5 July 2001
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Intro to Floer homology

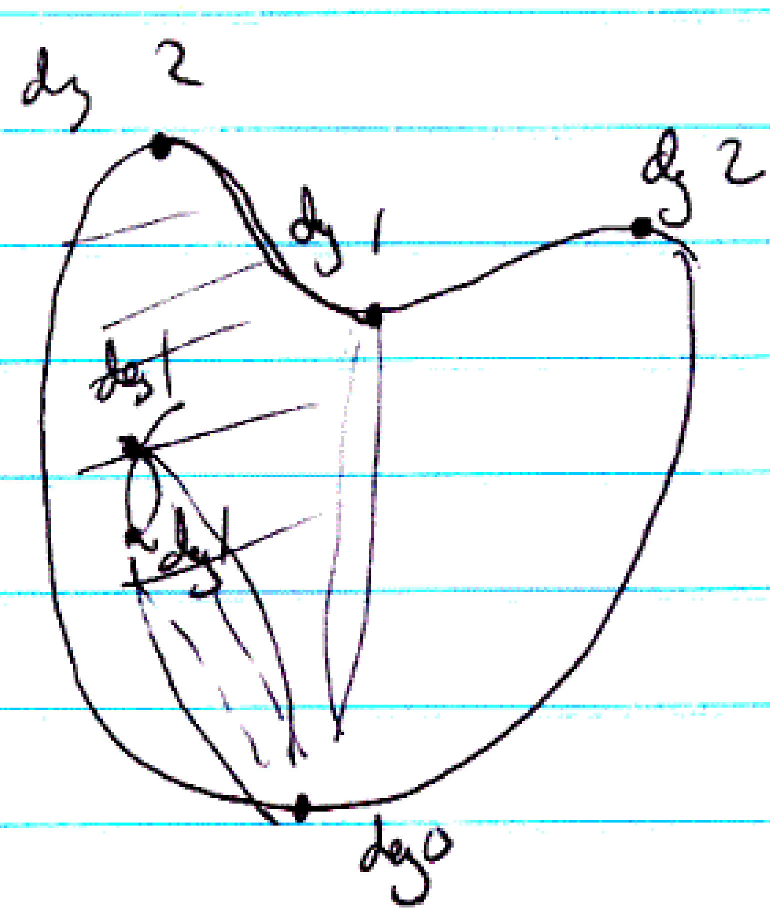
Morse theory $f: X \to \mathbb{R}$

$$\bigoplus_{p_i \in \mathrm{Crit}(f)} \mathbb{R}_i$$

grading degree $\mathrm{ind}_{p_i}(f)$

around p_i, $f = \sum_{i=1}^{n-p} x_i^2 - \sum_{n-p+1}^{n} x_i^2$

∂: $\deg_i \to \deg_{i-1}$ part ← counts gradient flow lines of f from deg i crit pts to deg $i-1$.



$$\begin{array}{ccc} 2 & 1 & 0 \\ \mathbb{R} \xrightarrow{1} & \mathbb{R} & \mathbb{R} \\ \mathbb{R} \xrightarrow{-1} & \mathbb{R} & \\ & \mathbb{R} & \end{array}$$

homology

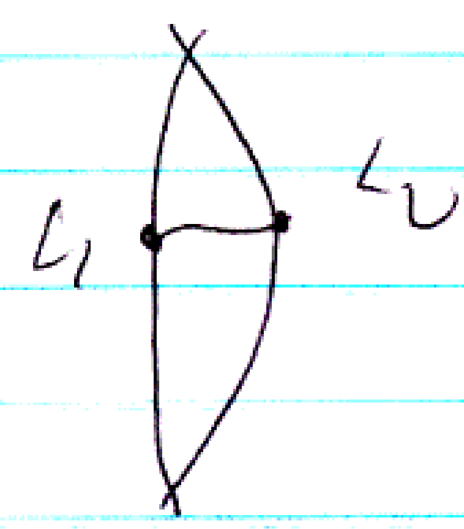
$$\begin{array}{ccc} \mathbb{R} & \mathbb{R} & \mathbb{R} \\ & \mathbb{R} & \\ H_2 & H_1 & H_0 \end{array}$$

Inf. dim $\Rightarrow$ Floer (1985?) homology for Lagrangian intersection

(X^{2n}, ω) symplectic manifold.

