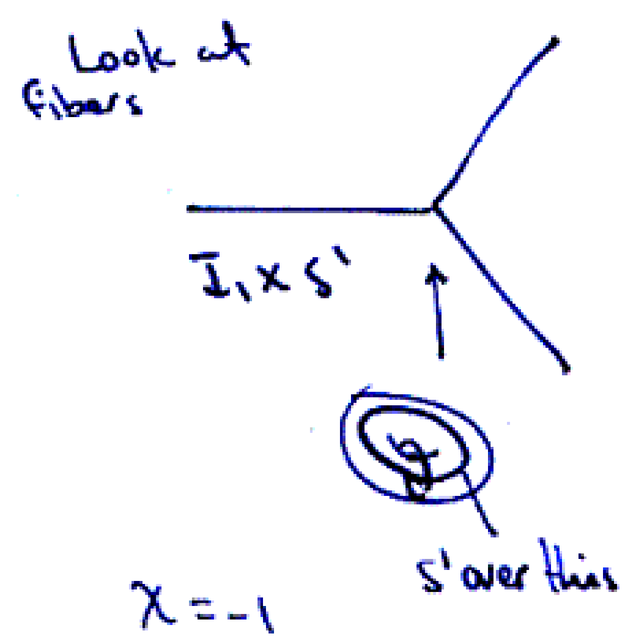
Take $S^1$ bundle $X \to Y$ w/ $c_1 = (0,1)$

$$\begin{array}{ccccc} & & X & \longrightarrow & Y \\ & & \uparrow & & \downarrow \\ X \amalg S & = & \bar{X} & \longrightarrow & \bar{Y} \\ & & & \searrow & \downarrow \\ & & & & B \end{array}$$

This is the fibration

S maps to S

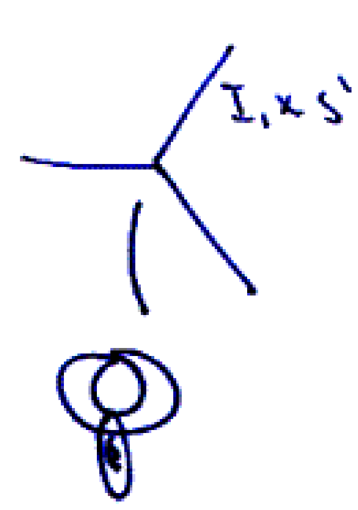
E.g. $\mathbb{R}^4 \to \mathbb{R}^3$ by $\mathbb{R}^1 \times$ Hopf map this is what's happening to transverse space




3) $X = \mathbb{C}^3 \setminus \{1 + z_1 z_2 z_3 = 0\}$

$f : X \to \mathbb{R}^3$

$f(z_1,z_2,z_3) = (|z_1|^2 - |z_2|^2, |z_1|^2 - |z_3|^2, \ln|1 + z_1 z_2 z_3|)$



Corollary of Examples:

Given $B$, $B_0 \subset B$ with $B \setminus B_0$ a graph, and a torus bundle $f_0 : X_0 \to B_0$ which only has the allowable monodromies, then there is a topological compactification $f : X \to B$.

Cor: If $f : X \to B$ is a well-behaved fibration with only semi-stable fibers, then you can dualize: $\check{f} : \check{X} \to B$ exists.

Pf Dualize $f_0 : X_0 \to B_0$ then compactify.

MAIN EX: QUINTIC.

We'll construct $B_0 \subset B$ and an affine structure on $B_0$

Let $\Xi \subseteq \mathbb{R}^4$ be the polytope which is the convex hull of

$(-1,-1,-1,-1)$, $(4,-1,-1,-1)$, ... $(-1,-1,-1,4)$

Let $B = \partial \Xi \cong S^3$

Affine coord charts

1) If $\sigma$ is a 3-face of $\Xi$, $\mathrm{Int}(\sigma)$ sits in an affine hyperplane of $\mathbb{R}^4$.