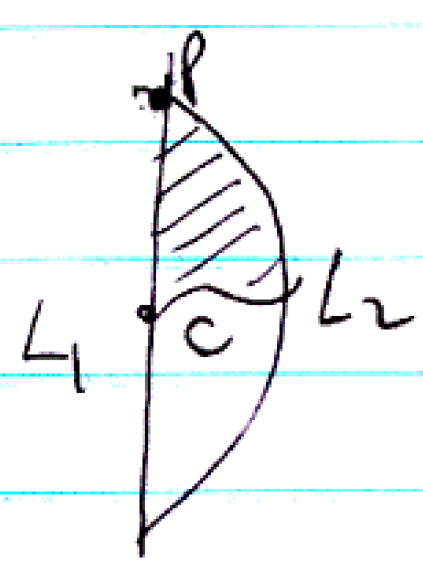
$L^n \subseteq X$ is Lagrangian $\iff \omega|_L \equiv 0$.

L_1, L_2



$M = \{$ curves in X, one boundary on L_1, one bdry on $L_2 \}$.

Symplectic action functional or symplectic area:



$$f(C) = \int_p^C \omega.$$

Gradient flow lines: Choose a metric on X, which gives almost complex structure J.

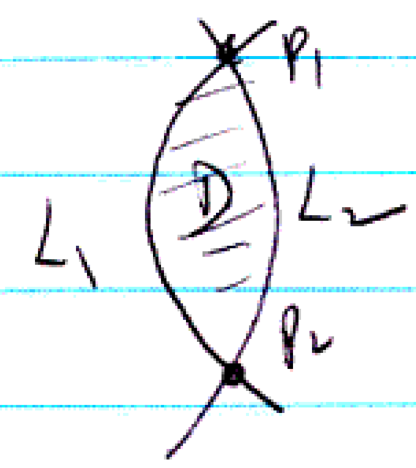
Gradient flow lines are holo discs.

Critical points of f: constant curves C i.e. intersection points of $L_1 \cap L_2$.

Relative indices

Fix index $p_1 \in L_1 \cap L_2$ to be zero

Define index p_2



(given a ~~holomorphic~~ disc D)

$TX|_D$ trivial $\Rightarrow$ trivialization of $K_X|_{\partial D}$
$$\| \quad \Lambda^n_{\mathbb{C}} TX$$

$\Lambda^n_{\mathbb{R}} TL_1 \subseteq \Lambda^n_{\mathbb{C}} TX$ winding no. = Maslov class.
= v. dim. $\mathcal{M}_{\text{holo discs}}(p_1, p_2)$

(vol. winds)

"Count" holo discs in moduli space of dim 0 to get boundary op between crit. pts. whose indices differ by 1.

Complex "computing" homology of space of open strings.

2 probs
- analysis – monotone mflds, Fanos...
  \ CY mflds: can be defined iff a certain obstruction $\in H^2_{\mathbb{R}}(L)$ vanishes (Superpotential is critical)
- grading – different homology classes of disc give different indices. $\Rightarrow HF^*$ is $\mathbb{Z}/N$ graded instead of $\mathbb{Z}$ graded.

<u>CY case</u> – use homology class of D to trivialize K_X;

in CY case this is canonically trivialized by Ω.

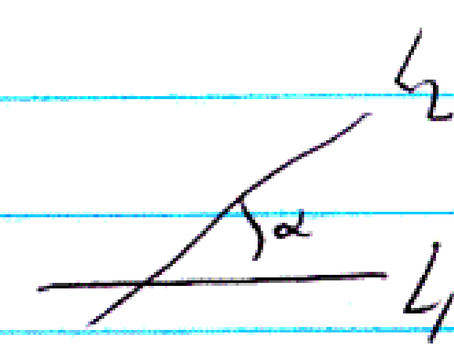
& for Lagrangians of $\underbrace{\text{Maslov class zero}}$, HF is $\mathbb{Z}$-graded.

$\Omega|_L = e^{i\theta}\mathrm{vol}_L$

$(d\theta) \in H^1(L; 2\pi\mathbb{Z})$ is the Maslov class

Graded Lagrangian is (L,θ), $\theta : L \to \mathbb{R}$ and $e^{i\theta}\mathrm{vol}_L = \Omega|_L$.

$\begin{pmatrix} L_1 \\ \theta_1 \end{pmatrix}, \begin{pmatrix} L_2 \\ \theta_2 \end{pmatrix}$, $p \in L_1 \cap L_2$



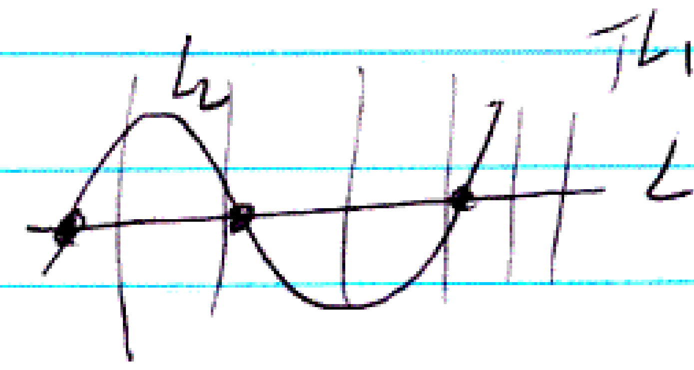
T_pX choose coords x_i, y_i s.t. $L_1 = \{y = 0\}$

$\Omega = dz_1 \cdots dz_n$ $\quad L_2 = \{y_i = (\tan\alpha_i)x_i\}$

$\theta_2 - \theta_1 = \sum \alpha_i + N\pi \quad N \in \mathbb{Z}$

<u>def</u> $\underset{\text{"}N\text{"}}{\mathrm{ind}_p}\big((L_1,\theta_1),(L_2,\theta_2)\big)$

$L_1 \subseteq T^*L_1 \qquad L_2 = \text{graph } df$



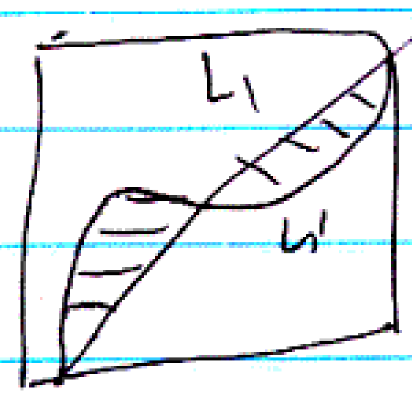
index = Morse index(f)

Holo discs $\longleftrightarrow$ gradient flow of f

(Oh) $HF^*_{\mathbb{R}}(L_1, L_2) = H^*_{\mathbb{R}}(L)$

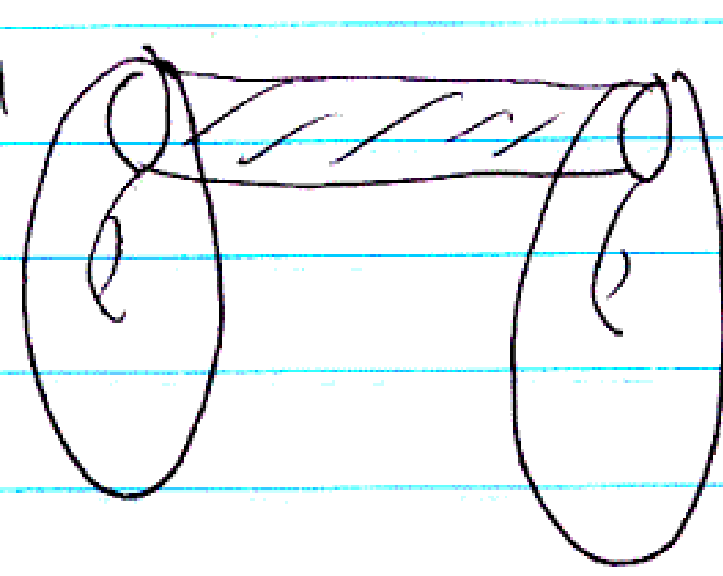
Globally $\exists$ Oh's spectral sequence $H^*(L, \mathbb{R}) \Rightarrow HF^*(L_1, L_2)$