So we can cover everything but 2-skeleton

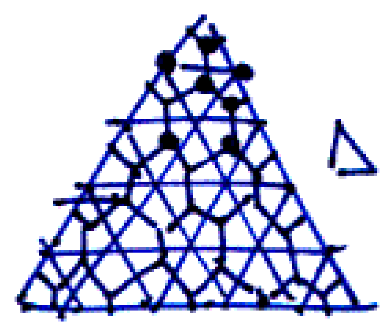
2) For each integral $v \in \partial \Xi$ let $U_v$ be a small open nbhd $U_v \cap U_{v'} = \phi$, $v \neq v'$.

On $U_v$ define $\psi_v : U_v \longrightarrow \mathbb{R}^4 / \mathbb{R}_v$ be the projection.

This is an embedding. Transition maps are linear.

Get affine str on $B_0 = \bigcup(\mathrm{Int}(\sigma)) \cup \bigcup(U_v)$

To choose $B_0$ precisely, choose triangulation of $\Xi$



$\Delta$

$B_0 = B \setminus \Delta$

$dy_1, \dots dy_n$

Compute monodromy of $T_B^*/\Lambda \longrightarrow B_0$.

Natural symp structure

Interior vertices negative.
Edge vertices positive

$X_0 \to B_0$ can be compactified to $X \to B$

We see $\chi(X) = \#\text{pos} - \#\text{neg}$
$= -10 \cdot 25 + 10 \cdot 5 = -200$

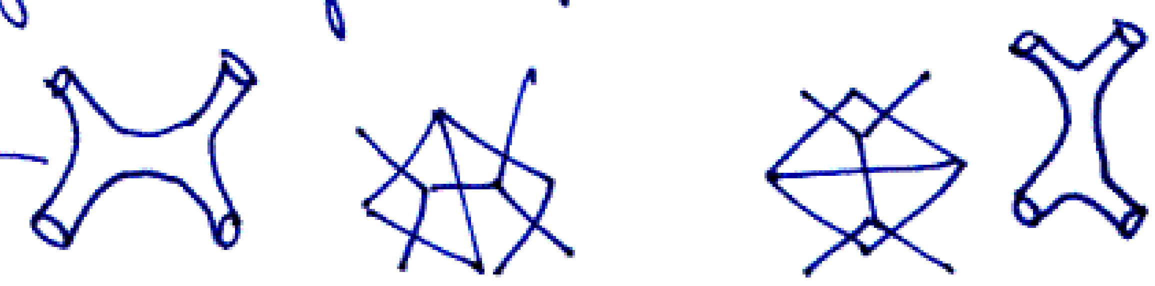
Thm $X$ is diffeomorphic to the quintic.
(Can use Wall's classification of 6-mflds)

Thm $\check{X}$ is diffeomorphic to the mirror quintic.

Ruan claims to construct Lagrangian $T^3$-fibrations on hypersurfaces in toric varieties.

Using diff'lt triangulations, get flop-related mirrors

critical locus



top same critical locus

Doesn't make difference to quintic, but does to mirror

Q: classification

Classification goes like this

sing fiber: $H^1(X_{b_0}, \mathbb{Z}) = H^1(X_b, \mathbb{Z})^{\mathrm{Im}\,\rho}$ follows from well-behaved

$H^i(X_{b_0}, \mathbb{Z}/n\mathbb{Z}) = H^1(X_b, \mathbb{Z}/n\mathbb{Z})^{\mathrm{Im}\,\rho}$ (spectral seqs.)

Suppose $H^i(X_{b_0}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z} & i=1 \\ \mathbb{Z} & i=2 \end{cases}$. (2-dim'l case)

say $e_1 \in H^1$ inv't $e_1, e_2$ basis

$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ Look mod $n$

If $|n| \neq 1$ $H^1(X_{b_0}, \mathbb{Z}/n\mathbb{Z})$
$= (\mathbb{Z}/n\mathbb{Z})^2$