$HF^*(L_1, L_2)$ indept. of hamiltonian defs of L_i



$\int_{L_1}^{L_1'} \omega = 0.$

(cotangent case $\Leftarrow$ closed 1-form is exact)



$\int_{\gamma_1}^{\gamma_2} \omega = 0 \quad \forall \gamma_i \in L_1$

E_1 E_2 vector bundles on CY

HYM HYM

$\updownarrow$

(poly)stable v. bundles

$$H^1(E_2^* \otimes E_1) = Ext^1(E_2, E_1)$$

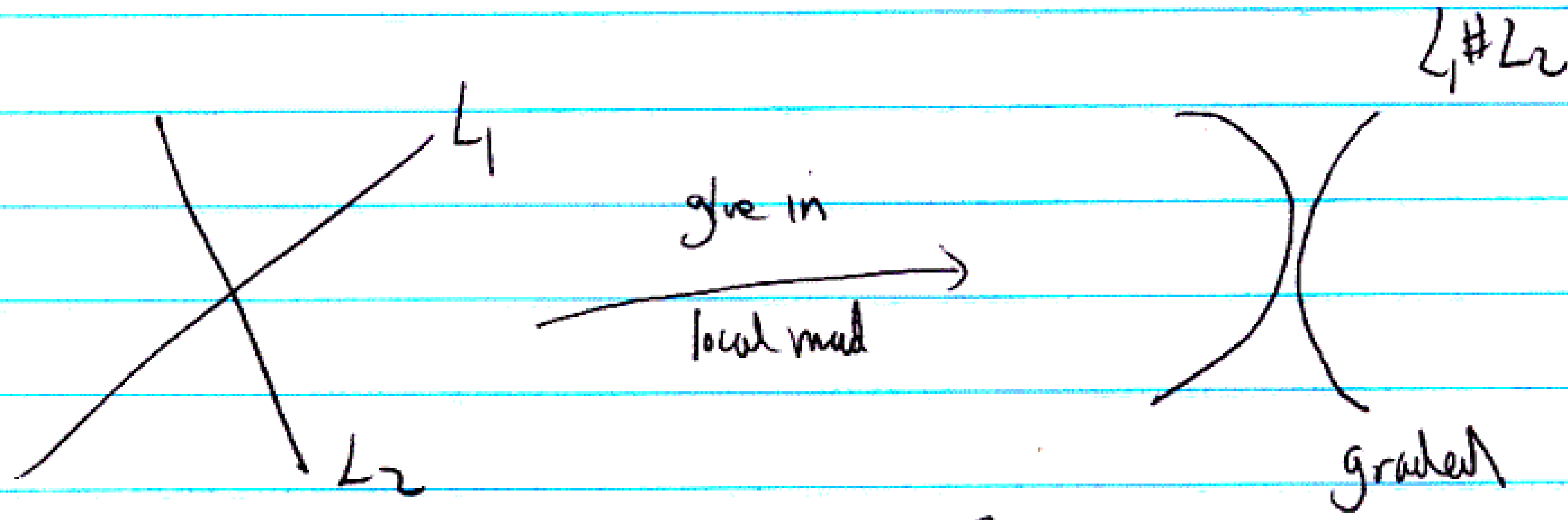
$$0 \to E_1 \to E \to E_2 \to 0$$

Conn. $\begin{pmatrix} A_1 & e \\ 0 & A_2 \end{pmatrix}$ $e \in \Omega^1(Hom(E_2, E_1))$

E stable $\Rightarrow \exists$ gauge transformation to HYM.

L_1, L_2 $\cap$ pts degree 1 HF^1

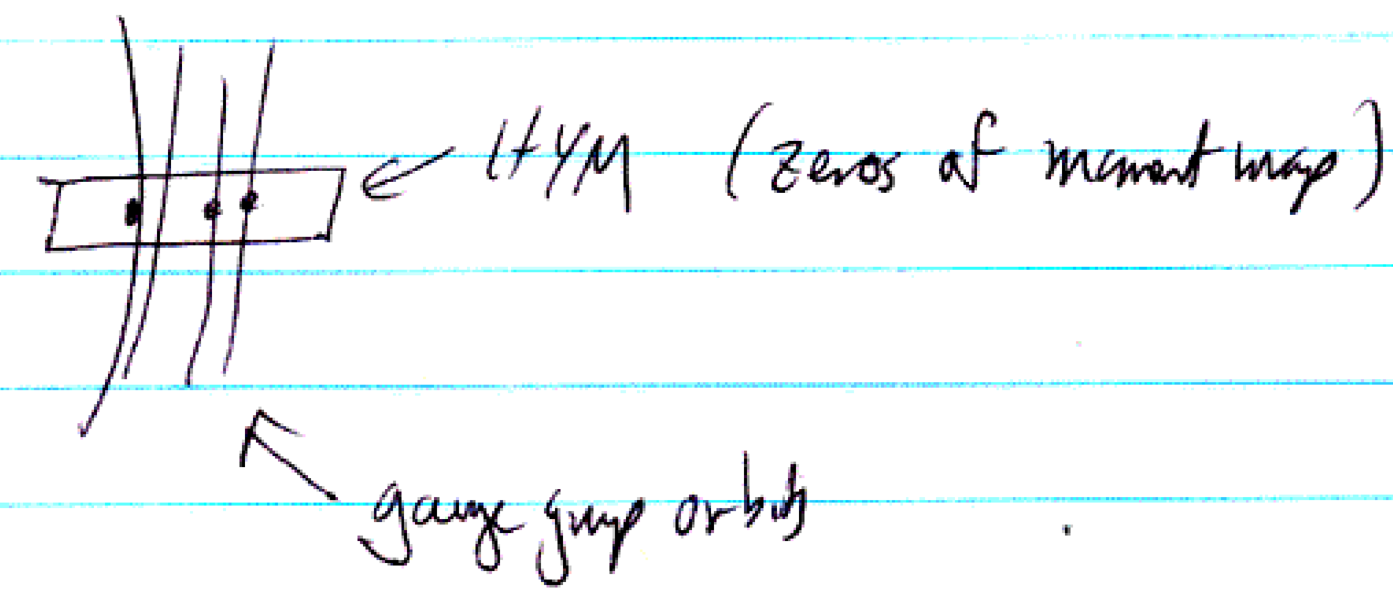
$\exists$ "Lagrangian #" $L_1 \# L_2$



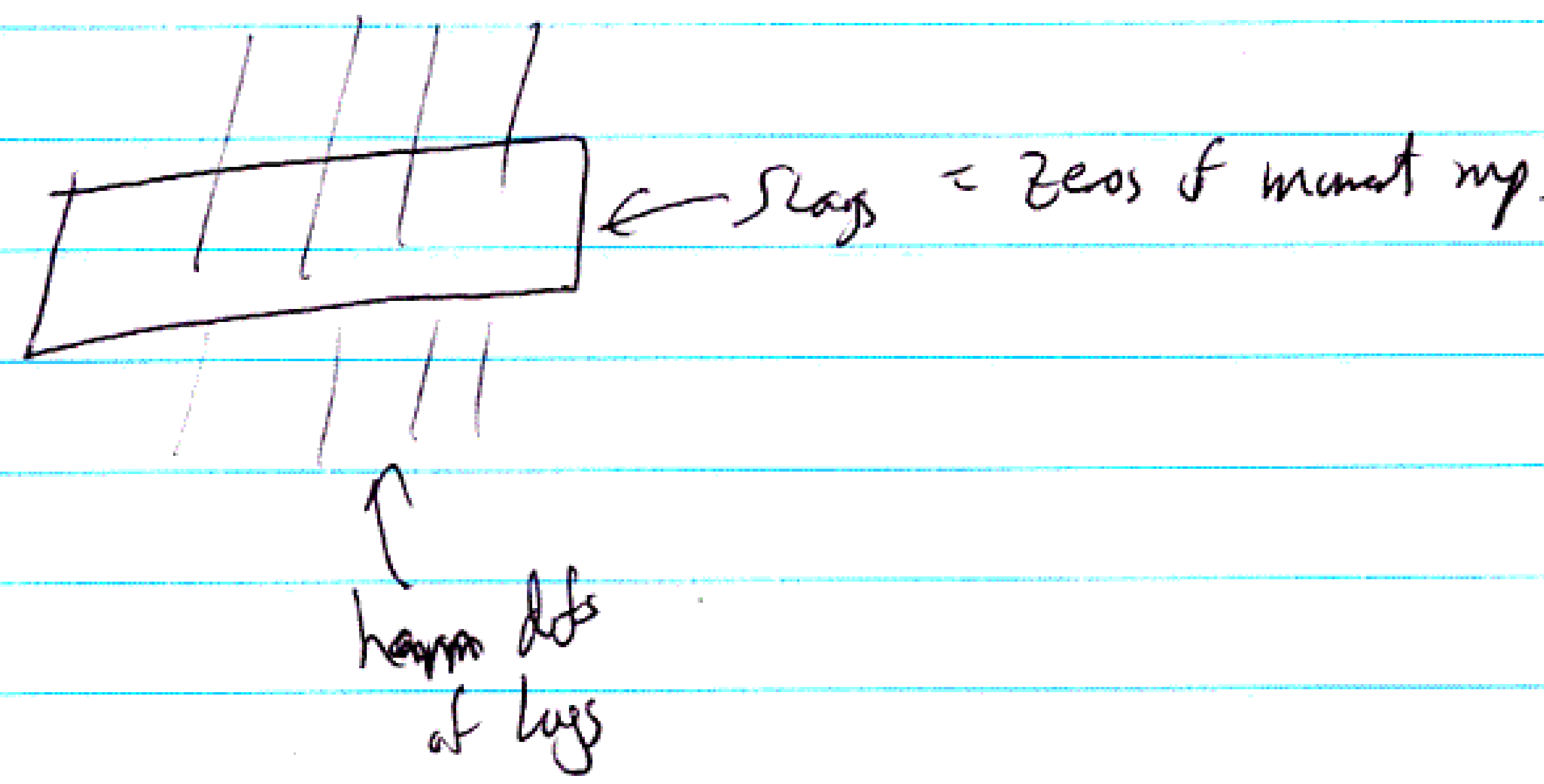
$\xrightarrow{\text{deform?}}$ SLag

HYM   Moment map (Atiyah-Bott, Donaldson)

$\mathcal{A}$ = conns



{Lags}



Thm If $HF^*(L,L)$ is defined then $\exists$ at most one Slag in its ham. def. orbit.

$\dfrac{\text{Lags}}{\text{ham. def}} \simeq$ Slags
↑
Stability

Pf L_1, L_2 smooth Slags, ham. def. equiv.

$HF^*(L_1,L_2) \Leftarrow H^*(L)$

[Fooo] $H^0(L) \cong$

$\exists \cap$ pt of L_1, L_2
f has a minimum

If transverse, deg 0 then locally L_2 = graph $df \subseteq T^*L_1$.
Slag $\Rightarrow d^*df = 0$ $\Rightarrow$ f doesn't have minimum by max principle.

$L_1 \cap L_2$ small conn. cpt of nbd of $L_1 \cap L_2$ in L_1

$X = T^* L_1$

$L_2 = \text{graph}(df)$

$d^\dagger df = 0$

f has min. on bdry of neighborhood (max principle)

Morse theory / handle cancellation

perturb f s.t. it has no interior + is Morse

Ham. def. of L_2 st. all $\cap$ pts. with L_1 have degree $\neq 0$.

$$\boxed{L_1 = L_2}$$